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ANALYSIS OF THRUST EFFECTS ON DIRECTIONAL STABILITY OF A SIMPLIFIED 2-DOF MODEL OF AN ELASTIC ROCKET STRUCTURE

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Abstract. A simplified model of a rocket structure by means of a 2-DOF, lumped parameter Beck column analogue was studied through stability analysis by Lyapunov's first method linearization of the equations of motion around equilibrium under different thrust intensities and spring constants. The equations of motion were generated from the analytical differentiation of the Lagrange's equations with dissipative effects lumped into a Rayleigh function and external forces represented by the action of a single follower force, which is tangent to the nozzle.

Keywords: Vibration, Rocket Thrust, Rocket Stability

1 Introduction

Rockets are tubular shaped, jet propelled systems whose task is to take payload from ground, watercraft or aircraft to another ground/sea location, air target or to the outer space. The payload might be a satellite, scientific instruments, explosives, passengers or cargo. The physical principle behind its motion is the law of conservation of linear momentum of the rocket-gas system, such that the high speed ejection of combustion gas from the nozzle results in propulsion of the rocket in the direction opposite to the gas flow. Diverse mechanical stresses appear on a rocket's structure during its operation such as those created by thrust force, combustion instability, pressure distribution and rigid-body acceleration, as mentioned in Sutton & Biblarz [1].

It is possible to model an elastic rocket structure by considering it on a reference frame stationary to its tip, analogous to a Beck column, under the effect of a follower force which is tangent to the nozzle and whose direction varies continuously as the structure deforms, as studied in Langtjhem [2]. Even though the model is bidimensional, reflects a worst-case scenario in terms of axial oscillation. Furthermore, the rocket structure is axisymmetric and axial stresses are dominant because of the great thrust force generated by the propulsion system.

A discretized model of three point masses with two rotational degrees-of-freedom (2-DOF) and constant follower force was generated as presented in Brejão & Brasil [3]. In that presentation, the equations of motion have been symbolically obtained by the use of Maclaurin polynomials of 3rd order in Lagrange's equation and numerically integrated by the use of a Runge-Kutta 4th and 5th order integrator. Variation of dissipation and thrust parameters resulted in marginal or asymptotic stability.

The present work furthered the investigation of the effects of the follower force in such 2-DOF model. The equations of motion have been obtained by the automated symbolic manipulation and solution of the Lagrange equations by the use of the input Lagrangian function in its full non-linear form as a way to preserve nonlinearity until the first-order ODE system could be obtained. Once obtained, such ODEs were subjected to Lyapunov's first method by linearization around the undeformed configuration and could also be integrated by the same Runge-Kutta 4th and 5th order integrator for any example case of state configuration.

2 Modelling

The studied model consists of three point masses m_1 located at the tip, $m_2 = 2m_1$ located at the middle and $m_3 = m_1$ located at the nozzle, as to reflect the mass distribution of the rocket. Each mass is linked to the next by an inextensible bar of length l and each rotational DOF has a spring k_i and damper c_i. A schematic of the model as given in [3] is shown next in Fig. 1:

