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USE OF MODAL ANALYSIS AS A METHOD TO ESTIMATE SECOND-ORDER GLOBAL EFFECTS IN REINFORCED CONCRETE STRUCTURES

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Abstract. Recent work shows that there is a relationship between the natural vibration period and the second-order global effects in the reinforced concrete frames, this link occurs because both depend essentially on the rigidity and mass matrix of the structure. Formulations are developed to demonstrate and validate that an amplification factor is able to approximate second-order global effects satisfactorily. Previous studies just consider the additional mass as a load increment. However, the second-order effects of external loads on the structure not beem take into account to calculate the internal forces in structural elements and their influence on the dynamic characteristics. Therefore, the second-order effects are included to analyze the influence of the dynamic characteristics (natural frequencies and vibration modes) and the amplification of the structure moment. Analyzes will be carried in a regular element of reinforced concrete. The amplification will be evaluated by means of natural frequencies and modes of vibration. Thus, this work generally aims at obtaining an alternative parameter of calculation, based on the natural frequency and mode of vibration of the structure, to be used as an indicator of the necessity or not of the consideration of the second-order global effects in reinforced concrete structures.

Keywords: Second-order effect, Modal analysis, Structural stability.

1 Introduction

The verticalization of cities to the detriment of the intensification of population density and the expansion of the urban network lead to the search for higher and slender structures, thus more susceptible to second-order effects. In addition, the development of structural calculation software and the execution of concrete with high compressive strength, have generated the conditions with which engineers deal with higher structures [1].

These higher structures are subjected to taller values of horizontal load due to the wind, and the combination of these loads with vertical loads, when evaluated in a deformed condition, causes the increase of internal moments due to the so-called geometric nonlinearity effects. These effects cause the emergence of additional that may compromise their stability [2].

In order to evaluate the stability in reinforced concrete structures, it is necessary to consider the effects of the physical and geometrical non-linearities. Since these structures also have a non-linear behavior, which results in a complex task [3].

Therefore, means were created to facilitate and evaluate second-order effects. As well as the approximate methods to evaluate or not the need to consider the global second-order effects demonstrated in ABNT NBR 6118 [4]. First, the instability parameter α is presented and then the parameter γ_Z, being an evaluative coefficient on the importance of second-order global forces in structures with at least four floors.

In this context, Feitosa and Alves [5], observed that amongthe the factors that most influence the research on the overall stability of a building are the lateral stiffness and the active weight in the structure. Other authors, such as Statler et al. [6], studied the relationship that could occur when analyzing this overall stability in metallic structures using their natural vibration period.

Already Reis et al. [7] demonstrated a method that is based on the definition of a simple parameter, called χ_T, to evaluate the stability of the structure through the natural vibration period to determine the final moments, taking into account the second-order global effects, basing in the amplification of first-order moments in reinforced concrete structures.

However, the analyzes made by Reis et al. [7] involves the use of the χ_T parameter in structures that have symmetry in their plane. In addition, Reis et al. [7] consider it dependent on a relation between the sum of the weight of all floors by the total weight of the structure that generates changes in the frequencies, but when the stresses are high, the structure presents a non-linear curve that represents the relation force-displacement. Thus, the natural frequencies suffer variation.

Recently, Leitão [2] studied the use of the χ_T parameter in reinforced concrete structures with different forms of structural configurations in the plane and along with its height, complementing the research of Reis et al. [7], checking that there is influence of the higher modes of vibration on the value of moments amplification due to the effects of second order. On the other hand, they considered as a participation factor a modal mass for the direction analyzed [2].

Therefore, to obtain an indicator, which refers to the susceptibility of the structure to suffer global effects of second-order, this work proposes the realization of the equation of a parameter of stability of structures of reinforced concrete. Based on the relationship that exists between the characteristics structure frequency (natural frequency and vibration mode) and stress increases due to the second-order global effects on structures, considering the effect of the loads on the elements in the change of natural frequency.

2 Theoretical fundamentals

To understand second-order effects is necessary to explore basic concepts of non-linear analysis of structures, such as physical nonlinearity and geometric nonlinearity.

In physical nonlinearity (FNL), the effect of the behavior of the material after the elastic regime is considered as being influenced by effects such as cracking, creep and yielding of the reinforcement [3].

One of the ways normally used in the analysis of the second-order global efforts to consider the FNL in concrete structures is with a factor of reduction of the stiffness of the structural elements. This factor varies according to the standard used.

When there are horizontal and vertical forces acting together on a structure, an increase of bending moments are generated due to the change in its geometry. The analysis that considers in its formulation the equilibrium of the structure in a deformed configuration is called geometric nonlinearity (GLN) [8].

Figure 1 shows the difference between the efforts obtained in the geometric linear analysis and the nonlinear geometric analysis:

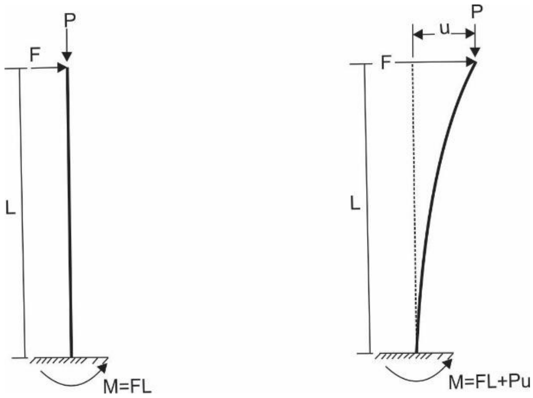


Figure 1: (a) Geometric linear analysis; (b) Nonlinear geometric analysis

The moment at the base is the result of the horizontal force, Fig. 1-a), applied to a certain height, not having an increase of moment due to a vertical load (P), since the equilibrium is effected in the initial position (undeformed). However, in Fig. 1-b there is a horizontal displacement (u) due to point load (F), in which, this new position of the element (deformed) generates an increase in moment due to the vertical force (P) horizontally displaced by a distance (u), known as second-order moment, which is added to the first-order moment, due to the horizontal load (F).

Structures subjected to loading and that are not rigid enough may be led to loss of stability [2].

2.1 Consideration of second-order effects

The simplified methods are those that allow by means of simple analyzes to predict the behavior of the structure in relation to the global effects of second-order.

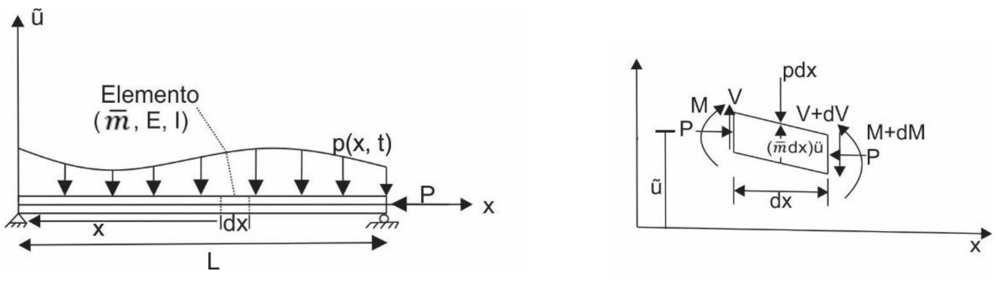
The ABNT NBR 6118 [4] classifies the structure as fixed nodes or free nodes, being classified as fixed nodes (γ_z less than or equal to 1.1) when the structure is subjected to horizontal forces that cause it to have small displacement horizontal. The calculation of γ_z is given by:

$$\gamma_z = \frac{1}{1-\frac{\Delta M}{M1}}, \tag{1}$$

where ΔM is the bending moment caused by the vertical loads acting on the deformed shape of the structure [4]; M1 is the moment of tipping, that is, the sum of the moments of all the horizontal forces of the combination considered, with their calculation values, in relation to the base of the structure [4].

