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MATHEMATICAL MODELING AND DYNAMICS OF TWO TETHERED SATELLITES: RIGID BODY APPROACH

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Abstract. A two-dimensional nonlinear mathematical model for two tethered satellites is developed. This complex system comprises of a long cable (also known as tether or, in this case, space tether) connecting two masses (satellites). Tethered satellites can be used in a variety of space applications such as electrodynamic propulsion, energy harvesting, momentum exchange, artificial gravity, etc. As a first rough mathematical model, the cable connecting the satellites is approximated by two connecting rod-like rigid bodies. If these rods are not aligned, it is assumed that the cable is not stretched (i.e. the cable is not under tension). This is an undesirable situation for this type of system. The whole system is allowed to rotate and translate only on a two-dimensional space. The set of ordinary differential governing equations of motion are obtained using the Lagrange´s equations approach. These nonlinear equations are numerically integrated and the dynamics of the system is investigated under several practical circumstances.

Keywords: nonlinear dynamics, rigid bodies, mathematical modeling, space tether, tethered satellites.

1 Introduction

Space tethers are nonlinear dynamical systems of great complexity in mathematical modeling and position and vibration control. In fact, the accuracy required in these mathematical models is directly related to the specific objectives to be achieved and the specific phenomena to be investigated [1]. A space tether model, for example, may consider nonlinear couplings between the deformations of the flexible tether, the rotational dynamics of the rigid bodies (in the tether extremities), and its orbital dynamics [2]. The stability of this class of dynamical system is also a very important issue to be investigated [3].

The space tether systems usually involve two rigid bodies or two point masses (two satellites, two rigid spacecraft) connected by a flexible cable as long as several thousands of kilometers. It was developed to transport payloads up and down without any propellant and many other specialised missions such as asteroid rendezvous and electrical generation in the upper atmosphere [4].

A mathematical model of some complexity based on a rigid body approach is developed in this paper. Only the tether dynamics is presented here.

2 Governing equations of motion of the tethered satellites: rigid body approach

The geometric model of the system investigated in this work is presented in Figure 1. This system comprises two rigid bodies (the satellites A and B) connected to a long cable (here represented by rods 1 and 2). The rods are treated as rigid bodies. The system is free to move in the horizontal plane (the gravity gradient are not considered here).

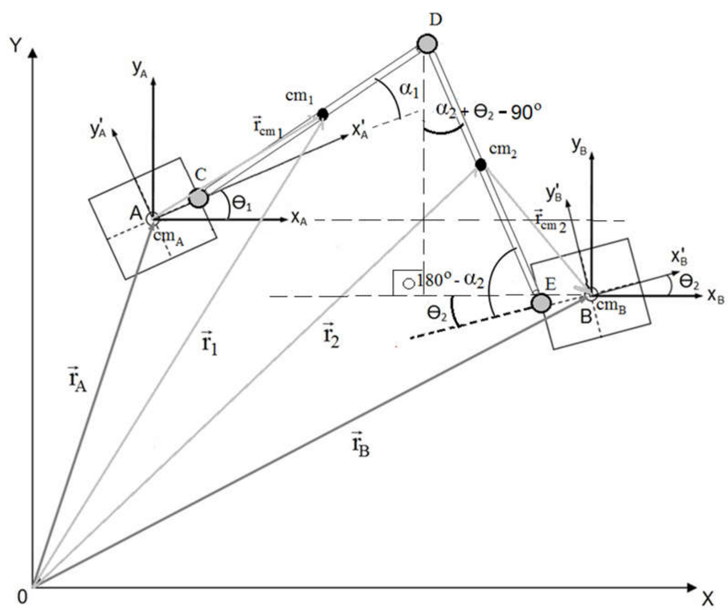

Figure 1. The space tether system.

In this figure, the inertial axis is represented by XY and all the other axis are moving axis (attached to the satellites).

The governing equations of motion are obtained through the energy method named Lagrange's equations [5]. In order to apply this method one needs to know the kinetic and potential (strain) energies stored in the space tether system (cable and satellites) during its time evolution.

The kinetic energy, T, of the space tether is given by:

$$T = T_A + T_B + T_1 + T_2 \,. \tag{1}$$

In Eq. (1), T_A is the kinetic energy of the body with the center of mass at point A, T_B is the kinetic energy of the body with the center of mass at point B, T_1 is the kinetic energy of the rigid body named rod 1 and T_2 is the kinetic energy of the rigid body named rod 2.

Expanding the terms in Equation (1) one obtains:

$$T = \frac{1}{2} m_A \left|\dot{\vec{r}}_A\right|^2 + \frac{1}{2} I_A \dot{\theta}_1^2 + \frac{1}{2} m_B \left|\dot{\vec{r}}_B\right|^2 + \frac{1}{2} I_B \dot{\theta}_2^2 + \frac{1}{2} m_1 \left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2} I_1 \left(\dot{\theta}_1 + \dot{\alpha}_1\right)^2 + \frac{1}{2} m_2 \left|\dot{\vec{r}}_2\right|^2 + \frac{1}{2} I_2 \left(\dot{\theta}_2 + \dot{\alpha}_2\right)^2 \,. \tag{2}$$

In Eq. (2), m_A is the mass of the body with the center of mass at point A, I_A is the moment of inertia around cm_A of the body with the center of mass at point A, m_B is the mass of the body with the center of mass at point B, I_B is the moment of inertia around cm_B of the body with the center of mass at point B, m_1 is the mass of the rigid body named rod 1, I_1 is the moment of inertia around cm_1 of the rigid body named rod 1, m_2 is the mass of the rigid body named rod 2 and I_2 is the moment of inertia around cm_2 of the rigid body named rod 2. The dot over the variables denotes differentiation with respect to the time t.