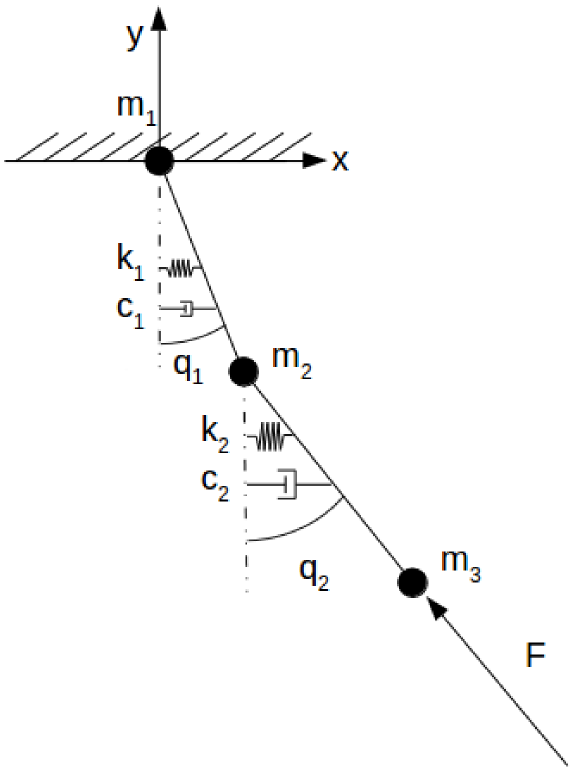

Figure 1. 2-DOF pendulum model of rocket under follower force

The required Lagrange's equations for the problem are therefore those who account for dissipative terms arising from dampening by means of a Rayleigh function and the follower thrust by means of an external, non-conservative generalized force. Usually, either the Rayleigh function or the generalized force approach is used in analytical mechanics, as in Lemos [4], but the present problem requires both to avoid resorting to Lagrange multipliers to treat coupling of the original four (or six, if the tip mass movement is taken into account) coordinates, a pair for each movable mass.

The i = 1,2 Lagrange equations have the format shown in Eq. 1:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i^{NC} \quad , \tag{1}$$

where L is the Lagrangian, R is the Rayleigh dissipation function and Q_i^{NC} are the external generalized forces which cannot be accounted for in R.

2.1 Further simplifying hypotheses

The bars were assumed as having equal length, the spring coefficients were assumed equal and constant and so were the viscous damping coefficients. Even though any $q_1 = q_2$ system could be thought of as axially stable, in practice there is some control over the rocket thrust as to direct its tip in a specific direction, in this case the y axis. The xy reference system is also assumed as only translating.

Gravitational effects were ignored, even though they might influence system stability, because the thrust-to-weight ratios of some rockets might achieve values as large as 10 to 100 as reported in Sutton & Biblarz [1], but not always, so low-thrust propulsive systems must take these effects in account.

Finally, mass, thrust and therefore their ratio are assumed constant. Even though typical large booster rockets have short burn rates of about 100 s and typical mass ratios of at least 0.5 as in Sutton & Biblarz [1], choosing appropriate mass values might account for worst-case scenarios. Constant thrust refers to choked flow on the nozzle, where the mass flow can only by increased by pressure and temperature differences [5].

2.2 Derivations of the equations of motion

Under the forementioned hypotheses, the Lagrangian for use in Eq. 1 can be defined as following for the j = 1,2,3 masses and i = 1,2 angular displacements in Eq. 2:

$$L = T - V = \frac{1}{2}\sum_{j} \boldsymbol{v_j}\cdot\boldsymbol{v_j} - \frac{1}{2}\sum_{i} k_i q_i^2 \quad , \tag{2}$$

where T is the kinetic energy term, V is potential energy term, **v** are the velocity vectors for each mass, k are the spring constants and q are the angular displacements for each DOF. The Rayleigh dissipation function can be defined as in Eq. 3:

$$R = \frac{1}{2}\sum_{i} c_i \dot{q}_i^2 \quad , \tag{3}$$

where c are the damping constants. The positions vectors are given by the equations shown in Eq. 4:

$$\begin{cases} \boldsymbol{r_1} = 0\boldsymbol{i} + 0\boldsymbol{j} \\ \boldsymbol{r_2} = l\sin q_1 \boldsymbol{i} - l\cos q_1 \boldsymbol{j} \\ \boldsymbol{r_3} = \boldsymbol{r_2} + l\sin q_2 \boldsymbol{i} - l\cos q_2 \boldsymbol{j} = l(\sin q_1 + \sin q_2)\boldsymbol{i} - l(\cos q_1 + \cos q_2)\boldsymbol{i} \end{cases} \quad , \tag{4}$$

where l is each bar length and q are the generalized coordinates. Velocity vectors for use in Eq. 2 can be obtained by simple differentiation of position vectors and are shown in Eq. 5:

$$\begin{cases} \boldsymbol{v_1} = 0\boldsymbol{i} + 0\boldsymbol{j} \\ \boldsymbol{v_2} = \dot{q}_1 l\cos q_1 \boldsymbol{i} + \dot{q}_2 l\sin q_1 \boldsymbol{j} \\ \boldsymbol{v_3} = l(\dot{q}_1\cos q_1 + \dot{q}_2\cos q_2)\boldsymbol{i} + l(\dot{q}_1\sin q_1 + \dot{q}_2\sin q_2)\boldsymbol{i} \end{cases} \quad . \tag{5}$$

