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STUDIES ON THE STATIONARY DYNAMIC RESPONSE OF A FOUNDATION SUPPORTED BY FLEXIBLE PILE AND SOIL TO A VERTICAL INCIDENT WAVE FIELD OR EXTERNAL FORCE

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Abstract. This paper investigates the vertical response of rigid circular foundation resting on the surface of a three-dimensional, transversely isotropic soil. In a previous study, the dynamic response of a surface foundation laying on a homogeneous transversely isotropic half-space was compared to the response of the same foundation supported by a pile embedded in the soil profile. External forces and vertical incident waves were considered as exciting energy sources in those studies. But the modelling presented in the previous analysis was subjected to many simplifying assumptions. The incident vertical wave field was assumed to impinge solely at the head of the pile. The response of the pile to an incident wave field impinging the whole length of the embedded pile was not considered. The present article intends to show a numerical methodology and results to overcome these mentioned simplifying assumptions. In the present case the surface foundation is rigid and the incident vertical wave field will be interacting with the pile throughout its entire length. The soil response was previously obtained through the synthesis of a series of Green's functions. The pile was modelled as a one dimensional, elastic finite element body. This pile-soil coupling was obtained by establishing direct kinematic compatibility and equilibrium at discrete points of the pile-soil contact. The numerical studies reported in this article investigate the role of the pile inertia ratio as well as the relative stiffness of the soil and the pile. The results show the influence of the pile presence on the vibration amplitude on the foundation.

Keywords: Dynamic soil-foundation interaction; Foundations Dynamics; Green's Functions

1 Introduction

As described by Das and Ramana [1] soil dynamics is one of the branches of soil mechanics that studies the behavior of soils subjected to dynamic loads. The emergence of this subject matter is due to the need to understand and decrease vibrations in foundations or structures due to different energy source, such as waves induced by the operation of heavy machinery, construction operations, earthquake, wind and wave actions.

Most of the structural engineering works involve some kind of component with direct contact with the soil. When a load, such as external forces or incident wave fields, act on these systems, soil and foundation displacement present coupled behavior. The process in which the vibration of the soil influences the structure response and vice-versa is called as dynamic soil-structure interaction (DSSI).

According to Menglin et al. [2], DSSI is one of the major subjects in earthquake engineering and the dynamic response of a structure supported on a flexible soil may differ significantly from the response of the same structure supported by a rigid base. One of the reason for this difference is that the vibrational energy of the structure supported by the soil may be dissipated by radiation of stress waves in the surround medium and by hysteretic action in the medium itself.

Computer evolution has enabled the development of numerical solutions to complex problems such as DSSI problem. Among the developed methods, the ones that are currently used the most are the Finite Element Method (FEM), Finite Difference Method (FDM) and the Boundary Element Method (BEM).

FEM is an efficient computing method widely used in engineering, discretizes a continuum into a series of elements with limited sizes to compute mechanics of the continuum. However, the discretization soil model via FEM leads a too large number of degree of freedom which requires a lot of internal computer memory and computing time (Menglin et al. [2]).

Differently from FEM, BEM only discretizes the boundary of the domain. Second Beskos [3], in the study of the DSSI, the use of the BEM is more advantageous than the FEM because it requires only a surface discretization and satisfies automatically the Sommerfeld radiation condition. Different studies were done using BEM, such as those done by Rajapakse and Wang [4], Romanini et al [5], Barros, Labaki and Mesquita [6].

The piles are most commonly subjected to vertical loads, but can also be subjected to other types of loads, such as horizontal loads and also momentum loads. According to Nazir and Azzam [7] the idea of using piles to support foundations and others structures on the soil is the increase the system bearing capacity.

The goal of the present paper is to introduce a method to study the coupled response of surface foundation interacting with an embedded pile. Adopted coupling methodology requires the determination of displacements and forces acting at the interface between the foundation and the pile due an external excitation or incident vertical wave field. The displacement and force responses of the foundation and pile must satisfy the criteria of kinematic compatibility and equilibrium of forces at the interface. It will show the response of the foundation supported by an embedded vertical pile of different lengths as well as different inertia properties.

2 Statement of the problem

The present study considers the problem of dynamic vertical response of a rigid, circular surface foundation supported solely by an embedded vertical pile with exterior and internal parts, as show in Fig.1. The foundation can be excited by an external vertical load or by a vertically-propagating seismic pressure wave field.