2.2 Dynamic analysis in a beam including the effect of the axial load on the element

To obtain a dynamic equation of a beam including the effect of the axial load, it was considered a uniform beam that vibrates freely with a constant axial load P, as shown in Figure 2-a), where $\tilde{u}(x,t)$ as displacement at any point x along the neutral axis, the internal shear V + dV, the moment M and M + dM, and the inertial forces $(\bar{m}dx)\ddot{u}$ acting on a dx element of length as shown in Figure 2-b), where $\bar{m} = \rho A$, mass per unit of length $\ddot{u} = \partial^2 \mathrm{u}(x,t)/\, \partial \mathrm{t}^2$, partial derivative to express acceleration.



(a) (b)

Figure 2: (a) Representation of the coordinates for vibration free in a column; (b) Convention of signals for forces and moment under constant axial load, acting in an arbitrary element. Font: Adapted from Paz, 1997.

In the beam model, the classical Euler-Bernoulli or pure bending theory is used, in which with the equilibrium equations and some combinations we have the partial differential equation that describes the moment of a beam element in Eq. (2) and its simplification by Eq. (3).

$$p(x,t) = EI\frac{\partial^4 u(x,t)}{\partial x^4} + P\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{2}$$

$$p(x,t) = EIu'^{v} + Pu'' + \bar{m}\ddot{u}. \tag{3}$$

Equation (3) represents the dynamic equation for a beam, where the roman indexes indicate derivatives with respect to (x) and the ($\ddot{}$) indicates derivatives with respect to time.

2.3 Analytical model for free vibration equation of motion

With the differential equation that represents the dynamic equation for a beam, one can consider the system is free and non-damped vibration with the appropriate boundary conditions and applying the method of separation of variables, we have:

$$\Phi(x)'^{v} + \frac{P\Phi(x)''}{EI} - \frac{w^2\bar{m}\Phi(x)}{EI} = 0, \tag{4}$$

$$w^2 f(t) + f(\ddot{t}) = 0, \tag{5}$$

$$D^4 + D^2\frac{P}{EI} - \frac{w^{2\bar{m}}}{EI} = 0, \tag{6}$$

in which it is particularly convenient to adopt:

$$k^2 = \frac{P}{EI}, \tag{7}$$

$$\beta^4 = \frac{w^2\bar{m}}{EI}, \tag{8}$$

where the roots of Eq. (6) are given by:

$$\lambda_1 = \sqrt{-\frac{k^2}{2} + \sqrt{\frac{k^4}{4} + \beta^4}}\ ;\ \lambda_3 = -\lambda_1, \tag{9}$$

$$i\lambda_2 = \sqrt{\frac{k^2}{2} + \sqrt{\frac{k^4}{4} + \beta^4}}\ ;\ \lambda_4 = -i\lambda_2. \tag{10}$$

The following equation can be found by mathematical means:

$$\Phi(x) = A\cos h(\lambda_1 x) + Bsenh(\lambda_1 x) + D\cos(\lambda_2 x) + Esen(\lambda_2 x), \tag{11}$$

where A, B, D, and E are integration constants. Considering the boundary conditions at the ends of the beam, these constant ones define the mode of free vibration.

Shaker and Center [12] have even come to the equation of Eq. (11), but the numerical realization of the effect of the axial load on the natural frequencies of the uniform beam for several boundary conditions has been obtained numerically. It is also worth remembering that the authors did not elaborate on the analytical development of obtaining the effect of an axial load on the natural vibration modes and frequencies, and in addition Eq. (11) depended on λ_1 and λ_2. Therefore, a development was required to evaluate only as a function of λ_1.

Considering a base fixe with a distributed mass ($\bar{m}$) and stiffness along the rod (EI), with free vibration and an axial load P, we can have an analytical understanding of the effect of the axial load on the frequencies and modes. Whit this, the boundary conditions for the described element are as follows: at x equal to zero, we have $\tilde{u}(0,t) = 0\ or\ \Phi(0) = 0$ and $\tilde{u}'(0,t) = 0\ ou\ \Phi'(0) = 0$ for x with a length

L we have:

$$M(L,t) = \frac{-EI\partial^2\tilde{u}(L,t)}{\partial x^2} = 0 \; ou \; \Phi''(L) = 0, \tag{12}$$

$$V(L,t) = -\frac{\partial^3\tilde{u}(L,t)}{\partial x^3} - \frac{P}{EI}\frac{\partial\tilde{u}(L,t)}{\partial x} = 0 \; ou \; \Phi'''(L) + k^2\Phi'(L) = 0. \tag{13}$$

Applying and developing Eq. 11 with substitution of the boundary conditions and the conditions, we have:

$$\Phi(x) = A[\cos h(\lambda_1 x) - \cos(\lambda_2 x)] + B[senh(\lambda_1 x) - \frac{\lambda_1}{\lambda_2} sen(\lambda_2 x), \tag{14}$$

$$A[{\lambda_2}^2 cos(\lambda_2 L) + {\lambda_1}^2 \cos h(\lambda_1 L)] + B[\lambda_1\lambda_2 sen(\lambda_2 L) + {\lambda_1}^2 senh(\lambda_1 L)] = 0, \tag{15}$$

$$A[-{\lambda_2}^3 \, sen(\lambda_2 L) + {\lambda_1}^3 sen\, h(\lambda_1 L) + k^2\lambda_2 \, sen(\lambda_2 L) + \lambda_1 sen\, h(\lambda_1 L)] + B\big[{\lambda_1}^3 cosh(\lambda_1 L) + \lambda_1{\lambda_2}^2 cos(\lambda_2 L) + k^2(\lambda_1 cos(\lambda_2 L) - \lambda_1 cosh(\lambda_2 L)\big] = 0. \tag{16}$$

Simplifying Eq. (15) and Eq. (16) as:

$$a_1 = {\lambda_2}^2 cos(\lambda_2 L) + {\lambda_1}^2 \cos h(\lambda_1 L), \tag{17}$$

$$b_1 = \lambda_1\lambda_2 sen(\lambda_2 L) + {\lambda_1}^2 senh(\lambda_1 L), \tag{18}$$

$$a_2 = -{\lambda_2}^3 \, sen(\lambda_2 L) + {\lambda_1}^3 sen\, h(\lambda_1 L) + k^2\lambda_2 \, sen(\lambda_2 L) + \lambda_1 sen\, h(\lambda_1 L), \tag{19}$$

$$b_2 = {\lambda_1}^3 cosh(\lambda_1 L) + \lambda_1{\lambda_2}^2 cos(\lambda_2 L) + k^2(\lambda_1 cos(\lambda_2 L) - \lambda_1 cosh(\lambda_2 L). \tag{20}$$

Thus, in matrix form (Eq. 17 to Eq. 20), we have:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}. \tag{21}$$

2.4 The natural frequency of vibration and mode of vibration

In this section, we demonstrate the equations to obtain a characteristic equation only as a function of λ_1, and then present a general equation that describes the natural vibration frequency considering the axial load on the analytical element, as well as a general equation describing the modes of vibrations.