The thrust force vector can be described as in Eq. 6:

$$\boldsymbol{F} = -F(t)\frac{(\boldsymbol{r_3}-\boldsymbol{r_2})}{|\boldsymbol{r_3}-\boldsymbol{r_2}|} = F(t)(\sin q_2\boldsymbol{i} - \cos q_2\boldsymbol{j}) \quad , \tag{6}$$

where $F(t)$ is the thrust intensity function, supposed dependent only on time and not on the configuration space. The generalized force that acts on the masses can be obtained by application of the principle of virtual work [4], as in Eq. 7:

$$Q_i^{NC} = \sum_{i} \boldsymbol{F_j}\cdot\frac{\partial \boldsymbol{r_j}}{\partial q_i} \quad , \tag{7}$$

the resulting generalized forces for this problem are given in Eq. 8:

$$Q_1^{NC} = F(t)\, l\sin(q_2-q_1);\ Q_2^{NC} = 0 \quad , \tag{8}$$

noting that **F** acts only on m_3.

The resulting equations of motion have been obtained by the symbolic manipulation and differentiation of Eq. 1 using MATLAB® [6] are shown in Eqs. 9:

$$\begin{aligned} &3ml^2\ddot{q}_1 + c\dot{q}_1 + kq_1 + ml^2[\sin(q_1-q_2) + \cos(q_1-q_2)]\dot{q}_2^2 = F(t)\, l\sin(q_2-q_1) \\ &ml^2\ddot{q}_2 + c\dot{q}_2 + kq_2 + ml^2[\sin(q_2-q_1)\dot{q}_1^2 + \cos(q_1-q_2)\ddot{q}_1] = 0 \end{aligned} \tag{9}$$

which can be readily seen to be non-linear, particularly it is strongly nonlinear on the second equation.

2.3 Lyapunov first method for nonlinear stability analysis

Lyapunov's first (or direct) method can be applied to a nonlinear system about an equilibrium vector state **x** = **x₀** as given by Kozlov & Furta [7] in Eq. 10:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \quad , \tag{10}$$

where A is a jacobian matrix evaluted at $\mathbf{x_0}$ of the full nonlinear functional relationship between the state vector and its time derivative, seen in Eq. 11:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t) \quad . \tag{11}$$

Therefore, A can be calculated by Eq. 12:

$$A_{ij}(\boldsymbol{x_0}) = \left(\frac{\partial f_i}{\partial x_j}\right)_{\boldsymbol{x}=\boldsymbol{x_0}} \tag{12}$$

and if the eigenvalues of A evaluated at a equilibrium point $\mathbf{x_0}$ have only negative real parts, then the nonlinear system can be said to be asymptotically stable about $\mathbf{x_0}$.

2.4 Assumed values for the damping constant (c)

Damping constants for structures are in the range of $\zeta = 0.01$ as pointed by Craig & Kurdila [8]. Assuming linearization around the only admissible stable point for the present analysis $q_1 = q_2 = 0$, the two uncoupled modes of axial oscillation have the same damping constant given by Eq. 13:

$$c = 2\sqrt{km}\zeta = 0.02\sqrt{km} \quad , \tag{13}$$

where k is the spring constant and m is a measure of inertia associated with each DOF and is naturally a function of the lumped masses.

2.5 Treatment of the spring constant (k)

Each DOF behaves modally around linearization as in Eq. 14:

$$ml^2\ddot{q}_i + c\,\dot{q}_i + k\,q_i = F(t)f(\boldsymbol{q})\,l \;\Rightarrow\; \ddot{q}_i + \frac{c}{ml^2}\dot{q}_i + \frac{k}{ml^2}q_i = \frac{F(t)}{ml}f(\boldsymbol{q}) \quad , \tag{14}$$

where f is a non-dimensional function of the DOFs. Choosing k^* as a measure of rigidity defined in Eq. 15 as:

$$k^* = \frac{k}{ml^2} \tag{15}$$

and it becomes possible to redefine c as in Eq. 16:

$$c^* = \frac{c}{ml^2} = 0.02\sqrt{\frac{k}{ml^4}} = 0.02\frac{\sqrt{k^*}}{l} \quad . \tag{16}$$

Finally, the thrust force is redefined in Eq. 17 :

$$F^* = \frac{F}{ml} = 0.02\sqrt{\frac{k}{ml^4}} = 0.02\frac{\sqrt{k^*}}{l} \tag{17}$$

and the configuration space is reduced to the analysis of F^* and k^*.

