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A COMPARISON BETWEEN THE CLASSICAL BEAM THEORIES BASED ON EQUIVALENT SINGLE LAYER AND MURAKAMI'S ZIGZAG THEORY.

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Abstract. The necessity for better properties materials has made advances in the composition of structural elements such as composite beams introduced in the fields of civil, marine, aerospace and mechanical engineering, which requires better material properties. However, the rapid increase in using composite materials required improvements in the kinematics of classical beam theories made to describe the mechanical properties of composited laminated beams. This paper presents a comparison between the Euler-Bernoulli Theory (EBT) and Timoshenko Beam Theory (TBT), applying in laminated composited beams associated with multilayer composite analysis theories: Equivalent Single Layer (ESL) and Murakami's ZigZag Theory (MZZ). The work reported here shows examples of different beam configuration and loading, subject to both theories. It was possible to compare the results with the exact solution derived by Pagano [1]. Defining the length/thickness relation in which these theories are valid. EBT presents precise results for low thickness beams; however, it presents low precision in the interlaminar stress analysis. To contour this limitation and get better results in the stress field this paper used the equilibrium equations of the elasticity theory. Meanwhile, TBT presents more accurate results for thicker laminate beams, and the effects of transverse shear deformation are more relevant in these cases. This article recognizes that TBT is more precise than EBT in all comparisons presented and can represent the mechanical properties with certain precision and simplicity.

Keywords: Laminated Composite Materials, Euler-Bernoulli Beam Theory, Timoshenko Beam Theory, Equivalent Single Layer Theory, Murakami's ZigZag Theory.

1 Introduction

The performance and weight advantages of laminated composite materials have led to the increase of the adoption of a great number of applications like aerospace vehicles, naval and civil structures. Designing efficient and reliable composite structures requires the use of improved analytical and computational methods because such structures have a larger influence on the effects of shear deformation, especially when the thickness and heterogeneity of the beams are increased.

Consequently, the use of the classical beam theories, Euler-Bernoulli Theory (EBT) and Timoshenko Beam Theory (TBT), demands the adoption of theories, such as Equivalent Single Layer Theory (ESL) and Murakami´s ZigZag Theory (MZZ), for the complete analysis of the mechanical behavior of laminated composite beams. In this paper, a comparative study between the application of those theories aims to show the difference among the obtained results. Also, the results of both theories are validated by the exact elasticity solution developed by Pagano [1].

2 Equivalent Single Layer (ESL)

The Equivalent Single Layer theory (ESL) adopts the consideration that all of the layers of the laminated beam are reduced to an equivalent single layer. Therefore, the ESL theory has inherent simplicity and low computational cost, often providing a sufficiently accurate description of the overall response of thin to moderately thick laminates. However, the ESL model has serious limitations. Such as low accuracy of the overall response of thick laminated beams, the lack of ability to describe the state of stresses and deformations along with the laminate thickness and it does not serve as a basis for modeling delamination kinematics.

The displacement field of the laminated beam is represented by:

$$\begin{aligned} u_x^{ESL}(x) &= u_0(x) + z\theta \\ u_z(x) &= w_0(x) \quad . \end{aligned} \tag{1}$$

where (u_x, u_z) are the components of the displacement along the x and z coordinate direction, respectively, of a point in the intermediate plane and θ is the rotation of the beam cross-section.

3 Euler-Bernoulli Theory (EBT)

For EBT the strain field is given by the strain displacements relations and becomes:

$$\varepsilon_x(x,z) = u_{,x}(x) \quad . \tag{2}$$

Which ε_x represents the normal strain in the x-axis.

4 Timoshenko Beam Theory (TBT)

The strain field for TBT has an increment of the transverse shear strain, then the strain field is given by:

$$\varepsilon_x^{(k)}(x,z) = u_{,x}(x) + z\theta_{,x}(x) \, . \tag{3a}$$

$$\gamma_{xz}^{(k)}(x) = w_{,x}(x) + \theta(x) \quad . \tag{3b}$$

Where $\gamma_{xz}^{(k)}$ represents the transverse shear strain measure and the superscript (k) refers to the layer k=1,2,…N for an N layered beam.