Recalling Eq. (17), (18), (19) and (20) and calculating the determinant of Eq. (21), we obtain:

$$\lambda_1{\lambda_2}^4 + {\lambda_1}^5 + k^2{\lambda_1}^3 - k^2\lambda_1{\lambda_2}^2 - {\lambda_1}^4\lambda_2 senh(\lambda_1 L)sen(\lambda_2 L) - k^2{\lambda_1}^3 \cosh(\lambda_1 L)\cos(\lambda_2 L) + 2{\lambda_1}^3{\lambda_2}^2 \cosh(\lambda_1 L)\cos(\lambda_2 L) + {\lambda_1}^2{\lambda_2}^3 senh(\lambda_1 L)sen(\lambda_2 L) + k^2\lambda_1{\lambda_2}^2 \cosh(\lambda_1 L)\cos(\lambda_2 L) - 2k^2{\lambda_1}^2\lambda_2 senh(\lambda_1 L)sen(\lambda_2 L) = 0. \tag{22}$$

With some simplifications we have the characteristic equation:

$$(-k^2\lambda_1)senh(L\lambda_1)sen\left(L\sqrt{k^2+{\lambda_1}^2}\right)\sqrt{k^2+{\lambda_1}^2} + cosh(L\lambda_1)\cos\left(L\sqrt{k^2+{\lambda_1}^2}\right)\left(k^4 + 2k^2{\lambda_1}^2 + 2{\lambda_1}^4\right) + 2{\lambda_1}^2\left(k^2+{\lambda_1}^2\right) = 0. \tag{23}$$

Thus, with the mathematical development, one can obtain the general equation that describes the natural vibration frequency of a beam considering the internal load of the element in a more simplified form, being the same as EI, $\bar{m}$, k^2 and λ_1, as shown:

$$w = \sqrt{\frac{EI}{\bar{m}}\beta^4} = \sqrt{\frac{EI}{\bar{m}}(k^2{\lambda_1}^2 + {\lambda_1}^4)}. \tag{24}$$

Given a unit value (A = 1) in Eq. (21), we obtain:

$$B = \frac{-{\lambda_2}^2 cos(\lambda_2 L) - {\lambda_1}^2 \cos h(\lambda_1 L)}{\lambda_1\lambda_2 sen(\lambda_2 L) + {\lambda_1}^2 senh(\lambda_1 L)}. \tag{25}$$

Substituting A and B in Eq. (14) we obtain the equation of the modes of vibrations:

$$\Phi(x) = [\cos h(\lambda_1 x) - \cos(\lambda_2 x)] + B[senh(\lambda_1 x) - \frac{\lambda_1}{\lambda_2} sen(\lambda_2 x)], \quad (26)$$

in which λ_2 is given by:

$$\lambda_2 = \sqrt{k^2 + {\lambda_1}^2}. \quad (27)$$

To normalize the mass modes, the modes are multiplied by an α factor. Therefore, the factor α in relation to the normalization of the mass modes is given by:

$$\alpha = \frac{1}{\sqrt{\int_0^L \Phi_i \Phi_i \bar{m} dx}}. \quad (28)$$

2.5 Parameter γ_z for a free-fixed column

Considering the compressed beam as shown in Fig. 3-(a), and considering the equilibrium equation in relation to the fixed support (Fig. 3-b), that is:

$$M(x) = F(L-x) + P\delta - Pu(x) + p\frac{x^2}{2} - pLx + p\frac{L^2}{2} + M. \quad (29)$$

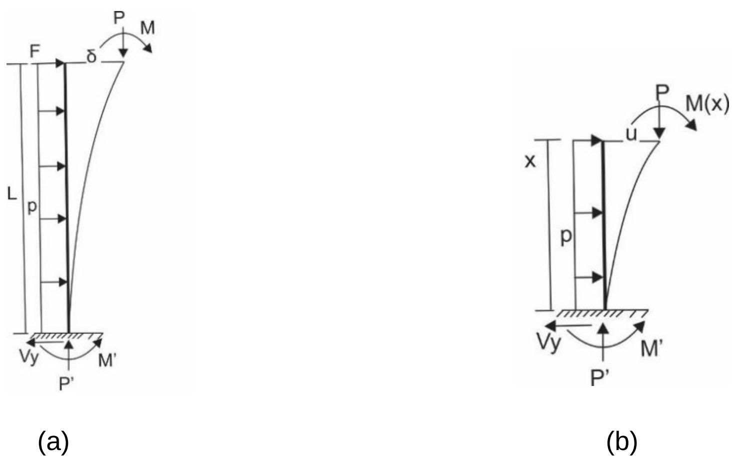


Figure 3: (a) Representation of the free-fixed column in the undisturbed condition (b) deformed condition of the free-fixed column

We can obtain a differential equation of simplified equilibrium using the relation between the curvature and the bending moment:

$$\frac{EI\partial^2 u(L,t)}{\partial x^2} + Pu(x) = F(L-x) + P\delta + p\frac{x^2}{2} - pLx + p\frac{L^2}{2} + M. \quad (30)$$

With the boundary conditions of the element shown in figure 4, applying the Laplace transform in equation (33) and showing u (s), we obtain:

$$u(s) = \frac{z^2\left(\frac{L}{s} - \frac{1}{s^2}\right)}{(s^2+k^2)} + \frac{k^2\delta\frac{1}{s}}{(s^2+k^2)} + \frac{d^2\left(\frac{L^2}{2s} - \frac{L}{s^2} + \frac{1}{s^3}\right)}{(s^2+k^2)} + \frac{\frac{j^2}{s}}{(s^2+k^2)}, \quad (31)$$

where:

$$z^2 = \frac{F}{EI} \;; k^2 = \frac{P}{EI} \;;\; d^2 = \frac{p}{EI} \;;\; j^2 = \frac{M}{EI}. \quad (32)$$

The solution of this equation is given by applying the inverse Laplace transform. Being plot A related to lateral load F, plot B to top displacement, plot C to M' and plot D to load p, the equation of lateral deflection of a compressed bar in plots is defined:

$$A(x) = z^2 \frac{kL - kx + sen(kx) - kL\cos(kx)}{k^3}, \quad (33)$$

$$B(x) = \delta[1 - cos(kx)], \tag{34}$$

$$C(x) = -\frac{1}{2}d^2\frac{(-k^2)L^2+2+2Lxk^2-x^2k^2+k^2L^2\cos(kx)-2\cos(kx)-2Lksen(kx)}{k^4}, \tag{35}$$

$$D(x) = (-j^2)\frac{(-1)+cos\,(kx)}{k^2}, \tag{36}$$

$$u(x) = A + B + C + D. \tag{37}$$

To obtain δ from Eq. (34) it is necessary to evaluate at x = L, considering that at x = L the value of u (L) must be equal to δ, thus obtaining the value of the same. However, to obtain the value of δ, an optimization method is necessary, considering that the equation is nonlinear.

$$u(L) = \delta = A(L) + B(L) + C(L) + D(L). \tag{38}$$

Evaluating at x = 0 to Eq. (29) and without the second-order effect, we obtain the static moment:

$$M1(0) = FL + p\frac{L^2}{2} + M. \tag{39}$$

Thus, by evaluating at x = 0 to Eq. (29) and with the second-order effect, we obtain:

$$M2(0) = FL + P\delta + p\frac{L^2}{2} + M. \tag{40}$$

Thus, having the increase given by:

$$\frac{M2}{M1} = X_z = \frac{FL+P\delta+p\frac{L^2}{2}+M}{FL+p\frac{L^2}{2}+M}, \tag{41}$$

where F is axial load at x, L is the length, P is the axial load at y, p is the distributed lateral load and M is the moment at the element.