3 Results

Graphs of the real part of the eigenvalues of matrix A (as defined in Eq. 12) for different constants F^* and k^* are shown below in Figs. 1, 2, 3 and 4:

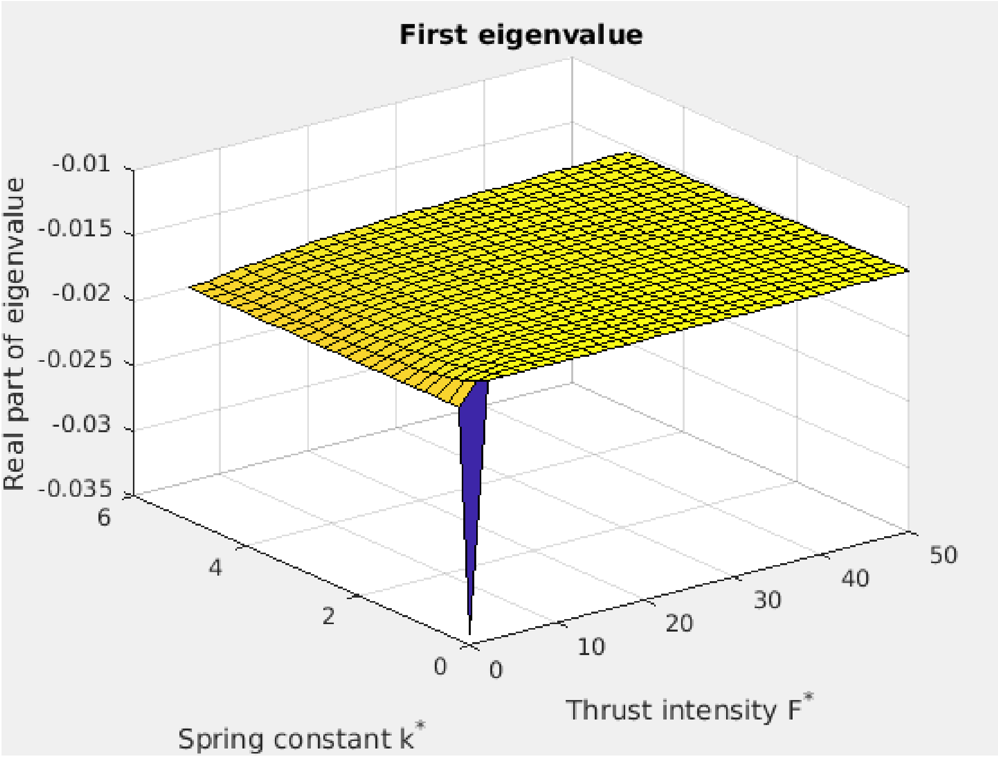

Figure 1. First eigenvalue of matrix *A* as a function of thrust intensity and spring constant