The index ‘soil’ is related to the soil, modeled here as a three dimensional homogeneous, transversely isotropic half-space, with Young’s modulus E_{soil}, Poisson’s ratio ν_{soil}, mass density ρ_{soil} and material damping coefficient η_{soil}. The index ‘pile’ is related to the pile, with Young’s modulus E_{pile}, density ρ_{pile}, radius a_{pile}, and length h_{pile}. The index ‘f’ is related to the foundation, with properties radius a_f and mass density ρ_f. The index ‘i’ is related to the internal part of the pile. The index ‘e’ is related to the external part of the pile.

The cylindrical coordinate system is placed so that the r-θ plane is aligned with the surface of the soil, and the pile is aligned along the z-axis. The center of the foundation coincides with the origin of the coordinate system (Fig. 1).

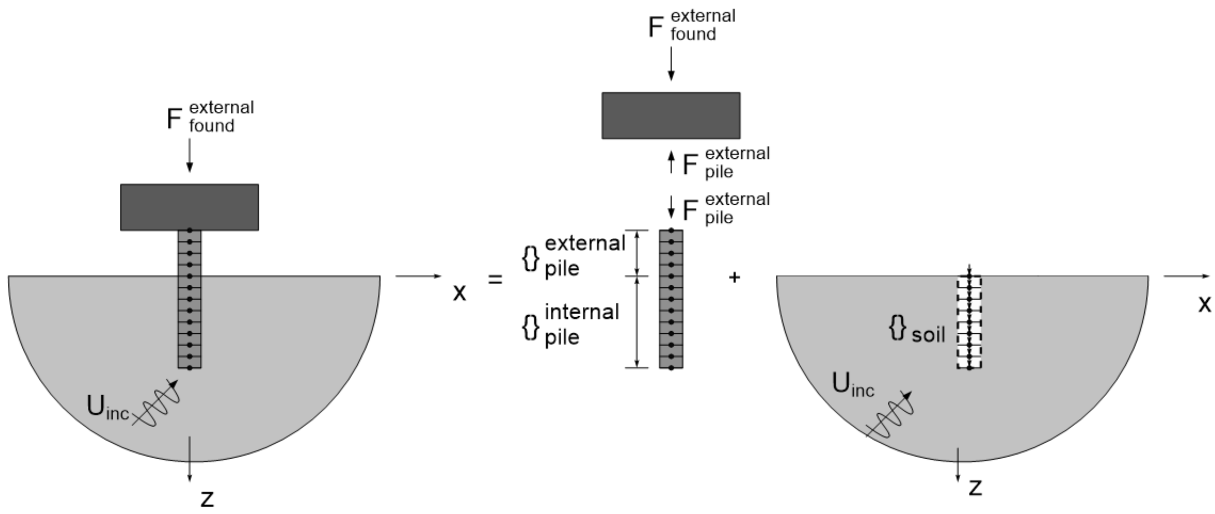


Figure 1. Foundation supported by a pile embedded in a soil with interior and exterior parts

In previous articles, Lima et. al [8,9] studied the dynamic response of a surface foundation laying on a homogenous transversely isotropic half-space. These results were compared to the ones obtained considering a foundation supported by a pile. From the engineering perspective those previous studies have cast light into the influence of a pile on the foundation response, due to external and internal excitations. External forces and wave excitation sources were considered in those studies. But the modelling presented in the previous analysis was subjected to many simplifying assumptions. The foundation was considered to be totally flexible, which leads to a soil-structure stress boundary value problem. Also, the incident vertical wave field was assumed to impinge solely the head of the pile. The response of the pile due to an incident wave field impinging the whole length of the embedded pile was not considered.

The present article intends to show a numerical methodology to overcome these mentioned simplifying assumptions. In the present case the surface foundation is rigid and the incident vertical wave field will be interacting with the pile throughout its entire length. The foundation is supported by a flexible pile embedded in soil, in which case there is no stress at the soil-foundation interface.

3 Numerical Model

This paper studies the effect of a pile in the vibration of the foundation supported by an elastic pile of radius a_p, length h_p, and Young's modulus E_p, which is in turn in bonded contact with the soil.