Expanding the velocity vectors and regrouping terms, Eq. (2) can be rewriten as:

$$\begin{aligned}
T = {} & \frac{1}{2} c_1 \dot{x}_A^2 + \frac{1}{2} c_1 \dot{y}_A^2 + \frac{1}{2} c_2 \dot{\alpha}_1^2 + \frac{1}{2} c_3 \dot{\theta}_1^2 + c_4 \dot{y}_A \dot{\theta}_1 - c_5 \dot{x}_A \dot{\theta}_2 + \frac{1}{2} c_6 \dot{\theta}_2^2 + c_7 \dot{\alpha}_2 \dot{\theta}_2 + \\
& c_8 \dot{y}_A \dot{\theta}_2 - c_9 \dot{x}_A \dot{\theta}_1 - c_{10} \dot{x}_A (\dot{\alpha}_1 + \dot{\theta}_1) + c_{11} \dot{\alpha}_1 \dot{y}_A + c_{12} \dot{x}_A (\dot{\alpha}_2 + \dot{\theta}_2) + c_{13} \dot{\alpha}_1 \dot{\theta}_1 + \\
& c_{14} \dot{y}_A (\dot{\alpha}_1 + \dot{\theta}_1) - c_{15} \dot{\alpha}_1 \dot{x}_A + \frac{1}{2} c_{16} \dot{\alpha}_2^2 - c_{17} \dot{y}_A (\dot{\alpha}_2 + \dot{\theta}_2) + c_{18} \dot{\theta}_1 (\dot{\alpha}_1 + \dot{\theta}_1) + c_{19} \dot{\theta}_1 \dot{\theta}_2 - \\
& c_{20} \dot{\theta}_1 (\dot{\alpha}_2 + \dot{\theta}_2) + c_{21} \dot{\theta}_2 (\dot{\alpha}_1 + \dot{\theta}_1) - c_{22} \dot{\theta}_2 (\dot{\alpha}_2 + \dot{\theta}_2) - c_{23} (\dot{\alpha}_1 + \dot{\theta}_1)(\dot{\alpha}_2 + \dot{\theta}_2) - \\
& c_{24} (\dot{\alpha}_1 \dot{\alpha}_2 + \dot{\alpha}_1 \dot{\theta}_2) + c_{25} (\dot{\alpha}_1 \dot{\alpha}_2 + \dot{\alpha}_2 \dot{\theta}_1) + c_{25} (\dot{\theta}_1 \dot{\theta}_2 + \dot{\alpha}_1 \dot{\theta}_2) - c_{26} (\dot{\theta}_1 \dot{\theta}_2 + \dot{\alpha}_2 \dot{\theta}_1) - \\
& c_{27} (\dot{x}_A \dot{\theta}_1 + \dot{x}_A \dot{\alpha}_1) - c_{28} (\dot{y}_A \dot{\theta}_2 + \dot{y}_A \dot{\alpha}_2) + c_{29} (\dot{y}_A \dot{\theta}_1 + \dot{y}_A \dot{\alpha}_1) + c_{30} (\dot{x}_A \dot{\theta}_2 + \dot{x}_A \dot{\alpha}_2)
\end{aligned} \tag{3}$$

where:

$c_1 = m_A + m_2 + m_B + m_1$

$c_2 = I_1 + m_1 d_{Ccm_1}^2 + m_2 L_1^2 + m_B L_1^2$

$c_3 = I_A + I_1 + m_1 (d_{Ccm_1}^2 + d_{cm_AC}^2 + 2 d_{cm_AC} d_{Ccm_1} \cos\alpha_1) + m_B (d_{AC}^2 + L_1^2 + 2 L_1 d_{AC} \cos\alpha_1) +$
$\quad m_2 (L_1^2 + d_{AC}^2 + 2 L_1 d_{AC} \cos\alpha_1)$

$c_4 = m_B d_{AC} \cos\theta_1 + m_2 d_{AC} \cos\theta_1 + m_1 [d_{cm_AC} \cos\theta_1 + d_{Ccm_1} \cos(\alpha_1 + \theta_1)]$

$c_5 = m_B d_{EB} \sin\theta_2$

$c_6 = I_B + I_2 + m_B (d_{EB}^2 + L_2^2) + m_2 d_{Dcm_2}^2$

$c_7 = I_2 + L_2^2 m_B + m_2 d_{Dcm_2}^2$

$c_8 = m_B d_{EB} \cos\theta_2$

$c_9 = m_B d_{AC} \sin\theta_1 + m_2 d_{AC} \sin\theta_1 + m_1 [d_{cm_AC} \sin\theta_1 + d_{Ccm_1} \sin(\alpha_1 + \theta_1)]$

$c_{10} = m_B L_1 \sin(\alpha_1 + \theta_1)$

$c_{11} = m_1 d_{Ccm_1} \cos(\alpha_1 + \theta_1)$

$c_{12} = m_B L_2 \sin(\alpha_2 + \theta_2)$

$c_{13} = I_1 + m_B L_1^2 + m_1 (d_{Ccm_1}^2 + d_{cm_AC} d_{Ccm_1} \cos\alpha_1) + m_2 (L_1^2 + L_1 d_{AC} \cos\alpha_1)$