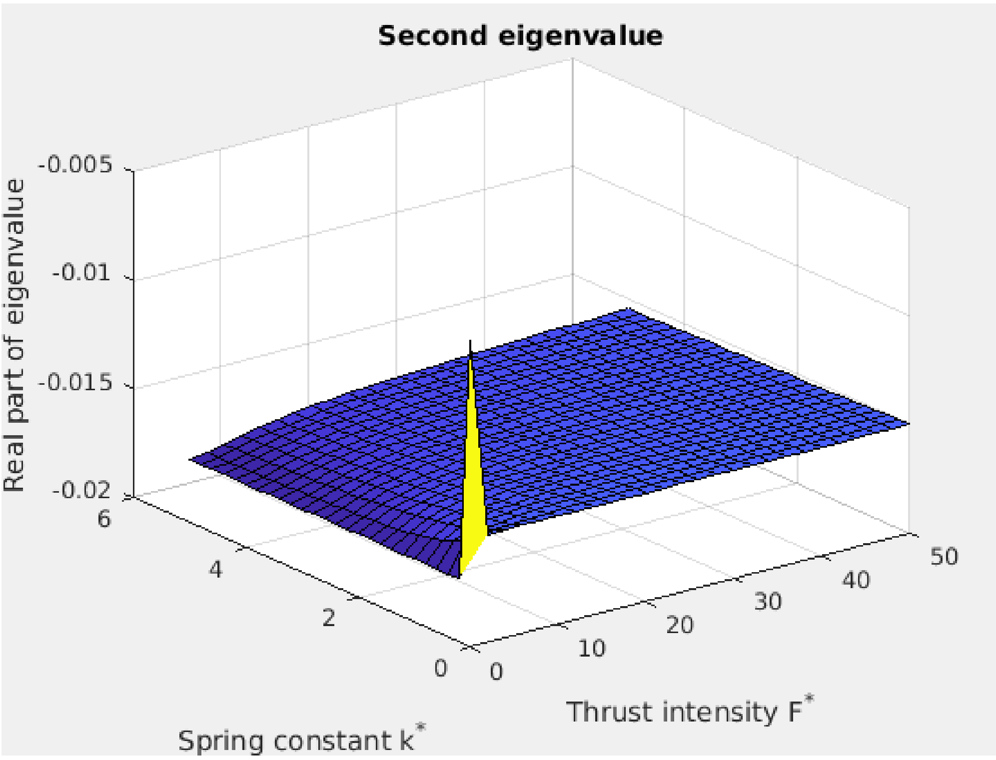

Figure 2. Second eigenvalue of matrix *A* as a function of thrust intensity and spring constant