2.6 Numerical solution for the free vibration equation

The Finite Element Method (FEM) is used to solve the differential equations. Galerkin's method is one of the weighted waste methods, being the same used in the development of this work [13].

It is necessary to adopt as form functions a polynomial of the third degree:

$$\tilde{u}(x) = \phi_1(x)v'_i + \phi_2(x)\theta_i + \phi_3(x)v'_j + \phi_4(x)\theta_j, \tag{42}$$

being:

$$\phi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, \tag{43}$$

$$\phi_2(x) = x - \frac{2x^2}{L} + \frac{x^3}{L^2}, \tag{44}$$

$$\phi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3, \tag{45}$$

$$\phi_4(x) = \frac{-x^2}{L} + \frac{x^3}{L^2}. \tag{46}$$

And the matrices are given as a result of the following integrals:

$$K_{ij} = EI\int_0^L \frac{\partial^2\phi_j(x)}{\partial x^2}\frac{\partial^2\phi_i(x)}{\partial x^2}dx = [K_{eviga}], \tag{47}$$

$$[K_{eviga}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \tag{48}$$

$$[K_{gbarra}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{49}$$

$$K_{gij} = -\int_0^L P(x)\frac{\partial\phi_i(x)}{\partial x}\frac{\phi_j(x)}{\partial x}dx = \left[K_{gviga}\right], \tag{50}$$

$$\left[K_{gviga}\right] = \begin{bmatrix} \frac{-6P}{5L} & \frac{-P}{10} & \frac{6P}{5L} & \frac{P}{10} \\ \frac{-P}{10} & \frac{-2PL}{15} & \frac{P}{10} & \frac{PL}{30} \\ \frac{6P}{5L} & \frac{P}{10} & \frac{-6P}{5L} & \frac{P}{10} \\ \frac{-P}{10} & \frac{PL}{30} & \frac{P}{10} & \frac{-2PL}{15} \end{bmatrix}, \tag{51}$$

where, in the local coordinate system, we have the elastic stiffness matrix of the beam $\left[K_{eviga}\right]$, the geometric stiffness matrix of the bar $\left[K_{gbarra}\right]$ and the geometric stiffness matrix of the beam $\left[K_{gviga}\right]$.

Thus, we obtain the total stiffness matrix of 2D frame in local coordinates $[K_T]$ by the combination of bar and beam members:

$$[K_T] = \begin{bmatrix} a_1 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 12a_2 - \frac{6P}{5L} & 6La_2 - \frac{P}{10} & 0 & -12a_2 + \frac{6P}{5L} & 6La_2 - \frac{P}{10} \\ 0 & 6La_2 + \frac{-P}{10} & 4L^2a_2 - \frac{2PL}{15} & 0 & -6La_2 + \frac{P}{10} & 2L^2a_2 + \frac{PL}{30} \\ -a_1 & 0 & 0 & a_1 & 0 & 0 \\ 0 & -12a_2 + \frac{6P}{5L} & -6La_2 + \frac{P}{10} & 0 & 12a_2 - \frac{6P}{5L} & -6La_2 + \frac{P}{10} \\ 0 & 6La_2 - \frac{P}{10} & 2L^2a_2 + \frac{PL}{30} & 0 & -6La_2 + \frac{P}{10} & 4L^2a_2 - \frac{2PL}{15} \end{bmatrix}. \tag{52}$$

where $a_1 = AE/L$ e $a_2 = EI/L^3$

Just as there is a need for knowledge of the mass matrix, given by:

$$M_{ij} = \int_0^L m\ddot{u}\,\phi_i\phi_j(x)dx = [M_1], \tag{53}$$

$$[M_1] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}, \tag{54}$$

$$[M_2] = \rho AL\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}, \tag{55}$$

where the consistent mass matrix of the beam $[M_1]$ and the consistent mass matrix of the bar $[M_2]$ is given in the local coordinate system.

Obtaining [M] by combining $[M_1]$ with $[M_2]$:

$$[M] = \begin{bmatrix} \frac{1}{3}a_3 & 0 & 0 & \frac{1}{6}a_3 & 0 & 0 \\ 0 & 156a_4 & 22La_4 & 0 & 54a_4 & -13La_4 \\ 0 & 22La_4 & 4L^2a_4 & 0 & 13La_4 & -3L^2a_4 \\ \frac{1}{6}a_3 & 0 & 0 & \frac{1}{3}a_3 & 0 & 0 \\ 0 & 54a_4 & 13La_4 & 0 & 156a_4 & -22La_4 \\ 0 & -13La_4 & -3L^2a_4 & 0 & -22La_4 & 4L^2a_4 \end{bmatrix}, \tag{56}$$

$$a_3 = \rho AL, \tag{57}$$

$$a_4 = \frac{\rho AL}{420}. \tag{58}$$

2.7 Modal analysis - natural frequencies and modes of vibration including the effect of axial loading

The dynamic equation, for the study of undamped free vibration is given by:

$$[M]\{\ddot{u}\} + [K_T]\{u\} = \{f_n\}, \tag{59}$$

where [M] is the mass matrix, $[K_T]$ is the total stiffness matrix, $[f_n]$ is the vector of external forces, {ü} is the acceleration and {u} is the vector of physical displacements.

Taking $[K_T]$ and [M] and with the boundary conditions, we can make the determinant of the coefficient matrix to be zero so that we have a solution of the eigenvalues:

$$det([K_T] - \lambda_n[M]) = 0. \tag{60}$$

To have a solution of the eigenvectors simply replace the eigenvalues in the original equation. The eigenvalues will then be associated with the eigenvectors found.

$$([K_T] - \lambda_n[M])\{\emptyset_n\} = 0, \tag{61}$$

$$\lambda_n = w^2. \tag{62}$$

Thus, the values of $\lambda_n = {w_n}^2$ are the eigenvalue results and the results of $\{\emptyset_n\}$ are the corresponding eigenvectors, where w_n is the natural vibration frequency of the structure.

The orthogonality properties of a undamped system of various degrees of freedom can be manifested from a coefficient α as follows:

$$\alpha_i = \frac{1}{\sqrt{\{\emptyset_i\}^T[M]\{\emptyset_i\}}}. \tag{63}$$

The normalized mass mode $\{\overline{\emptyset}_i\}$ can be obtained as:

$$\{\overline{\emptyset}_i\} = \frac{\{\emptyset_i\}}{\sqrt{\{\emptyset_i\}^T[M]\{\emptyset_i\}}}. \tag{64}$$

2.8 Displacement as a function of natural frequency and vibration mode

In order to calculate the displacement from the vibration mode of the structure and its natural frequency, the following development is necessary:

$$\{u\} = [\emptyset]\{q\}, \tag{65}$$

having:

$$u_1 = \emptyset_{11}q_1 + \emptyset_{12}q_2 + \cdots + \emptyset_{1n}q_n, \tag{66}$$

Relating to the dynamic equation for the study of undamped vibration and with some reorganizations we have:

$$u_1 = \frac{{\emptyset_{11}}^2 f_1(t) + \emptyset_{11}\emptyset_{21}f_2(t) + \cdots + \emptyset_{11}\emptyset_{n1}f_n(t)}{w^2{}_{n1}} + \frac{{\emptyset_{12}}^2 f_1(t) + \emptyset_{12}\emptyset_{22}f_2(t) + \cdots + \emptyset_{12}\emptyset_{n2}f_n(t)}{w^2{}_{n2}} + \cdots + \frac{\emptyset_{1n}\emptyset_{1n}f_1(t) + \emptyset_{1n}\emptyset_{2n}f_2(t) + \cdots + \emptyset_{1n}\emptyset_{nn}f_n(t)}{w^2{}_{n2}q_n}, \tag{67}$$

Assuming for example with two degrees of freedom - Fig. (4).