$c_{14} = m_B L_1 \cos(\alpha_1 + \theta_1)$

$c_{15} = m_1\, d_{Ccm1} \sin(\alpha_1 + \theta_1)$
$c_{16} = I_2 + m_2 d_{Dcm2}^2 + m_B L_2^2$
$c_{17} = m_B L_2 \cos(\alpha_2 + \theta_2)$
$c_{18} = m_B L_1 d_{AC} \cos\alpha_1$
$c_{19} = m_B d_{AC} d_{EB} \cos(\theta_1 - \theta_2)$
$c_{20} = m_B L_2 d_{AC} \cos(\theta_1 - \alpha_2 - \theta_2)$
$c_{21} = m_B L_1 d_{EB} \cos(\theta_2 - \alpha_1 - \theta_1)$
$c_{22} = m_B L_2 d_{EB} \cos\alpha_2$
$c_{23} = m_B L_1 L_2 \cos(\alpha_1 + \theta_1 - \alpha_2 - \theta_2)$
$c_{24} = m_2 L_1 d_{Dcm2}\left[\sin\alpha_1 \sin\theta_2 \cos(\alpha_2 - \theta_1) + \cos\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2)\right]$
$c_{25} = m_2 L_1 d_{Dcm2}\left[\cos\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \cos\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2)\right]$
$c_{26} = m_2 d_{Dcm2}\left[L_1 \cos\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2) + d_{AC} \cos(\alpha_2 - \theta_1 + \theta_2)\right]$
$c_{27} = m_2 L_1 \sin(\alpha_1 + \theta_1)$
$c_{28} = m_2 d_{Dcm2} \cos(\alpha_2 + \theta_2)$
$c_{29} = m_2 L_1 \cos(\alpha_1 + \theta_1)$
$c_{30} = m_2 d_{Dcm2} \sin(\alpha_2 + \theta_2)$

No potential energy is considered in this work. The lagrangian of the system, L, therefore, is given by:

$$L = T\,. \tag{4}$$

The Lagrange´s equations of motion for this problem are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_A}\right) - \frac{\partial L}{\partial x_A} = 0 \tag{5}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_A}\right) - \frac{\partial L}{\partial y_A} = 0 \tag{6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}_1}\right) - \frac{\partial L}{\partial \alpha_1} = 0 \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}_2}\right) - \frac{\partial L}{\partial \alpha_2} = 0 \tag{10}$$

For the variables x_B and y_B one has respectively the constraint equations given by:

$$x_B = x_A + d_{AC}\cos\theta_1 + L_1 \cos(\alpha_1 + \theta_1) + d_{EB} \cos\theta_2 - L_2 \cos(\alpha_2 + \theta_2) \tag{11}$$

$$y_B = y_A + d_{AC}\sin\theta_1 + L_1 \sin(\alpha_1 + \theta_1) + d_{EB} \sin\theta_2 - L_2 \sin(\alpha_2 + \theta_2) \tag{12}$$

Substituting L given by Eq. (4) into Eqs. (5) to (10), a set of nonlinear ordinary differential governing equations of motion for the space tether is obtained and is given by:

$$
\begin{aligned}
&c_1\ddot{x}_A-\left(c_{10}+c_{15}+c_{27}\right)\ddot{\alpha}_1-\left(c_{10}+c_{15}+c_{27}+c_{32}\right)\ddot{\theta}_1+\left(c_{12}+c_{30}-c_5\right)\ddot{\theta}_2+\\
&\left(c_{12}+c_{30}\right)\ddot{\alpha}_2-\left(c_{11}+c_{14}+c_{29}\right)\dot{\alpha}_1^2-2\left(c_{11}+c_{14}+c_{29}\right)\dot{\alpha}_1\dot{\theta}_1-\\
&\left(c_{11}+c_{14}+c_{29}+c_{31}\right)\dot{\theta}_1^2+\left(c_{17}+c_{28}\right)\dot{\alpha}_2^2+2\left(c_{17}+c_{28}\right)\dot{\alpha}_2\dot{\theta}_2+\\
&\left(c_{17}+c_{28}-c_8\right)\dot{\theta}_2^2=0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&c_1\ddot{y}_A+\left(c_8-c_{17}-c_{28}\right)\ddot{\theta}_2+\left(c_{11}+c_{14}+c_{29}\right)\ddot{\alpha}_1-\left(c_{17}+c_{28}\right)\ddot{\alpha}_2+\\
&\left(c_{11}+c_{14}+c_{29}+c_{31}\right)\ddot{\theta}_1-\left(c_{32}-c_{10}-c_{15}-c_{27}\right)\dot{\theta}_1^2-\left(c_{12}+c_{30}-c_5\right)\dot{\theta}_2^2+\\
&2\left(c_{12}+c_{30}\right)\dot{\alpha}_2\dot{\theta}_2-\left(c_{10}+c_{15}+c_{27}\right)\dot{\alpha}_1^2-2\left(c_{10}+c_{15}+c_{27}\right)\dot{\alpha}_1\dot{\theta}_1+\\
&\left(c_{12}+c_{30}\right)\dot{\alpha}_2^2=0
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&c_3\ddot{\theta}_1+\left(c_{19}-c_{20}+c_{21}-c_{23}+c_{25}-c_{26}\right)\ddot{\theta}_2+\left(c_{76}+c_2\right)\ddot{\alpha}_1-\left(c_{20}+c_{23}+c_{75}\right)\ddot{\alpha}_2-\\
&c_{74}\ddot{x}_A+c_{73}\ddot{y}_A+\left(c_{72}-c_{34}-c_{35}-c_{23}\right)\dot{\theta}_2^2-c_{71}\dot{\alpha}_1^2+\left(c_{70}-c_{23}\right)\dot{\alpha}_2^2-c_{69}\dot{\alpha}_1\dot{\theta}_1+\\
&\left(c_{68}-2c_{23}-2c_{35}\right)\dot{\alpha}_2\dot{\theta}_2+c_{67}\dot{\alpha}_1\dot{\alpha}_2+c_{67}\dot{\alpha}_1\dot{\theta}_2=0
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\left(c_6-c_{22}\right)\ddot{\theta}_2+\left(c_{66}-c_{20}-c_{23}\right)\ddot{\theta}_1+c_{65}\ddot{x}_A+c_{64}\ddot{y}_A+\left(c_{63}+c_{21}-c_{23}\right)\ddot{\alpha}_1+\\
&c_{62}\ddot{\alpha}_2+\left(c_{61}+c_{36}+c_{33}\right)\dot{\theta}_1^2+\left(c_{36}+c_{33}+c_{60}\right)\dot{\alpha}_1^2+c_{43}\dot{\alpha}_2^2+c_{59}\dot{\alpha}_1\dot{\alpha}_2+\\
&c_{58}\dot{\alpha}_1\dot{\theta}_1+c_{43}\dot{\alpha}_2\dot{\theta}_2+c_{57}\dot{\alpha}_2\dot{\theta}_1=0
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&c_2\ddot{\alpha}_1+\left(c_{56}-c_{23}-c_{24}\right)\ddot{\alpha}_2+c_{55}\ddot{\theta}_1+\left(c_{54}-c_{23}\right)\ddot{\theta}_2+c_{53}\ddot{x}_A+c_{52}\ddot{y}_A+\\
&c_{51}\dot{\theta}_1^2+\left(c_{50}-c_{33}\right)(\dot{\alpha}_2\dot{\theta}_2+\dot{\theta}_2^2)+c_{49}\dot{\theta}_2^2+\left(c_{48}-c_{33}\right)\dot{\alpha}_2^2+c_{47}\dot{\theta}_1\dot{\theta}_2+\\
&\left(c_{45}-2c_{33}\right)\dot{\alpha}_2\dot{\theta}_2-c_{46}\dot{\alpha}_1\dot{\alpha}_2+c_{44}\dot{\alpha}_2\dot{\theta}_1=0
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&c_7\ddot{\alpha}_2+\left(c_{25}-c_{20}-c_{23}-c_{26}\right)\ddot{\theta}_1+\left(c_7-c_{22}\right)\ddot{\theta}_2+\left(c_{41}-c_{23}\right)\ddot{\alpha}_1+\\
&\left(c_{12}+c_{30}\right)\ddot{x}_A-\left(c_{17}+c_{28}\right)\ddot{y}_A+\left(c_{33}+c_{42}\right)\dot{\alpha}_1^2+\left(c_{33}+c_{40}\right)\dot{\theta}_1^2+c_{43}\dot{\theta}_2^2+\\
&\left(2c_{33}-c_{39}\right)\dot{\alpha}_1\dot{\theta}_1+\left(c_{37}-c_{38}\right)\dot{\alpha}_1\dot{\theta}_2+c_{37}\dot{\theta}_1\dot{\theta}_2=0
\end{aligned}
\tag{18}
$$