The models of soil in this paper require the derivation of Green's functions corresponding to vertical loads applied on the surface or within the half-space.

Consider an elastic, transversely isotropic, three-dimensional full-space. The problem is governed by the Cauchy-Navier differential equations, that are described in terms of the displacement components ui=ui(r,θ,z,ω) (i=r,θ,z). Rajapakse and Wang (1993) proposed a solution for this coupled problem in terms of Hankel integral transforms and series expansion. The solution is written in terms of arbitrary functions, the values of which are determined from the boundary and continuity conditions of a given problem.

The analyses of foundation–pile–soil dynamic interaction presented in this work are built upon previous formulations of a rigid plate laying over the soil, an embedded elastic bar to model the pile embedded within an homogeneous, transversely isotropic half-space. These formulations are summarized below.

3.1 Model of surface foundation

This section summarizes a model of the vibratory response of a rigid, massless, circular plate of zero thickness, presented by Labaki, Mesquita and Rajapakse [10]. The axisymmetric vertical displacement w_z of the plate–soil interface can be related to its interface contact traction t_j through an integral equation that cannot be solved directly, since the distribution of the traction field t_j is not known (Labaki, Mesquita and Rajapakse, 2013). A numerical piece-wise constant approximation of the traction field can be obtained by considering that the plate–soil interface consists of M_f concentric annular disc elements of inner and outer radii s_{1k} and s_{2k} (k=1,M_f) (Lysmer [11]):

$$\sum_{k=1}^{M_f} u_{zz}\left(r_i, r_k, \omega\right) t_z\left(r_k, \omega\right) = \Delta \,, \tag{1}$$

in which Δ is the vertical displacement of the rigid plate, which is the same for all disc elements of the plate–soil interface, $t_z(r_k,\omega)$ is the time-harmonic, spatially constant traction distribution over disc element k, with $r_k=(s_{1k}+s_{2k})/2$, ω is the frequency of excitation, and $u_{zz}(r_i,r_k,\omega)$ is the axisymmetric vertical displacement of disc element i due to a unit vertical load uniformly distributed over disc element k (i,k=1,M_f). The total vertical force at the plate–soil interface is obtained from the resulting traction field $t_z(r_k)$ by

$$F_Z\left(\omega\right) = \sum_{k=1}^{M_f} A_k t_z\left(r_k, \omega\right) = \sum_{k=1}^{M_f} \pi\left(s_{2k}^2 - s_{1k}^2\right) t_z\left(r_k, \omega\right), \tag{2}$$

in which A_k is the area of the annular disc element k. The vertical dynamic flexibility of the rigid plate–soil system is obtained by dividing the resulting rigid-plate displacement Δ by $F_z(\omega)$.

3.2 Model of pile

The pile is modeled using the Finite Element Method (FEM), using bar elements of length l_e, with two nodes and one degree of freedom per node. The pile is divided into n_p elements, which results in $n_n= n_p+1$ is the number of nodes in the pile model.

By assuming a linear interpolation function along the bar elements, the stiffness and mass matrices of the pile elements are given respectively by Eq. (3) and Eq. (4).

$$\left[K_e\right] = \frac{\pi a_{pile}^2 E_{pile}}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3}$$

$$\left[M_e\right] = \frac{\pi a_{pile}^2 \rho_{pile} l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4}$$

The global stiffness [K] and mass [M] matrices are obtained from the matrices of elements by the standard FEM assembly process. In the present paper, the dynamic compliance of the pile Sp is the inverse of the response of the dynamic equilibrium equation for the pile.

3.3 Model of soil

In this work, the soil is modeled as a three-dimensional, homogeneous, transversely isotropic, unbounded full-space under stationary loads of circular frequency ω. The Green's function describing such medium has been derived by Rajapakse and Wang [4] in terms of Hankel transforms. The general solution is expressed in terms of a set of arbitrary functions, whose values depend on the boundary and continuity conditions of a given problem.

A free surface is introduced in the full-space. An annular area of the surface of the half-space (z=0, radii s_{1k} and s_{2k}, $s_{2k}>s_{1k}$) is subjected to a uniformly distributed vertical stationary load. The resulting axisymmetric vertical displacement at a ring of radius r_i at z=0, $u_{zz}(r_i,r_k,\omega)$, is the solution incorporated in Eq. (1). In the present paper, the vertical displacement of the rigid plate resting on the surface of the half-space, due to a unit load, is the dynamic compliance of the soil S_{soil}.