5 Murakami's ZigZag Theory (MZZ)

Murakami's Zigzag Theory (MZZ) was first introduced by Murakami [2] and is one of the subclasses of ZZ theories that offers zigzag behavior through geometric parameters of axial displacement absent in ESL.

The displacement field by MZZ is presented in Gherlone [3] in the form of:

$$\begin{aligned} u_x^{ZZ(k)}(x,z) &= u(x) + z\theta(x) + \phi^{(k)}(z)\psi(x) \\ u_z^{ZZ}(x,z) &= w(x) \end{aligned} \quad . \tag{4}$$

where $u_x^{(k)}$ and u_z are the displacements in the *x*- and z-axis directions, respectively. The axial displacement is obtained by superposing a through-the-thickness zigzag contribution to the constant and linear terms of EBT and TBT, u_x *and* $z\theta(x)$, respectively. $\phi^{(k)}(\text{z})$ is the zigzag function of Murakami and $\psi\left(x\right)$ represents the magnitude of the local layerwise rotation.

According to Gherlone [3], laminated structures show periodic behavior. And the proposed zigzag function has a periodic nature:

$$\phi^{(k)}(z) = (-1)^k \frac{z - z_m^{(k)}}{h^{(k)}} . \tag{5}$$

Which $\phi_{Mur}^{(k)}(\text{z})$ alternatively assume values between -1 and +1 on laminate interfaces, $z_m^{(k)}$ is the coordinate of the middle plane of the layer interface and $h^{(k)}$ represents half of the corresponding layer thickness.

Therefore, MZZ is based only on geometry, rather than a material based function as it is not derived from any physical condition.

6 Example Problem and Numerical Results

6.1 Problem Description and Solutions

In this paper, the problem consists of a simple supported laminated beam under a sinusoidal load, which is given by:

$$q(x) = q_0 \sin\left(\pi x / L\right). \tag{6}$$

Where L is the total length of the beam and q_0 is a quantity that represents the amplitude of the loading (units of force/length).

The governing equations are given by the principle of virtual work (PVW), the stresses are obtained by the constitutive equations and the equilibrium equations and boundary conditions are obtained by an integration over the cross-section Area.

6.2 Numerical Results

For the determination of numerical results, it was used a 3-layer composite laminate with a (0/90/0) configuration, the material quantities are given by Table 1. The laminate has a total thickness of $2h$ and each layer has the same corresponding thickness of $2h^{(k)} = 2h/3$.

Table 1. Material properties of the laminate.

E_1	E_2	G_{12}	G_{23}	ν_{12}	ν_{23}
25	1	0.5	0.2	0.25	0.25

The numerical values are expressed in a non-dimensional form, followed by:

$$\bar{u}_x(0,z)=\frac{E_2 b}{hq_0}u_x \qquad \bar{w}\left(\frac{L}{2},x\right)=\frac{100E_2bh^3}{q_0L^4}u_z \qquad \bar{\sigma}_x\left(\frac{L}{2},z\right)=\frac{b\sigma_x}{q_0} \qquad \bar{\tau}_{zx}(0,z)=\frac{b\tau_{zx}}{q_0} \quad .(7)$$

The slenderness ratio of the beam is shown in the form of $S = L/2h$.

The maximum transverse displacement (deflection) as the S parameter increases is shown in Figure 1. The Exact solution is expressed by Pagano in the graph.