where:

$c_{31} = m_B d_{AC} \cos\theta_1 + m_2 d_{AC} \cos\theta_1 + m_1 d_{cm_AC} \cos\theta_1$

$c_{32} = m_B d_{AC} \sin\theta_1 + m_2 d_{AC} \sin\theta_1 + m_1 d_{cm_AC} \sin\theta_1$

$c_{33} = m_B L_1 L_2 \sin(\alpha_1 + \theta_1 - \alpha_2 - \theta_2)$

$c_{34} = m_B L_1 d_{EB} \sin(\theta_2 - \alpha_1 - \theta_1)$

$c_{35} = m_B L_2 d_{AC} \sin(\theta_1 - \alpha_2 - \theta_2)$

$c_{36} = m_B L_1 d_{EB} \sin(\theta_2 - \alpha_1 - \theta_1)$

$c_{37} = m_2 L_1 d_{Dcm_2}\left[\cos\alpha_1 \sin(\alpha_2 + \theta_1 - \theta_2) + \cos\theta_1 \sin(\theta_2 + \alpha_1 - \alpha_2) - \sin\theta_1 \cos(\alpha_2 + \alpha_1 - \theta_2)\right]$

$c_{38} = m_2 L_1 d_{Dcm_2}\left[\sin\alpha_1 \cos(\theta_2 - \alpha_2 + \theta_1)\right]$

$c_{39} = m_2 L_1 d_{Dcm_2}\left[\sin\alpha_1 \sin\theta_2 \sin(\alpha_2 - \theta_1) - \cos\theta_2 \sin(\alpha_1 + \theta_1 - \alpha_2) + \sin\alpha_2 \cos(\alpha_1 + \theta_1 - \theta_2) - \right.$
$\left. \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2) - \cos\alpha_2 \cos\theta_1 \sin(\alpha_1 - \theta_2)\right]$

$c_{40} = m_B L_2 d_{AC}\left[\sin(\theta_1 - \alpha_2 - \theta_2)\right] - m_2 d_{Dcm_2} d_{AC}\left[\sin(\alpha_2 - \theta_1 + \theta_2)\right] +$
$m_2 L_1 d_{Dcm_2}\left[\cos\alpha_2 \cos\theta_1 \sin(\alpha_1 - \theta_2) - \cos\alpha_1 \sin\alpha_2 \cos(\theta_1 - \theta_2) + \sin\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2)\right]$

$c_{41} = m_2 L_1 d_{Dcm_2}\left[\cos\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \cos\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2) - \cos\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2) - \right.$
$\left. \sin\alpha_1 \sin\theta_2 \cos(\alpha_2 - \theta_1)\right]$

$c_{42} = m_2 L_1 d_{Dcm_2}\left[\cos\theta_1 \cos\theta_2 \sin(\alpha_1 - \alpha_2) - \cos\alpha_1 \sin\theta_2 \cos(\alpha_2 - \theta_1) + \sin\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2) + \right.$
$\left. \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2)\right]$