3.4 Kinematic compatibilities and equilibrium equations

Since the pile has parts outside and inside the half-space, the displacements and the forces of the pile will be divided on the external and internal, as shown in Eqs.5 and 6.

$$\left\{U_{pile}(\underline{x})\right\}=\begin{Bmatrix}\left\{U_{pile}^{external}(\underline{x})\right\}\\ \left\{U_{pile}^{internal}(\underline{x})\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{U_{pile}^{E}(\underline{x})\right\}\\ \left\{U_{pile}^{I}(\underline{x})\right\}\end{Bmatrix}. \tag{5}$$

$$\left\{F_{pile}(\underline{x})\right\}=\begin{Bmatrix}\left\{F_{pile}^{external}(\underline{x})\right\}\\ \left\{F_{pile}^{internal}(\underline{x})\right\}\end{Bmatrix}=\begin{Bmatrix}\left\{F_{pile}^{E}(\underline{x})\right\}\\ \left\{F_{pile}^{I}(\underline{x})\right\}\end{Bmatrix}. \tag{6}$$

The pile response is determined through a flexibility matrix, S_p, as shown in Eq. (7):

$$\left\{U_{pile}(\underline{x})\right\}=\left[S_{pile}\right]\left\{F_{pile}(\underline{x})\right\}. \tag{7}$$

Subdividing eq.7 into the external and internal parts of the pile:

$$\begin{Bmatrix}\left\{U_{pile}^{E}(\underline{x})\right\}\\ \left\{U_{pile}^{I}(\underline{x})\right\}\end{Bmatrix}=\begin{bmatrix}\left[S_{pile}^{EE}\right] & \left[S_{pile}^{EI}\right]\\ \left[S_{pile}^{IE}\right] & \left[S_{pile}^{II}\right]\end{bmatrix}\begin{Bmatrix}\left\{F_{pile}^{E}(\underline{x})\right\}\\ \left\{F_{pile}^{I}(\underline{x})\right\}\end{Bmatrix}. \tag{8}$$

By writing eq. (8) explicitly:

$$\left\{U_{pile}^{E}(\underline{x})\right\}=\left[S_{pile}^{EE}\right]\left\{F_{pile}^{E}(\underline{x})\right\}+\left[S_{pile}^{EI}\right]\left\{F_{pile}^{I}(\underline{x})\right\}. \tag{9}$$

$$\left\{U_{pile}^{I}(\underline{x})\right\}=\left[S_{pile}^{IE}\right]\left\{F_{pile}^{E}(\underline{x})\right\}+\left[S_{pile}^{II}\right]\left\{F_{pile}^{I}(\underline{x})\right\}. \tag{10}$$

The soil response obtained through the Green's functions, S_{soil}, as shown in Eq. (11):

$$\left\{U_{soil}\right\}=\left[S_{soil}\right]\left\{F_{soil}\right\}. \tag{11}$$

The equilibrium equation of the foundation with mass m_f:

$$\left\{F_{found}^{E}(\underline{x})\right\}-\left\{F_{pile}^{E}(\underline{x})\right\}=-\omega^2 m_f\left\{U_f\right\}. \tag{12}$$

The total displacement field of the soil is:

$$\left\{U_{Total}(\underline{x})\right\}=\left\{U_{Inc}(\underline{x})\right\}+\left\{U_{Scattter}(\underline{x})\right\}. \tag{13}$$

The equilibrium equation at the interface soil-internal part of the pile is:

$$\left\{F_{pile}^{I}(\underline{x})\right\}+\left\{F_{soil}\right\}=\left\{0\right\}. \tag{14}$$

The kinematic compatibility in the interface soil-internal pile part is:

$$\left\{U_{Total}(\underline{x})\right\}=\left\{U_{pile}^{I}(\underline{x})\right\}. \tag{15}$$

Assuming that the scattered field is the field generated by the forces in the half-space.