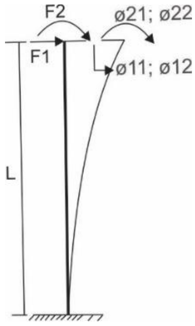


Figure 4: Free-fixed column bar with the representation of the boundary conditions at the top

We can rewrite Eq. (67) as:

$$u_1 = \frac{\phi_{11}{}^2 f_1(t) + \phi_{11}\phi_{21} f_2(t)}{w^2{}_{n1}} + \frac{\phi_{12}{}^2 f_1(t) + \phi_{12}\phi_{22} f_2(t)}{w^2{}_{n2}}. \quad (68)$$

Without considering the effect of the moment we can have an equation capable of calculating the displacements from the modes of vibrations and the natural frequencies of vibration:

$$u_1 = \frac{\phi_{11}{}^2 f_1(t)}{w^2{}_{n1}} + \frac{\phi_{12}{}^2 f_1(t)}{w^2{}_{n2}}. \quad (69)$$

Considering the contribution of a single frequency it is possible to calculate the displacement from the mode of vibration and the natural frequency of vibration, as follows:

$$u_1 \approx \frac{\phi_{11}{}^2 f_1(t)}{w^2{}_{n1}}. \quad (70)$$

3 Methodology

The research will develop in the verification of the susceptibility of structures in reinforced concrete to second-order effects from the analysis of natural frequency and mode of vibration. For this, analyzes will be performed in a free-fixed column.

The amplification factors and the natural frequencies of vibration of the model will have their due analysis, as well as the analysis of the influence of the loads in the natural frequencies.

Therefore, the following steps will be established for the study:

1) Analytically develop the equations for the natural frequencies and modes of vibration for a free-fixed column;

2) To compare analytically and numerically the natural frequencies and modes of vibration for a free-fixed column;

3) To analyze the results of natural frequencies and modes of vibration of a free-fixed column with and without axial load;

4) Analyze the relationship between load versus natural frequency of vibration in a free-fixed column;

5) Obtain the relationship between the stability parameter and the natural vibration frequency;

6) Verify, from the analyzes, that the natural frequency can be used as an indicator for the second-order effects, serving as the basis for the overall instability check-in a reinforced concrete free-fixed column.

4 Results

For the numerical and analytical modeling, a reinforced concrete pillar model was chosen, being free-fixed in the base, with a distributed mass $\overline{m}$ and stiffness along the bar (EI), with free vibration and an axial load P(x), as in Fig. 6 and with properties presented in Table 1.

Table 1 Free-fixed column properties

Length (L)	3	(m)
$\overline{m}$ =ρA	100	(kg/m²)
Modulus of elasticity (E)	1.00E+07	(N/m²)
Transversal section	0.04	(m²)
Inertia (I)	0.000133	(m^4)
Critical Load (Pcrít.)	365	(N)

The unknown represented by $\overline{m}$ is equal to the specific mass of the concrete (ρ) multiplied by the area of the cross-section (A) of the chosen element.

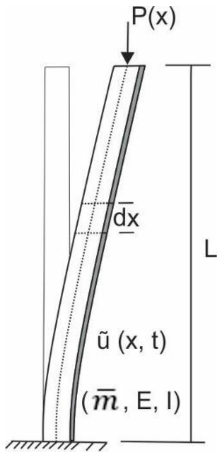


Figure 5: Free-fixed column $\tilde{u}(x,t)$

4.1 Graphic λ_1

To determine the natural frequency analytically, Eq. (24), there is a need for prior knowledge of the values of λ_1, obtained from Eq. (23) with the values of the zeros of the function. Figure 6 shows the function of Eq. (23) in logarithmic coordinates in the (y) axis, where zeros of the function are identified which identify the λ_1 without the addition of charge for the first 6 natural frequencies in the (x) axis.

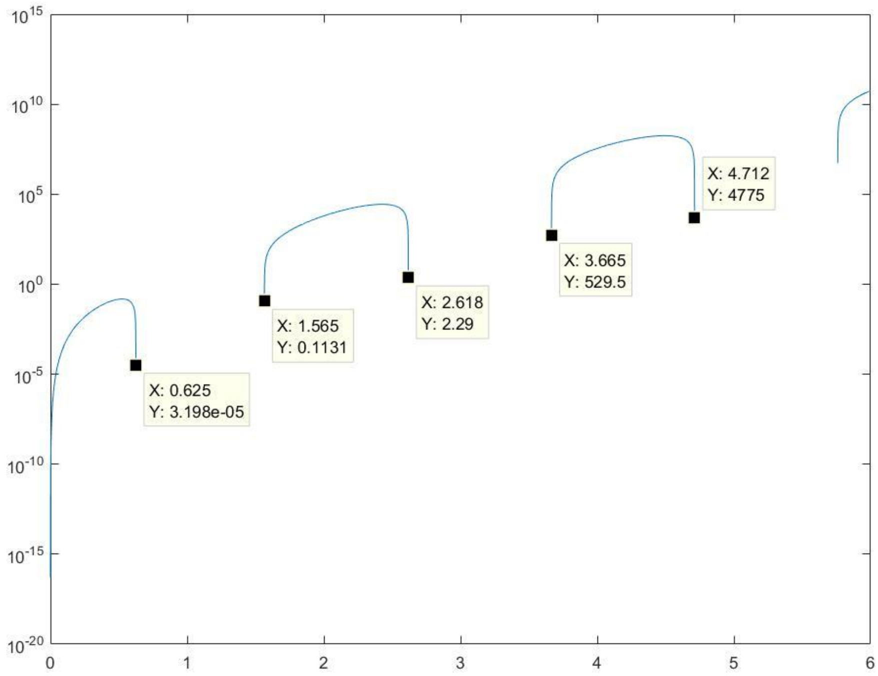


Figure 6: λ_1 no load

As the critical load of the studied free-fixed column of 365N, Figure 7 presents the function of Eq. (23) in logarithmic coordinates in the (y) axis, where we can see zeros values of the function that identify the values of λ_1 of the structure with the addition of half the critical load of the studied structure. Therefore, for the studied free-fixed column the value of the axial load P is 182N for the first 6 natural frequencies in the (x) axis.