$c_{43} = m_B L_2 d_{EB} \sin\alpha_2$

$c_{44} = m_2 L_1 d_{Dcm_2}\left[\sin\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2) - \sin\alpha_1 \sin\theta_2 \sin(\alpha_2 - \theta_1) + \cos\alpha_2 \cos\theta_1 \sin(\alpha_1 - \theta_2) - \right.$
$\cos\alpha_1 \sin\alpha_2 \cos(\theta_1 - \theta_2) - \cos\theta_1 \cos\theta_2 \sin(\alpha_1 - \alpha_2) - \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2) +$
$\left. \sin\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2)\right]$

$c_{45} = m_2 L_1 d_{Dcm_2}\left[\cos\alpha_1 \sin\alpha_2 \cos(\theta_1 - \theta_2) - \sin\alpha_1 \cos\theta_2 \cos(\alpha_2 - \theta_1) - \cos\alpha_1 \cos\alpha_2 \sin(\theta_1 - \theta_2) - \right.$
$\sin\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2) + \sin\alpha_1 \sin\theta_2 \sin(\alpha_2 - \theta_1) +$
$\left. \cos\theta_1 \sin\theta_2 \cos(\alpha_1 - \alpha_2) - \cos\theta_1 \cos\theta_2 \sin(\alpha_1 - \alpha_2)\right]$

$c_{46} = 2 m_2 L_1 d_{Dcm_2} \sin\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2)$

$c_{47} = m_2 L_1 d_{Dcm_2}\left[\cos\alpha_2 \cos\theta_1 \sin(\alpha_1 - \theta_2) - \sin\alpha_1 \sin\theta_2 \sin(\alpha_2 - \theta_1) + \sin\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2) - \right.$
$\left. \cos\alpha_1 \sin\alpha_2 \cos(\theta_1 - \theta_2) - \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2) + \sin\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2) - \cos\theta_1 \cos\theta_2 \sin(\alpha_1 - \alpha_2)\right]$

$c_{48} = m_2 L_1 d_{Dcm_2}\left[\sin\alpha_1 \sin\theta_2 \sin(\alpha_2 - \theta_1) - \cos\alpha_1 \cos\alpha_2 \sin(\theta_1 - \theta_2) - \sin\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \right.$
$\left. \cos\theta_1 \cos\theta_2 \sin(\alpha_1 - \alpha_2)\right]$

$c_{49} = m_2 L_1 d_{Dcm_2} \cos\alpha_1 \sin\alpha_2 \cos(\theta_1 - \theta_2)$

$c_{50} = m_2 L_1 d_{Dcm_2}\left[\cos\theta_1 \sin\theta_2 \cos(\alpha_1 - \alpha_2) - \sin\alpha_1 \cos\theta_2 \cos(\alpha_2 - \theta_1) - \cos\alpha_2 \sin\theta_1 \cos(\alpha_1 - \theta_2)\right] -$
$m_B L_1 d_{EB} \sin(\theta_2 - \alpha_1 - \theta_1)$

$c_{51} = \left(m_1 d_{cm_AC} d_{Ccm_1} + m_2 L_1 d_{AC} + m_B L_1 d_{AC}\right) \sin\alpha_1$

$c_{52} = \left(m_1 d_{Ccm_1} + m_B L_1 + m_2 L_1\right) \cos(\alpha_1 + \theta_1)$

$c_{53} = \left(-m_B L_1 - m_2 L_1 - m_1 d_{Ccm_1}\right) \sin(\alpha_1 + \theta_1)$

$c_{54} = m_B L_1 d_{EB} \cos(\theta_2 - \alpha_1 - \theta_1) + m_2 L_1 d_{Dcm_2}\left[\cos\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \sin\alpha_1 \sin\theta_2 \cos(\alpha_2 - \theta_1) - \right.$
$\left. \cos\theta_1 \cos\theta_2 \cos(\alpha_1 - \alpha_2) - \cos\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2)\right]$

$c_{55} = I_1 + m_B L_1^2 + m_1 d_{Ccm_1}^2 + m_2 L_1^2 + \left(m_1 d_{cm_AC} d_{Ccm_1} + L_1 m_2 d_{AC} + m_B L_1 d_{AC}\right) \cos\alpha_1$

$c_{56} = m_2 L_1 d_{Dcm_2}\left[\cos\alpha_2 \sin\theta_1 \sin(\alpha_1 - \theta_2) - \cos\alpha_1 \sin\alpha_2 \sin(\theta_1 - \theta_2)\right]$

$$c_{57} = m_2 L_1 d_{Dcm2}\left[\sin\theta_1\cos(\alpha_2+\alpha_1-\theta_2) - \cos\alpha_1\sin(\alpha_2+\theta_1-\theta_2) - \cos\theta_1\cos(\theta_2+\alpha_1-\alpha_2)\right]$$

$$c_{58} = 2m_B L_1 d_{EB}\sin(\theta_2-\alpha_1-\theta_1) + 2m_B L_1 L_2\sin(\alpha_1+\theta_1-\alpha_2-\theta_2) + m_2 L_1 d_{Dcm2}\left[\cos\theta_2\sin(\theta_1+\alpha_1-\alpha_2) - \sin\alpha_1\sin\theta_2\sin(\alpha_2-\theta_1) - \sin(\alpha_2-\theta_1-\alpha_1+\theta_2)\right]$$

$$c_{59} = m_2 L_1 d_{Dcm2}\left[\sin\theta_1\cos(\alpha_2+\alpha_1-\theta_2) - \cos\alpha_1\sin(\alpha_2+\theta_1-\theta_2) + \sin\alpha_1\cos(\theta_2-\alpha_2+\theta_1) - \cos\theta_1\sin(\theta_2+\alpha_1-\alpha_2)\right]$$

$$c_{60} = m_2 L_1 d_{Dcm2}\left[\cos\theta_1\cos\theta_2\sin(\alpha_1-\alpha_2) - \cos\alpha_1\sin\theta_2\cos(\alpha_2-\theta_1) + \sin\alpha_1\sin\alpha_2\sin(\theta_1-\theta_2) + \cos\alpha_2\sin\theta_1\cos(\alpha_1-\theta_2)\right]$$