$$\left\{U_{Scatter}(\underline{x})\right\}=\left\{U_{solo}\right\}. \tag{16}$$

The kinematic compatibility at the interface foundation- external part of the pile is:

$$\left\{U_{pile}^{E}(\underline{x})\right\}=\left\{U_{f}\right\}. \tag{17}$$

Substituting Eqs. 5 through 17, the following equations are found:

$$\left\{U_{p}^{e}\right\}=\left[S_{pile}^{EE}\right]\left[\left\{F_{found}^{E}(\underline{x})\right\}+\omega^{2}m_{f}\left\{U_{p}^{e}\right\}\right]-\left[S_{pile}^{EI}\right]\left[\left[S_{soil}\right]^{-1}\left\{U_{p}^{i}\right\}-\left[S_{soil}\right]^{-1}\left\{U_{inc}\right\}\right]. \tag{18}$$

$$\left\{U_{p}^{i}\right\}=\left[S_{pile}^{IE}\right]\left[\left\{F_{found}^{E}(\underline{x})\right\}+\omega^{2}m_{f}\left\{U_{p}^{e}\right\}\right]-\left[S_{pile}^{II}\right]\left[\left[S_{soil}\right]^{-1}\left\{U_{p}^{i}\right\}-\left[S_{soil}\right]^{-1}\left\{U_{inc}\right\}\right]. \tag{19}$$

Writing Eqs. 18 and 19 in matrix form:

$$\begin{bmatrix}1-S_{pile}^{EE}\omega^{2}m_{f} & S_{pile}^{EI}S_{soil}^{-1}\\ -S_{pile}^{IE}\omega^{2}m_{f} & S_{pile}^{II}S_{soil}^{-1}+[I]\end{bmatrix}\begin{Bmatrix}U_{p}^{e}\\ U_{p}^{i}\end{Bmatrix}=\begin{Bmatrix}S_{pile}^{EE}F_{f}+S_{pile}^{IE}S_{soil}^{-1}U_{inc}\\ S_{pile}^{IE}F_{f}+S_{pile}^{II}S_{soil}^{-1}U_{inc}\end{Bmatrix}. \tag{20}$$

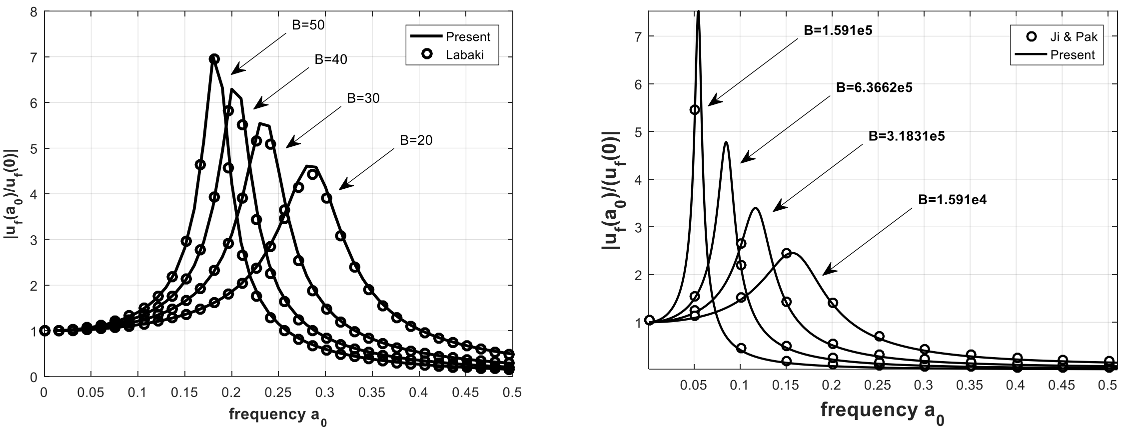
Equation (20) gives the displacement of the pile that can be subjected to an incident vertical wave field or an external excitation.

4 Numerical Results

This section presents numerical results to investigate the displacement of the foundation shown in Fig. 1 and study the displacement of the pile (Eq.20). For the purpose of presentation of numerical results, the following normalizations are defined: ratio of modulus of elasticity $E'=E_{pile}/E_{soil}$, ratio of density $\rho'=\rho_{pile}/\rho_{soil}$ and mass ratio $B=M_f/M_{soil}$, in which M_f is the mass of the foundation and M_{hs} is the mass of the soil. The soil mass, M_{hs}, is defined as the mass comprised by a volume formed by the area of the soil-foundation interface possessing a unit depth.