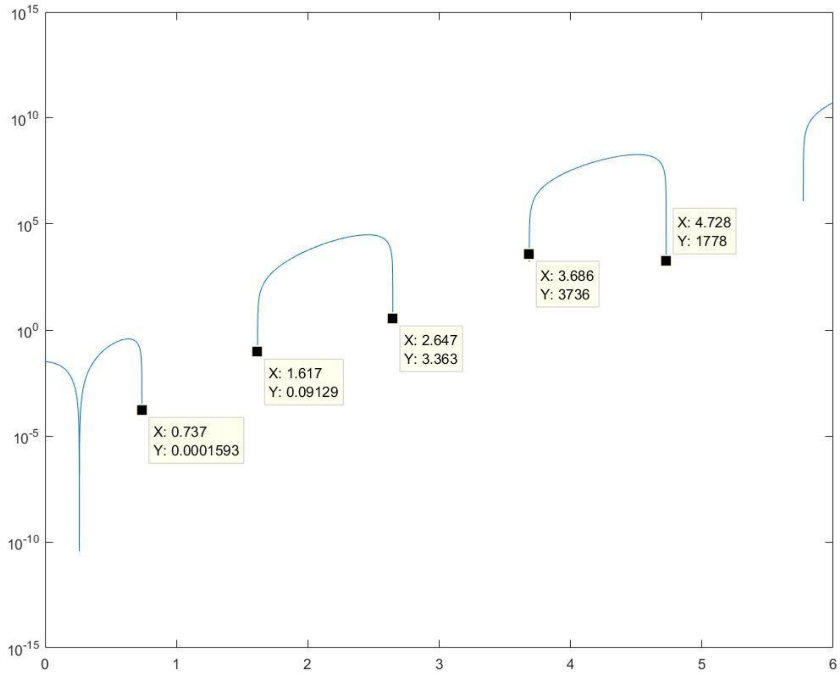


Figure 7: λ_1 with load

Note that it is possible to identify the values of λ_1 of a structure with no axial load (Fig. 6) and with the addition of axial load P (Fig. 7) with Eq. (23). The prior knowledge process of λ_1 becomes necessary since Eq. (24) depends on λ_1, the properties of the material (EI and $\overline{m}$) that have already been given in Table 1 and the axial load P, k^2 = P / EI.

4.2 Comparison between the analytical solution and numerical solution

Taking Eq. (24), the properties in Table 1 and the values of λ_1 shown in Fig. 6 and Fig. 7, it is possible to obtain the natural vibration frequencies of the column analytically without axial load P as shown in Table 2 and with the addition of half the critical load of the chosen element (P equal to 182N) as shown in Table 3. The value of the natural frequencies is given by w_n, where the values from the first to the fifth frequency natural vibration.

The numerical results were obtained with the matrix analysis methodology, where it was shown that the values of $\lambda_n = {w_n}^2$ are the results of the eigenvalues and the results of $\{\emptyset_n\}$ are the corresponding eigenvectors, where w_n is the natural frequency of vibration of the structure.

Table 2 Comparison between the analytical solution and the approximate solution with the discretization of the elements (without load)

n	$\boldsymbol{\lambda_1}$	$\boldsymbol{w_n}$			$\boldsymbol{w_n}$		$\boldsymbol{w_n}$	
		Analytical (Hz)	Numeric 3 discret. (Hz)	% Error	Numeric 5 discret. (Hz)	% Error	Numeric 25 discret. (Hz)	% Error
1	0.63	1.426519	1.4266	0.0101	1.4265	0.0014	1.4265	0.0000
2	1.57	8.939843	8.9692	0.3284	8.9443	0.0500	8.9398	0.0001
3	2.62	25.03182	25.3437	1.2460	25.1217	0.3592	25.0319	0.0007
4	3.67	49.05238	57.0731	16.3514	49.6276	1.1727	49.0536	0.0026
5	4.78	81.08709	107.4117	32.4647	82.3694	1.5815	81.0927	0.0070

It is observed that the analytical and numerical solution for natural frequency in the condition without axial load P at the top and with the discretization of the elements are compared in Table 2. The error percentage was made of the numerical solution with three, five and twenty and five discretizations of the free-fixed column relative to the analytical solution. The analytical and numerical method showed a frequency of 33 Hz longitudinal, which does not change with the loading, being disregarded.

Note that for the first frequency the solution obtained by means of the finite element model is very close to the analytical solution. From the second frequency, the numerical model has errors, being reduced with the discretization method as seen in Table 2.

Being the element subjected to half of the critical load, that is, 182N, the analytical and numerical solution with the discretization of the elements are compared in Table 3.

Table 3 Comparison between the analytical solution and the approximate solution with the discretization of the elements (with load)

x	λ_1	w_n			w_n		w_n	
		Analytical (Hz)	Numeric 3 discret. (Hz)	% Error	Numeric 5 discret. (Hz)	% Error	Numeric 25 discret. (Hz)	% Error
1	0.737	1.71651	1.71680	0.0170	1.71655	0.0022	1.71651	0.0000
2	1.617	9.298253	9.32937	0.3346	9.30298	0.0509	9.29826	0.0001
3	2.647	25.3421	25.65635	1.2400	25.43274	0.3577	25.34227	0.0007
4	3.685	49.34605	57.35255	16.2252	49.92218	1.1675	49.34731	0.0026
5	4.728	81.37106	107.67510	32.3260	82.65189	1.5741	81.37673	0.0070

The first frequency of the solution obtained by means of the finite element model is very close to the analytical solution. Already from the second frequency, the numerical model has errors, being reduced with the discretization process as seen in Table 3.

4.3 Graphic frequency vs. load

In order to understand the behavior of the first natural frequency of the free-fixed in an analytical and numerical way, for the different levels of load, the graph of Figure 10 was generated.

This plot (Fig. 8) was obtained by means of Eq. (24) of the properties presented in Table 1 and the values of λ_1 by means of Eq. (23). Numerically the graph (Fig. 8) was obtained from the total stiffness matrix $[K_T]$ of the structure by varying the load value and obtaining the natural frequency for each load applied.

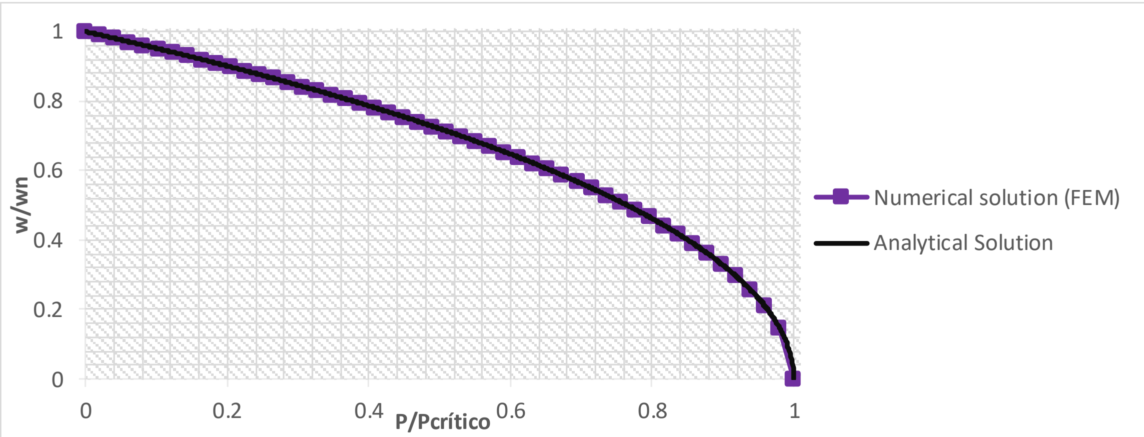


Figure 8: Relation between the axial compression load and the first natural frequency for a free-fixed column, analytical and numerical

The expression for the axially loaded natural frequency Eq. (24) clearly shows that an axial force reduces the natural frequency of the structure. Figure 8 shows that the natural frequency of a free-fixed column is decreased by an axial compressive load, having a considerable change in the first frequency. There is little change comparing analytically and numerically.