$$c_{61} = m_B L_2 d_{AC}\sin(\theta_1-\alpha_2-\theta_2) - m_B d_{AC} d_{EB}\sin(\theta_1-\theta_2) - m_2 d_{Dcm2} d_{AC}\sin(\alpha_2-\theta_1+\theta_2) + m_2 L_1 d_{Dcm2}\left[\cos\alpha_2\cos\theta_1\sin(\alpha_1-\theta_2) + \sin\theta_1\cos\theta_2\cos(\alpha_1-\alpha_2) - \cos\alpha_1\sin\alpha_2\cos(\theta_1-\theta_2)\right]$$

$$c_{62} = I_2 + L_2^2 m_B + m_2 d_{Dcm_2}^2 - m_B L_2 d_{EB}\cos\alpha_2$$

$$c_{63} = m_2 L_1 d_{Dcm2}\left[\cos\alpha_2\sin\theta_1\sin(\alpha_1-\theta_2) - \sin\alpha_1\sin\theta_2\cos(\alpha_2-\theta_1) - \cos\theta_1\cos\theta_2\cos(\alpha_1-\alpha_2) - \cos\alpha_1\sin\alpha_2\sin(\theta_1-\theta_2)\right]$$

$$c_{64} = m_B d_{EB}\cos\theta_2 - m_B L_2\cos(\alpha_2+\theta_2) - m_2 d_{Dcm2}\cos(\alpha_2+\theta_2)$$

$$c_{65} = m_B L_2\sin(\alpha_2+\theta_2) - m_B d_{EB}\sin\theta_2 + m_2 d_{Dcm2}\sin(\alpha_2+\theta_2)$$

$$c_{66} = m_B d_{AC} d_{EB}\cos(\theta_1-\theta_2) - m_2 d_{Dcm2} d_{AC}\cos(\alpha_2-\theta_1+\theta_2) + m_B L_1 d_{EB}\cos(\theta_2-\alpha_1-\theta_1) + m_2 L_1 d_{Dcm2}\left[\cos\alpha_2\sin\theta_1\sin(\alpha_1-\theta_2) - \cos\theta_1\cos\theta_2\cos(\alpha_1-\alpha_2) - \cos\alpha_1\sin\alpha_2\sin(\theta_1-\theta_2)\right]$$

$$c_{67} = m_2 L_1 d_{Dcm2}\left[\sin\alpha_1\sin\theta_2\sin(\alpha_2-\theta_1) + \cos\alpha_2\sin(\theta_1-\alpha_1+\theta_2) - \cos\theta_2\sin(\theta_1-\alpha_1+\alpha_2) + \sin\alpha_2\cos(\alpha_1-\theta_1+\theta_2)\right]$$

$$c_{68} = m_2 L_1 d_{Dcm2}\left[\cos\theta_1\sin(\alpha_2-\alpha_1+\theta_2) + \cos\alpha_1\sin(\alpha_2-\theta_1+\theta_2) - \sin\theta_1\cos(\alpha_2+\alpha_1-\theta_2)\right] + 2m_2 d_{Dcm2} d_{AC}\sin(\alpha_2-\theta_1+\theta_2)$$

$$c_{69} = \left(2m_1 d_{cmAC} d_{Ccm1} + 2m_2 L_1 d_{AC} + 2m_B L_1 d_{AC}\right)\sin\alpha_1$$

$$c_{70} = m_2 d_{Dcm2} d_{AC}\sin(\alpha_2-\theta_1+\theta_2) - m_B L_2 d_{AC}\sin(\theta_1-\alpha_2-\theta_2) - m_2 L_1 d_{Dcm2}\left[\cos\alpha_1\cos\alpha_2\sin(\theta_1-\theta_2) + \sin\alpha_2\sin\theta_1\sin(\alpha_1-\theta_2) + \cos\theta_1\cos\theta_2\sin(\alpha_1-\alpha_2)\right]$$

$$c_{71} = \left(m_1 d_{cmAC} d_{Ccm1} + L_1 m_2 d_{AC} + m_B L_1 d_{AC}\right)\sin\alpha_1$$

$$c_{72} = m_B d_{AC} d_{EB}\sin(\theta_1-\theta_2) + m_2 d_{Dcm2} d_{AC}\sin(\alpha_2-\theta_1+\theta_2) + m_2 L_1 d_{Dcm2}\left[\cos\alpha_1\sin\alpha_2\cos(\theta_1-\theta_2) - \cos\alpha_2\sin\theta_1\cos(\alpha_1-\theta_2) + \cos\theta_1\sin\theta_2\cos(\alpha_1-\alpha_2)\right]$$

$$c_{73} = \left(m_B L_1 + m_1 d_{Ccm1} + m_2 L_1\right)\cos(\alpha_1+\theta_1) + \left(m_B d_{AC} + d_{AC} m_2 + m_1 d_{cmAC}\right)\cos\theta_1$$

$$c_{74} = \left(m_B L_1 + m_1 d_{Ccm1} + m_2 L_1\right)\sin(\alpha_1+\theta_1) + \left(m_B d_{AC} + d_{AC} m_2 + m_1 d_{cmAC}\right)\sin\theta_1$$

$$c_{75} = m_2 L_1 d_{Dcm2}\left[\cos\alpha_1\sin\alpha_2\sin(\theta_1-\theta_2) - \cos\alpha_2\sin\theta_1\sin(\alpha_1-\theta_2) + \cos\theta_1\cos\theta_2\cos(\alpha_1-\alpha_2)\right] + m_2 d_{Dcm2} d_{AC}\cos(\alpha_2-\theta_1+\theta_2)$$

$$c_{76} = m_B L_1 d_{AC}\cos\alpha_1 + \left(m_1 d_{cmAC} d_{Ccm1} + L_1 m_2 d_{AC}\right)\cos\alpha_1$$

Equation (13) is the governing equation for the variable x_A, Eq. (14) is the governing equation for the variable y_A, Eq. (15) is the governing equation for the variable θ_1, Eq. (16) is the governing equation for the variable θ_2, Eq. (17) is the governing equation for the variable α_1 and Eq. (18) is the governing equation for the variable α_2.