Normalized frequency a_0, is defined as $a_0=a_f\omega(\rho_{soil}/G_{soil})^{1/2}$. The soil-pile-foundation system is subjected to an external axial load, F^{e}_{fund} or to an incident wave field $U_{inc}=U_0\cos((\omega z_p)/c_p)$, where U_0 is the displacement at the half-space surface, z_p is the vector of the nodal coordinates of the internal pile in the z-direction and c_p is pressure wave propagation velocity in vertical direction. In this work, the external axial load, F^{e}_{fund}, and the displacement at the half-space surface, U_0, have unit amplitude.

In this section, the following parameters are considered: for the half-space, E_{hs}=2.5Pa, η_{hs}=0.01, υ_{hs}=0.25, ρ_{hs}=1kg/m³; for the foundation, a_f=1m. The proposed formulation in which the foundation interacts with the pile embedded in the soil (Fig. 1), have been compared with a solution by Labaki, Mesquita and Rajapakse [12] for an external excitation and by Ji and Pak [13] for an incident vertical wave.



(a) (b)

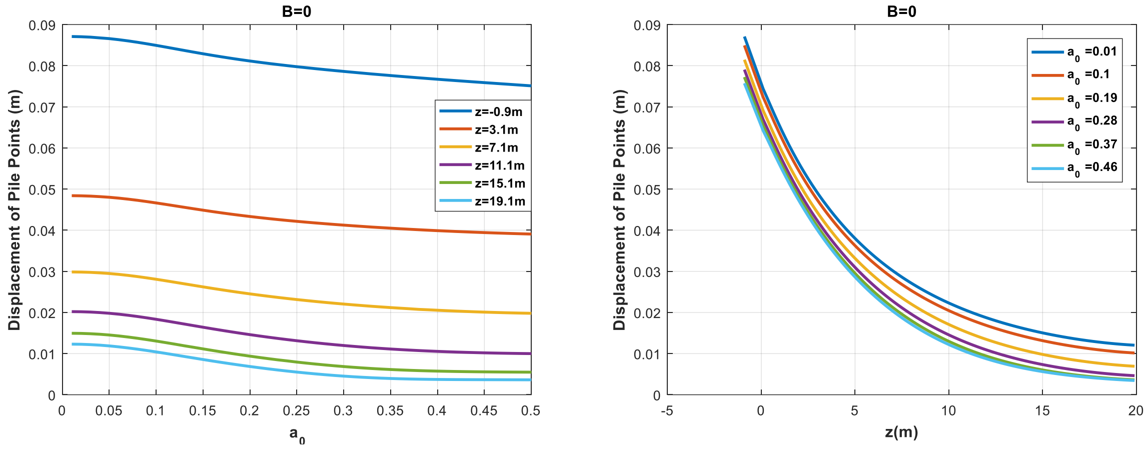
Figure 2. Validation of the displacement of the foundation with previous results submitted to: (a) external force, (b) incident wave field.

4.1 Case 1: External excitation

Figure 2a considers the case of external excitation of a rigid plate at the surface of the half-space. An analysis of this problem can be obtained with the present formulation by considering the dimension of the pile (including internal and external parts) are much smaller than those of the plate and the material properties of the pile are equal of the soil. Figure 2b considers the case of an incident wave field of a piled raft, in which a pile properties are: h_p/a_p=5, ρ'=4 and E'=1000 and the dimension of the external part of the pile are much smaller than internal part of the pile. Both comparisons in Fig. 2 show a good agreement of the present formulation with their alternative equivalents for different sets of data.

Figure 3 shows a pile displacement for a massless foundation supported by a pile embedded in a half-space subjected to external excitation applied in the foundation. In this study pile properties are: h_p/a_p=20, ρ'=1 and E'=10 and the pile was divided into 105 points (100 internal points and 5 external points).