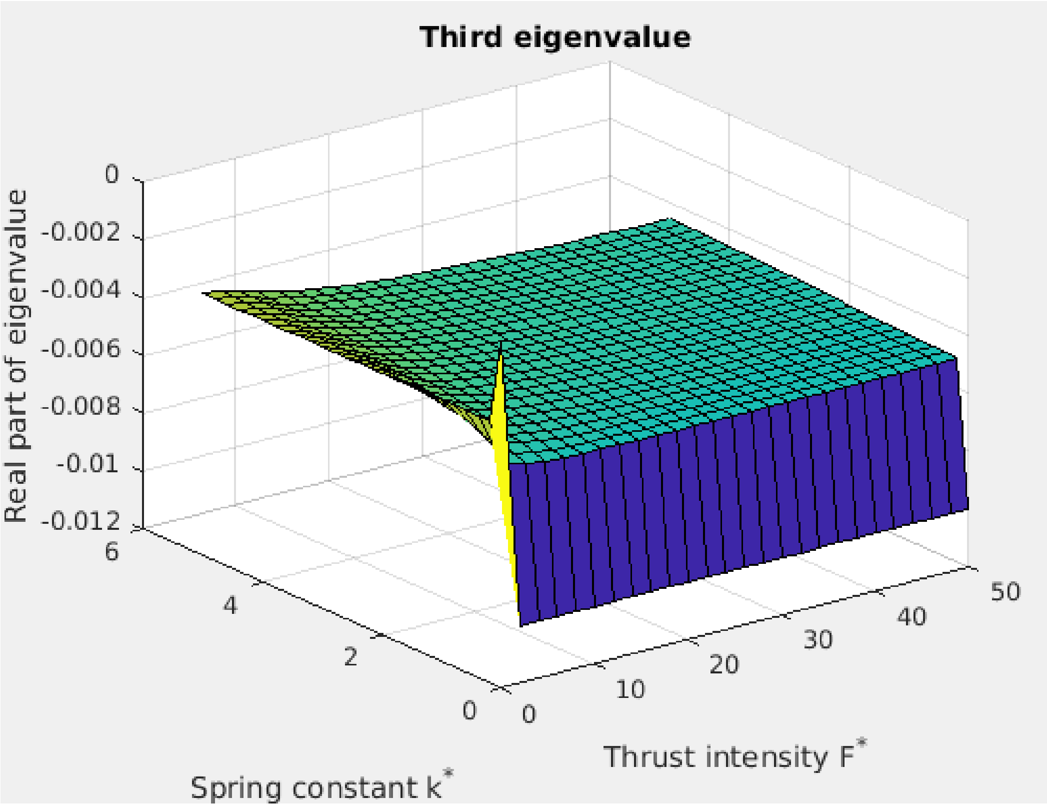

Figure 3. Third eigenvalue of matrix A as a function of thrust intensity and spring constant

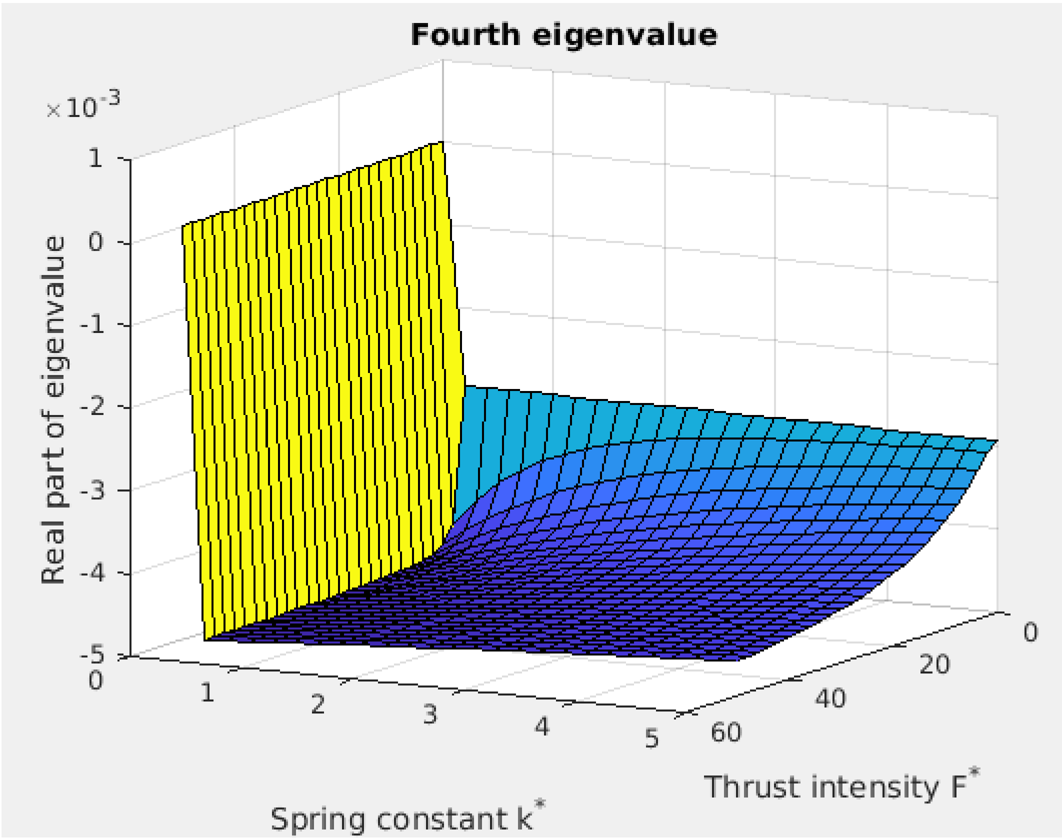

Figure 4. Fourth eigenvalue of matrix A as a function of thrust intensity and spring constant

It can be thus seen that the thrust force actually contributes to the stability about the zero point. That could be explained by the fact that any deviation of the $\mathbf{x_0}$ = (0,0,0,0) is corrected by the opposing action of the thrust force. The constant thrust contribution to stabilization vanishes very quickly with increasing intensity and seems to be largely independent of spring constant values. In the limit of zero thrust, the system behaves as a damped double pendulum with a very low damping coefficient (as seen by the "spikes" in the origin of the graphs).