This set is rearranged, put in state space form and numerically integrated using the fourth order Runge-Kutta method. No external forces are considered here. The dynamics of the space tether starts with some initial conditions.

3 Numerical simulations

The parameters used in the numerical simulations are presented in Table 1. The time step used in the numerical integration of the governing equations of motion is 0.001s. The Runge-Kutta 4th order method is used in the numerical integration.

Table 1. Parameters used in the numerical simulation.

Parameter	Nomenclature	Value and units
mass of satellite A	m_A	1000 kg
mass of satellite B	m_B	1000 kg
mass of rod 1	m_1	400 kg
mass of rod 2	m_2	400 kg
distance from point C to the center of mass of rod 1	d_{Ccm1}	4 m
distance from point A to point C (the same as d_{cmAC})	d_{AC}	1 m
distance from point E to point B	d_{EB}	1 m
distance from point D to the center of mass of rod 2	d_{Dcm2}	4 m
moment of inertia of satellite A (relative to the center of mass of this body)	$I_A = \frac{2m_A d_{AC}^2}{3}$	667 $kg.m^2$
moment of inertia of rod 1 (relative to the center of mass of this body)	$I_1 = \frac{m_1 L_1^2}{12}$	83333 $kg.m^2$
moment of inertia of satellite B (relative to the center of mass of this body)	$I_B = \frac{2m_B d_{EB}^2}{3}$	667 $kg.m^2$
moment of inertia of rod 2 (relative to the center of mass of this body)	$I_2 = \frac{m_2 L_2^2}{12}$	83333 $kg.m^2$
length of rod 1	L_1	50 m
length of rod 2	L_2	50 m

A set of initial conditions is considered and the resulting dynamics of the space tether is ploted in Figs. 2 to 8. The initial conditions are presented in Table 2.

Table 2. Initial conditions used in the numerical simulation.

Parameter	Value and units
x_A	0 m
$\dot{x}_A$	1.7 m/s
y_A	0 m
$\dot{y}_A$	0.1 m/s
θ_1	22.5^o
$\dot{\theta}_1$	0.1 rad/s
θ_2	-22.5^o
$\dot{\theta}_2$	0 rad/s
α_1	22.5^o
$\dot{\alpha}_1$	0 rad/s
α_2	157.50^o
$\dot{\alpha}_2$	0.1 rad/s
x_B	72.6 m
$\dot{x}_B$	-5.4 m/s
y_B	0 m
$\dot{y}_B$	7.3 m/s

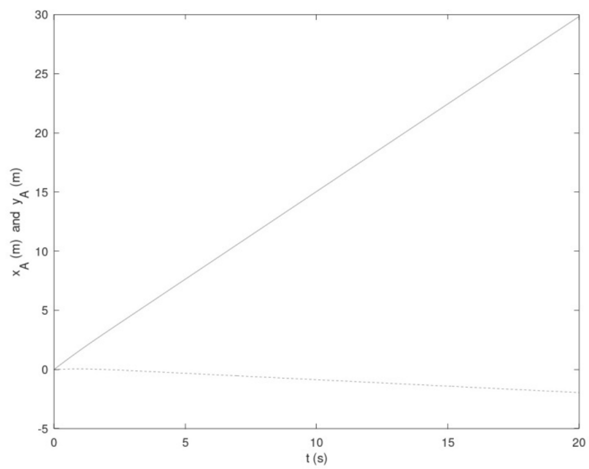

Figure 2. Position of satellite with center of mass in A: variables x_A (——) and y_A (----).

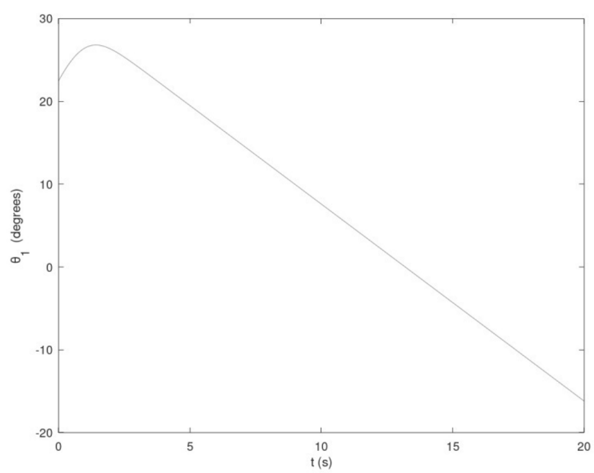

Figure 3. Angular position of satellite with center of mass in A.

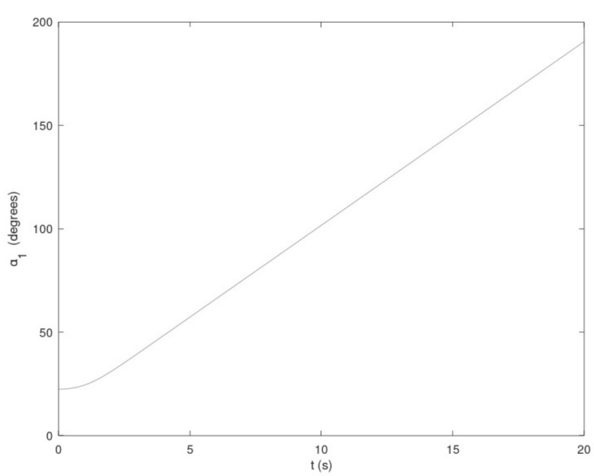

Figure 4. Angular position of rod 1 (with center of mass in cm_1).