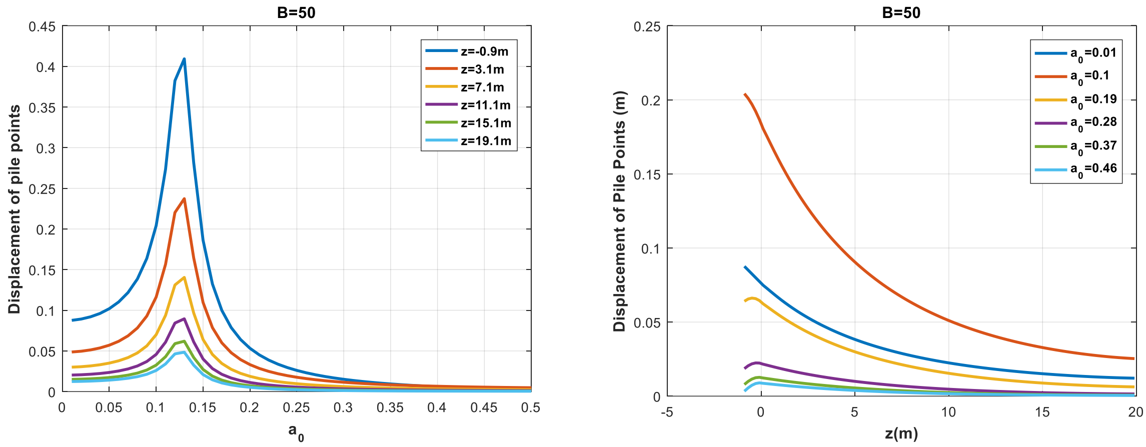
Figure 3a shows as the pile gets deeper, its displacement decreases as the frequency a_0 increases. Figure 3b shows the same trend, as the frequency a_0 increases, the displacement of the pile points decreases when the depth of the pile increases.



(a) (b)

Figure 3. Displacement of the pile points for a massless foundation supported by a pile embedded in a half-space subjected to external excitation: (a) along the frequency for a pile point, (b) along the pile point for a frequency.

Figure 4 shows the same case presented in Fig. 3, but now the foundation has mass of B = 50.



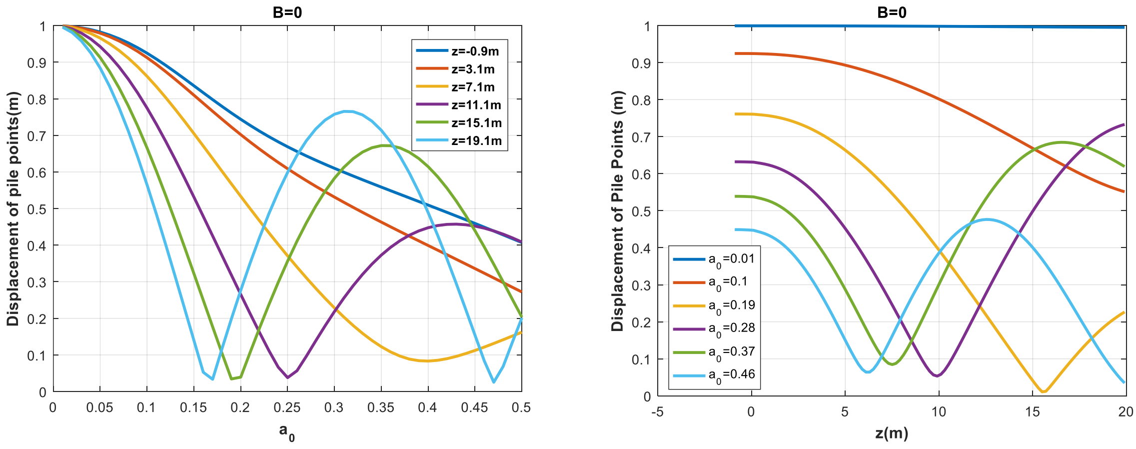
(a) (b)

Figure 4. Displacement of the pile points for a foundation with mass supported by a pile embedded in a half-space subjected to external excitation: (a) along the frequency for a pile point, (b) along the pile point for a frequency.

Figure 4a shows a well-defined resonance region for all cases considered. The greater displacement of the pile occurs in the first case (external part of the pile). As the depth of the pile increases, its displacement decreases. Figure 4b shows the displacement along the pile points for a fixed frequency.

4.2 Case 2: Incident wave field

In the studies of Figs. 3 and 4, the foundation was excited by an external force. In the studies of Figs. 5 and 6, the same system is used, but now the foundation is excited by an incident wave field.