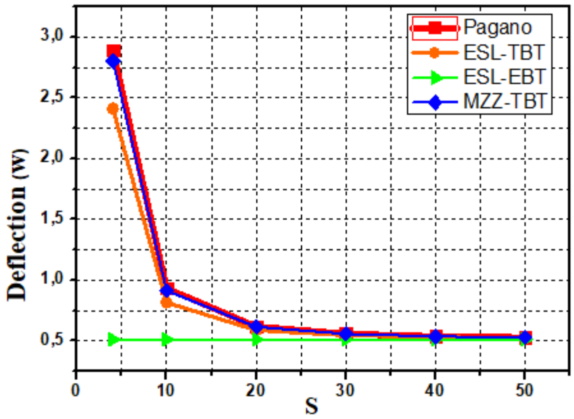


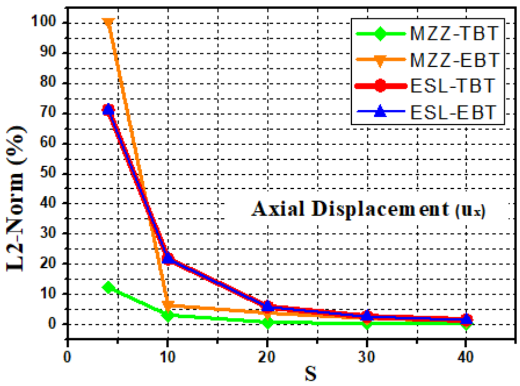
Figure 1. Maximum deflection along with different slenderness ratios.

In Figure 1, both of the MZZ based on TBT and ESL based on TBT express the distribution of the deflection in a curve with a similar shape as the exact solution of Pagano. However, the EBT theory only exhibits a curve in the aspect of a constant function. Thus, it is possible to see that EBT can only express good results for thin beams, having large mismatches among thicker beams when compared with the Exact solution.

Since axial displacements and normal stresses have different values along with the beam thickness coordinate, for S = 4, 10, 20, 30 and 40 it was compared the difference between all of the theories presented in this paper with the exact solution using the L2-norm Eq. (8). These results are illustrated in Figure 2a and Figure 2b.

$$L2-Norm(\%)=100\sqrt{\frac{\sum_{i=1}^{N}(Vr_i-Vc_i)^2}{\sum_{i=1}^{N}(Vr_i)^2}}\,. \tag{8}$$

Which Vr are the values of the exact solution and Vc are the values calculated for each theory presented in this paper.



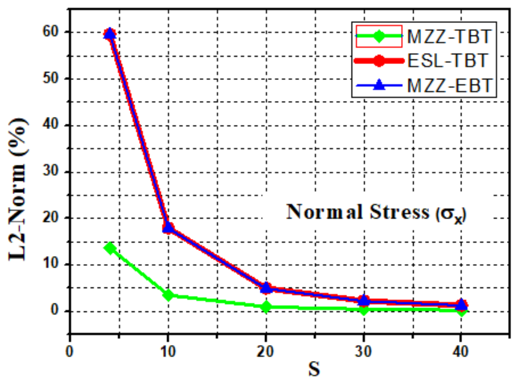


Figure 2a. L2-norm along with different slenderness ratios for the axial displacement and Figure 2b. L2-norm along with different slenderness ratios for the normal stress.

Analyzing the axial displacement in Figure 2a it is possible to observe that the results of the normalized error decreases as the beam goes thinner for all of the theories, the MZZ based on TBT performed well, with the results of the error for nearly all points are closer to 0. The MZZ for EBT shows an enormous error value for the case of S=4, but the results become better for S=10 and beyond. Both of ESL based on EBT and TBT gives similar results. Consequently, the plot of both curves is almost indistinctive. In ESL, the results are below 10% for beams with S=20 and beyond.

Regarding the normal stress analyzed in Figure 2b, the MZZ based on Timoshenko presents values below 10% for beams with S=6 and beyond and values below 5% for beams with S=10 and ahead. The ESL results for both TBT and EBT were again very similar and can only predict accurate results for beams with Slenderness ratio bigger than 30.

The results of the error for the transverse shear stress distribution have almost the same value of approximately 42% for MZZ based on TBT and around 80% for ESL based on TBT. Since EBT does not have a shear angle component in the stain distribution of Eq. (2), it cannot give results for the transverse shear stress.

7 Conclusions

The lightweight properties of laminated composite beams raised the increase of these structures in various applications. However, it requires improved analytical methods. This paper presents a comparative study between MZZ, and ESL both based on EBT and TBT. The MZZ based on TBT presents more accurate results fora all comparisons made on this paper. However, it produces discrepant results for the transverse shear stress. the MZZ based on EBT only produces distinguishing results from ESL for the axial displacement. Still, it cannot produces accurate results for thick beams. Finally, this work can conclude that the MZZ theory provides a refinement on the ESL theory with the increment of a simple piecewise function. Also, the use of TBT generates better results compared with EBT. Though, the analysis of the shear stress still presents high discrepancies and requires further studies on this theme.

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