From the results of Fig. 10, an approximation is given by:

$$f(x) = \left(1 - \frac{P}{Pcrit}\right)^{\frac{1}{2.091}}, \tag{71}$$

$$W_p = W_n \left(1 - \frac{P}{Pcrit}\right)^{\frac{1}{2.091}}. \tag{72}$$

When plotting the second natural frequency of vibration (Fig. 9), it is observed that the second natural frequency is decreased by an axial compressive load, however, having little change. There is little change comparing analytically and numerically.

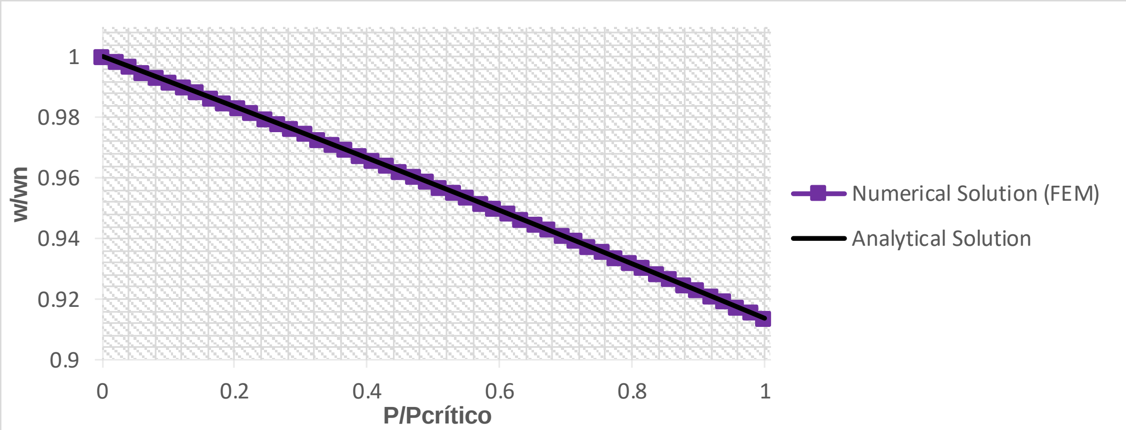


Figure 9: Relation between the axial compression load and the second natural frequency for a free-fixed column, analytical and numerical

Thus, by measuring the natural frequency of a column, it is theoretically possible to calculate the axial load. Note that there is a direct relationship between free vibration and buckling of the column.

4.4 Vibration modes

For a better understanding of the displacement along the length of the free-fixed column, subjected to an axial load P of 182N, the first mode of vibration obtained was analyzed analytically and numerically. The results obtained can be observed in Fig. 10.

The analytical vibration modes were obtained by means of Eq. (26), previously requiring Eq. (25), as well as the properties presented in Table 1 and the values of λ_1 shown in Fig. numerical form modes were obtained by the corresponding eigenvectors of Eq. (61). The modes obtained numerically and analytically were normalized to mass.

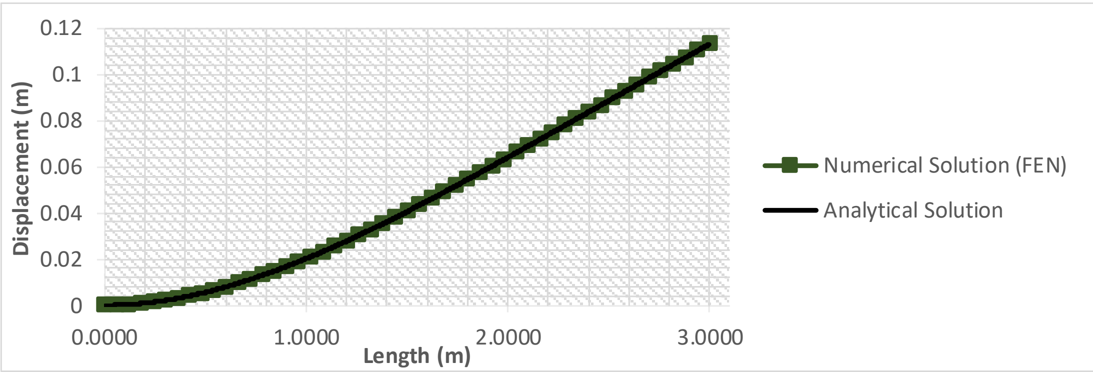


Figure 10: Comparison between the 1° mode of analytical and numerical vibration. (With a load of 182 N)

It is observed that there is little change comparing analytically and numerically the first vibration mode of the free-fixed column subjected to half the critical load thereof.

Thus, Fig. 11 illustrates the first five modes of vibration, obtained analytically. It was observed both in the analytical method and in the numerical method that there was a longitudinal vibration mode corresponding to a frequency of 33 Hz, which does not change with loading and is disregarded.

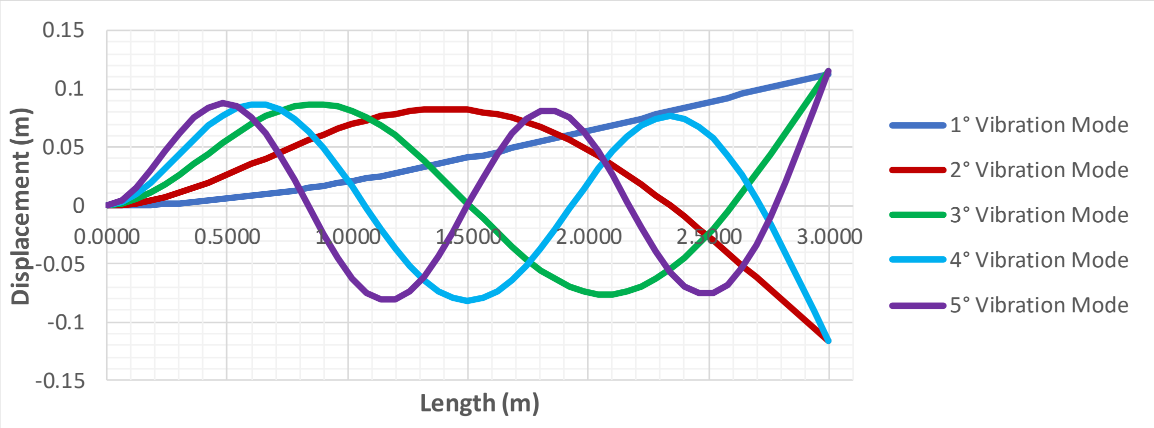


Figure 11: Vibration modes. (With a load of 182 N)

In order to explore the behavior of the first mode of vibration as a function of the modification of the axial load P at the top of the column, the critical load of approximately 365N was divided into five, adopting for each analysis a different axial load P, with an axial load P of 91.38N, 182.77N, 274.16N and 365.54N as shown in Fig. 12.