As an example, the coordinate space curve is shown for $k^* = 1$, $F^* = 1$, starting conditions $q_1 = q_2 = 0.01$, zero angular velocities and time between 0 and 100 s in Fig. 5:

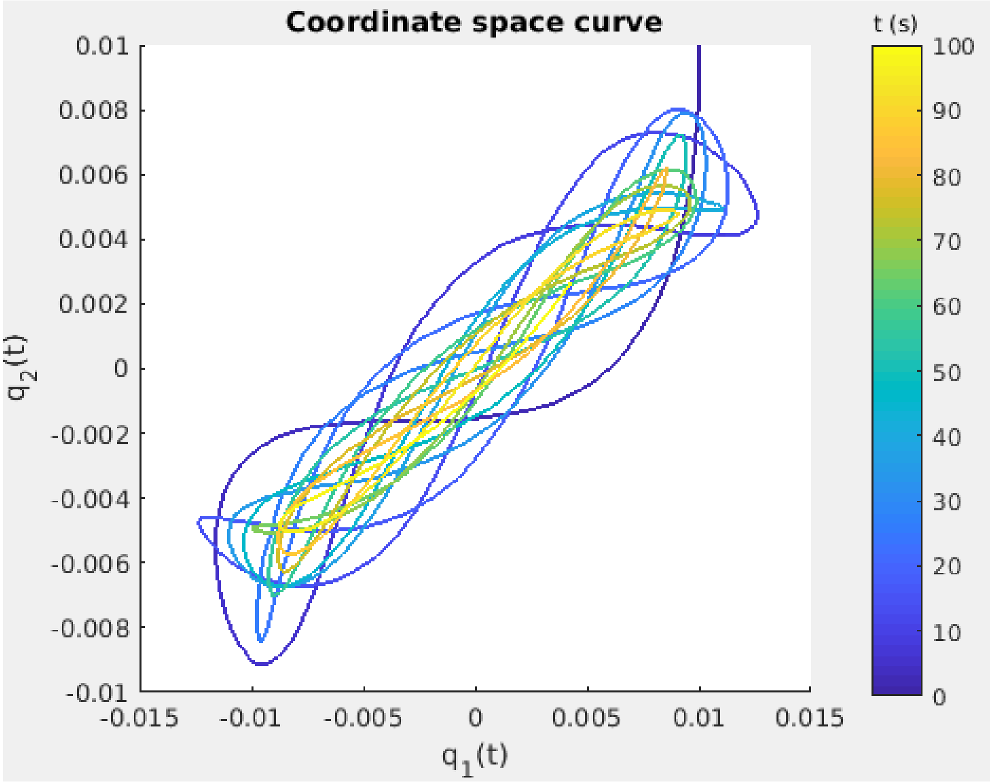

Figure 5. Coordinate space curve for example parameters

The graph shows that the curve tends not only to stabilize around the origin with increasing time, but the coordinate curve tends to $q_1 = q_2$, that is, the "pendulum" swings as a whole. The effects of this "whole" structure vibration have not been studied in the present paper and might be a topic of future research.

Phase state graphs for both generalized coordinates could also be generated, as shown in Figs. 6 and 7:

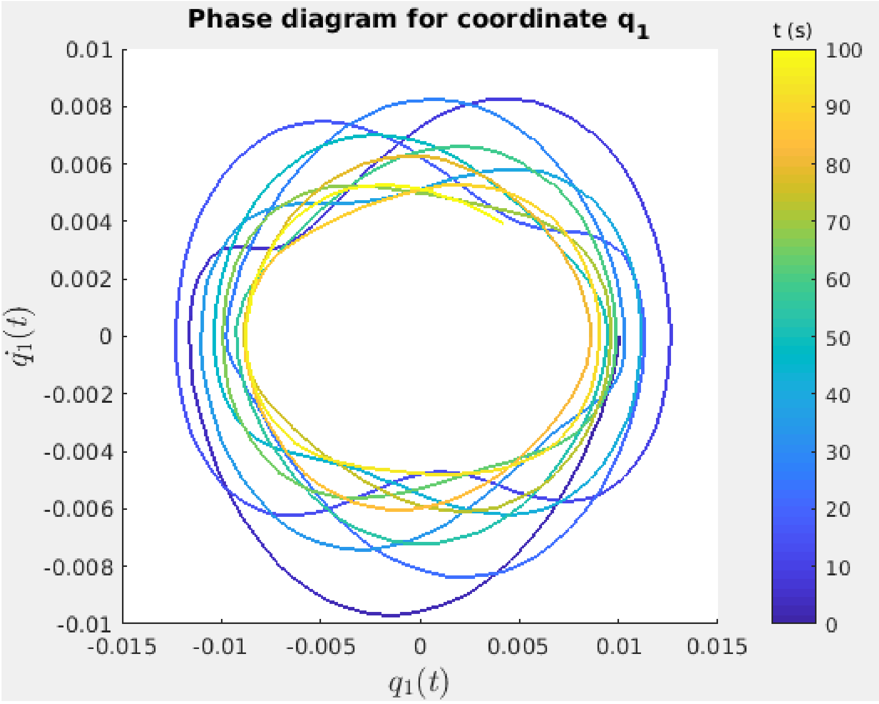

Figure 7. Phase diagram for coordinate q_1

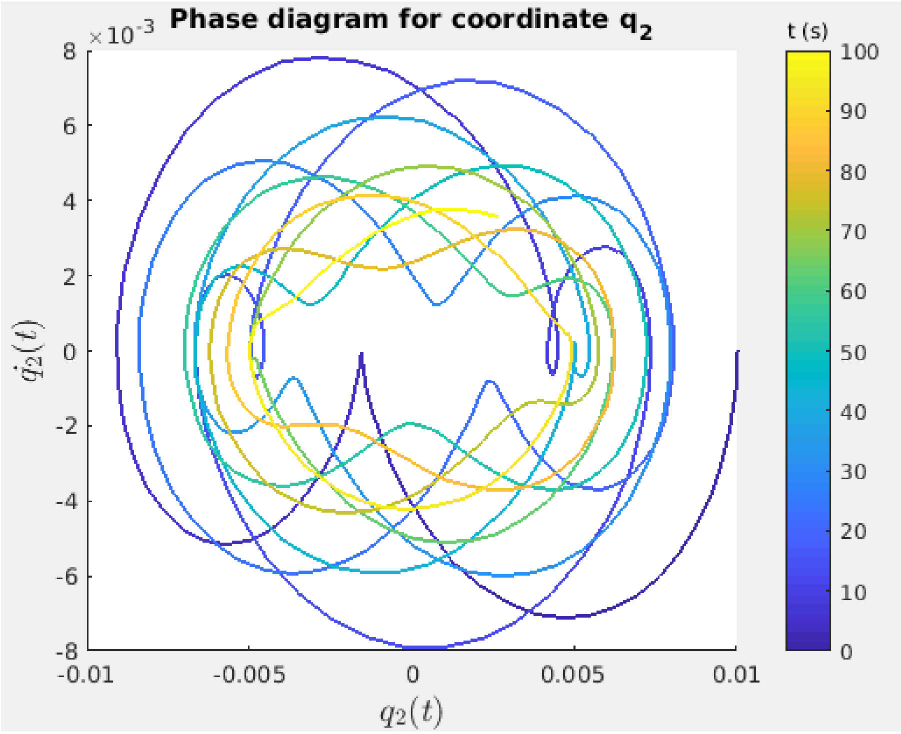

Figure 8. Phase diagram for coordinate q_2

The preceding phase diagrams show tendencies to asymptotic stability as they converge slowly to the equilibrium position of the system.

4 Conclusions

We present a study of stability of a 2-DOF nonlinear lumped parameters Beck's Column model of a rock structure excited by its thrust, considered as a follower force. The equations of motion were derived via Lagrange's Equations and symbolic computation. Stability was investigated using Lyapunov's first method and results show that thrust acts as a stabilizer in axial oscillations, although its frequency effects on structure have not been yet studied.

Future research could also incorporate the effects of variable mass and variable thrust intensities in the stability analysis, considering gravitational, combustion, aerodynamic and aeroelastic effects (and their variation in time) as well as increasing the number of angular DOF or even considering a continuous bar model for the rocket.

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