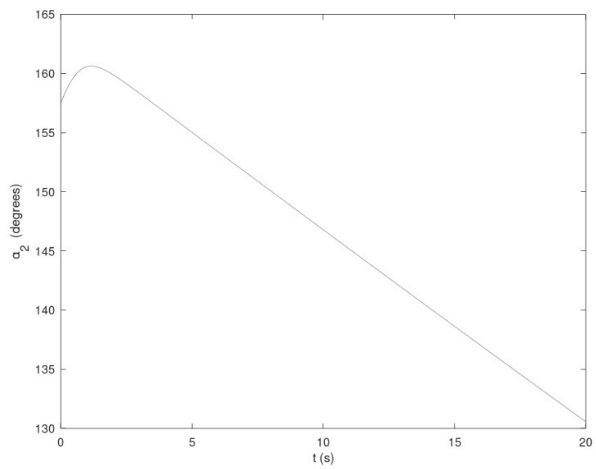

Figure 5. Angular position of rod 2 (with center of mass in cm_2).

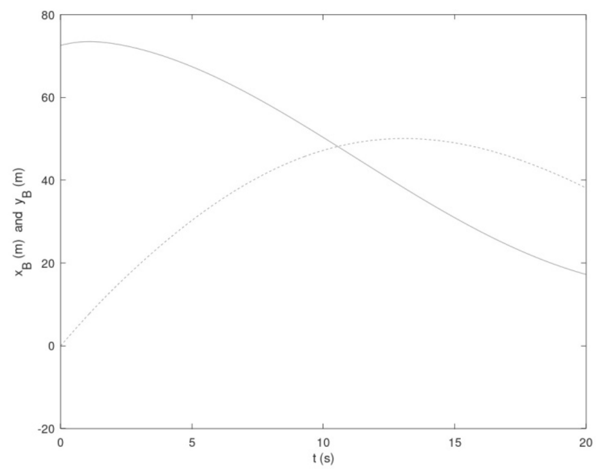

Figure 6. Position of satellite with center of mass in B: variables x_B (——) and y_B (- - - -).

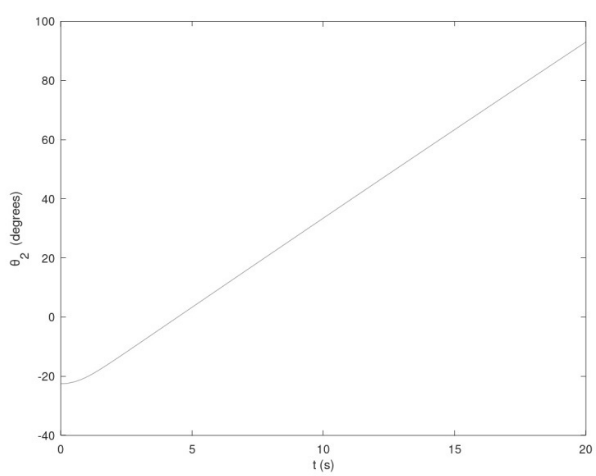

Figure 7. Angular position of satellite with center of mass in B.

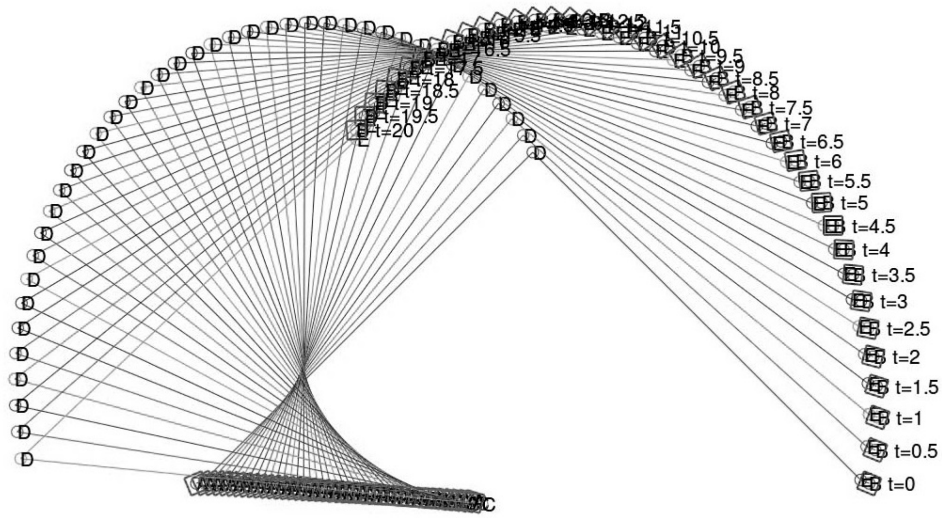

Figure 8. The general dynamics of the space tether.

Figures 2 to 7 show the time behaviour of the six independent variables and the two dependent variables (x_B and y_B) for this problem. Figure 8 shows the general dynamics of the space tether, where all the eight variables and the tether geometry are ploted together during the time evolution of the system.

4 Conclusions

The governing equations of motion for the space tether were obtained using the Lagrange´s equations approach. The numerical integration of the governing equations was performed using the Runge-Kutta 4th order method with fixed time step. The results presented here are coherent and proves that the numerical integration of such a complex set of nonlinear ordinary differential equations is stable. A set of initial conditions is given to the system and it evolves in time accordingly. Many other cases were tested and are not presented here. With the results presented here it is possible to add a nonlinear control law to these governing equations in order to drive this system through some desired angles and trajectories.

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