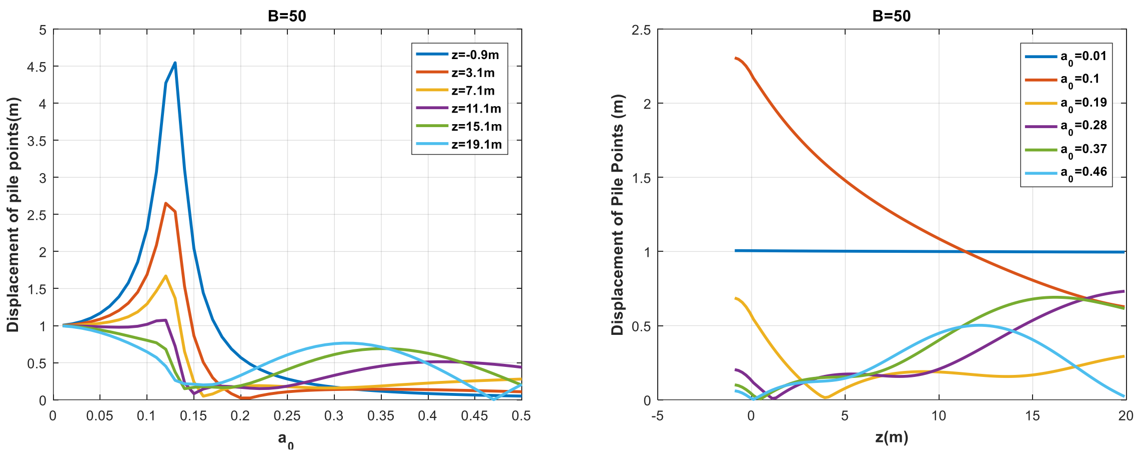


(a) (b)

Figure 5. Displacement of the pile points for a massless foundation supported by a pile embedded in a half-space subjected to incident wave field: (a) along the frequency for a pile point, (b) along the pile point for a frequency.

Figures 5a and 5b show that the incident wave field has a greater effect on pile displacement compared to the effect that an external excitation causes in displacement of the same pile. A deeper pile point may have a greater displacement than a pile point closer to the half-space surface, as shown in Fig. 5a. The same pile point may have a larger displacement to a higher frequency, as shown in Fig 5b.

Figure 6 shows the same case presented in Fig. 5, but now the foundation has mass of B = 50.



(a) (b)

Figure 6. Displacement of the pile points for a foundation with mass supported by a pile embedded in a half-space subjected to incident wave field: (a) along the frequency for a pile point, (b) along the pile point for a frequency.

Figure 6a shows a well-defined resonance region for pile points near to the half-space surface. As the depth of the pile increases, the resonance is no longer perceived. The greater displacement of the pile occurs in the first case (external part of the pile) for a low frequency. Figure 5b shows the displacement along the pile points for a fixed frequency.

Figure 7 shows the wave field profile that is acting along the pile for some frequencies.

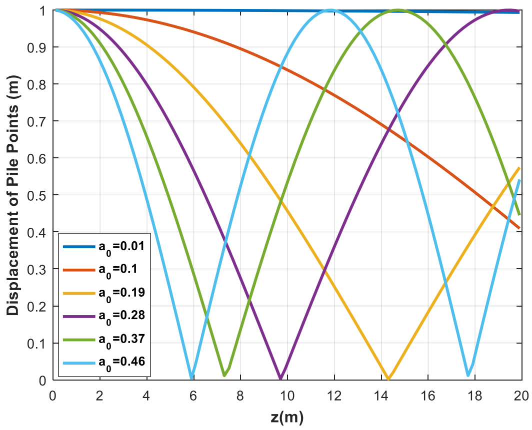


Figure 7. Foundation supported by a pile embedded in a soil with interior and exterior parts

Analyzing Fig. 6 and Fig. 7 notices that the pile displacement profile is influenced by the behavior of the incident wave acting on it.

5 Conclusion

This paper presented a formulation for analyzing the dynamic response of rigid foundations interacting with an underlying flexible pile embedded a homogenous half-space, subjected to incident vertical waves fields or external excitation. The pile is partially embedded with degree of freedoms inside the soil and above the soil surface. The results showed the displacement of the pile points along its length for different frequency of the incident waves. All the results show that the frequency of the incident wave have remarkable influences on the flexible pile response.

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