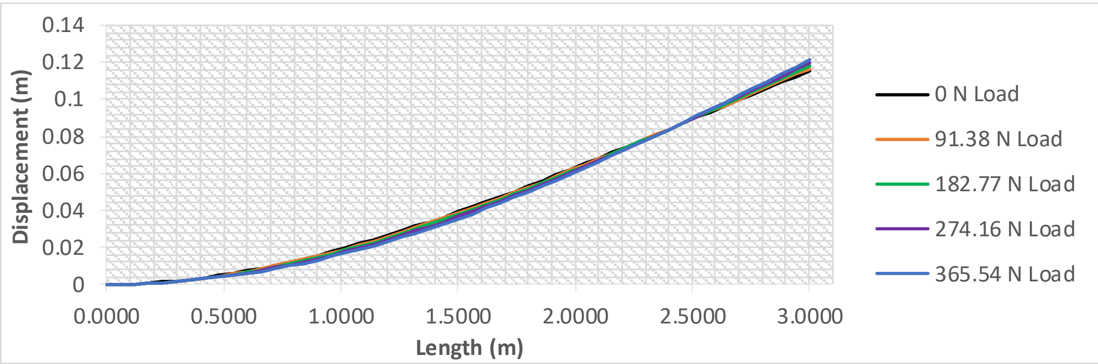


Figure 12: 1° Vibration mode. (Varying the load)

It can be seen from Figure 12 that the axial load P in which the free-fixed column is subjected modifies the first mode of vibration of the structure.

4.5 Majority of the moment

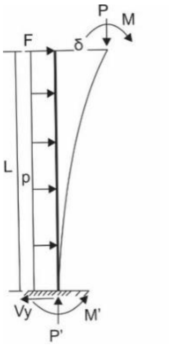
A reinforced concrete free-fixed column as shown in Fig. 13, with properties shown in Table 4, is modeled analytically in order to understand the increse of first-order stresses as a function of the first-order moment for the consideration of second-order effects.

Table 4 Free-fixed column properties

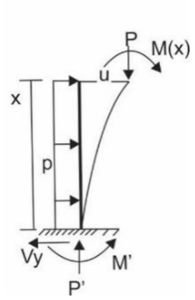
Length (L)	3	(m)
Modulus of elasticity (E)	1.00E+07	(N*m²)
Section	0.04	(m²)

Inertia (I)	0.000133	(m^4)

The calculation of the M2 / M1 amplification as a function of axial load increase (P) at the top of the element, that was demonstrated in Eq. (44), is dependent on the axial load at x (represented by F), the length of the free-fixed column (L), the axial load at y (represented by P), the distributed lateral load (represented by p) and the moment at element (M), as can be seen in Fig. 13.



(a)



(b)

Figure 13: (a) Representation of the free-fixed column in the undisturbed condition (b) deformed condition of the free-fixed column

However, to exemplify the behavior of the free-fixed column shown in Fig. 14 is calculated to the amplification M2 / M1 as a function of the increase of axial load (P) at the top of the element to the condition of an element with the same properties shown in Table 4 with or without influence of the lateral load (F), the distributed load (p) and the moment at the top (M), as shown in Fig. 14.

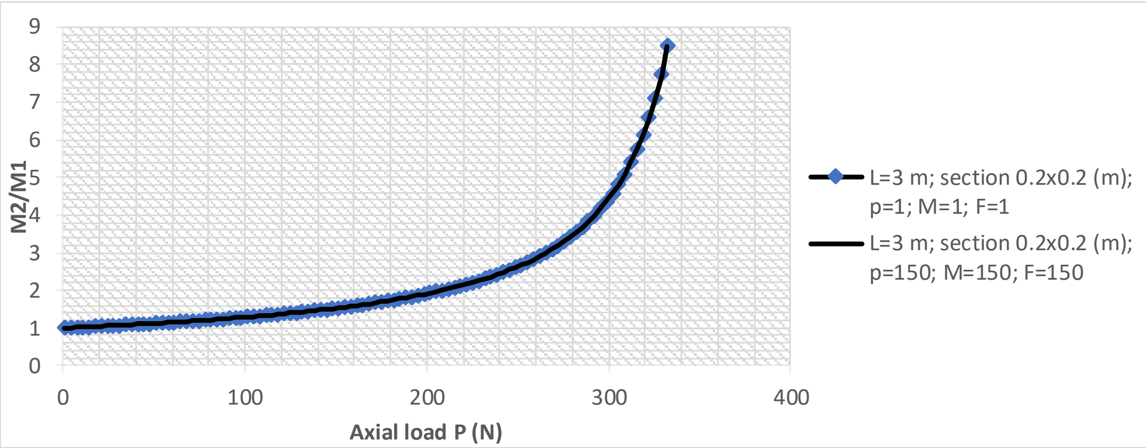


Figure 14: Amplification due to the increase in load - M2/M1

It is observed from the analysis shown in Fig. 14 that the amplification ratio M2 / M1 as a function of the increase of the axial load (P) at the top of the element independent of the lateral load (F), of the distributed load (p) and of the moment at the top (M).

It is also calculated to the magnification M2 / M1 as a function of increasing the axial load (P) at the top of the element to have the behavior of the free-fixed column shown in Fig. 13, but modifying the length (L) and section, as shown in Table 5 and Fig. 14.

Table 5 Free-fixed column properties

	Free-fixed column 01	**Free-fixed column 02**	**Free-fixed column 03**	**Units of measure**
Length (L)	3	4	3	(m)

Modulus of elasticity (E)	1.00E+07	1.00E+07	1.00E+07	(N*m²)
Section	0.04	0.075	0.075	(m²)
Inertia (I)	0.000133	0.0005625	0.0005625	(m^4)

Three elements were adopted in order to verify the influence of the change of properties of the element, such as section and length (L) in the behavior of the vibration modes of the structure.

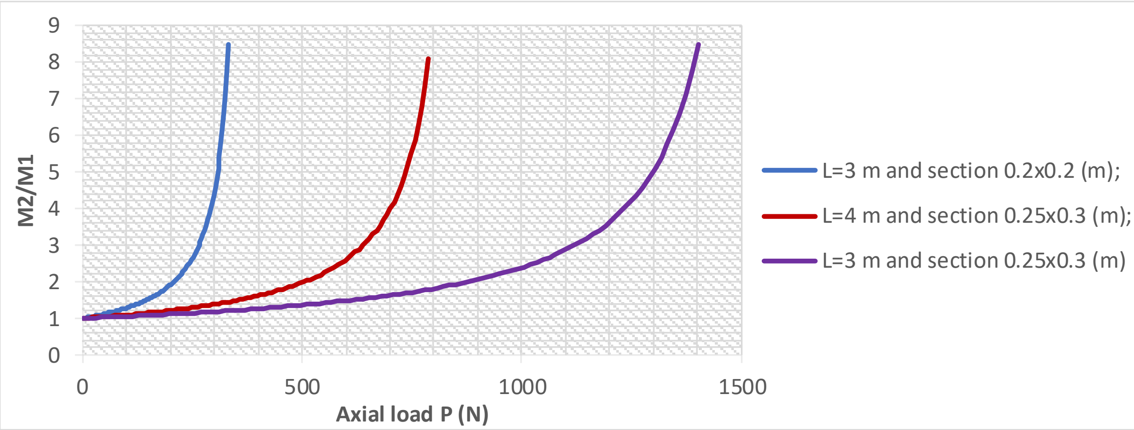


Figure 15: Amplification M2/M1 as a function of the increase of load with the change of length and section of the structural element

It may be noted that the amplification of M2/M1 is directly associated to changing the geometries of the free-fixed column, but are little influenced by lateral loads. It is worth remembering that the amplification M2/M1 as a function of increasing the axial load (P) at the top of the element shown so far (Fig. 15) uses a rather complex methodology.

It is also noted that considering the contribution of a single frequency it is possible to calculate the displacement from natural frequency and the vibration mode, as shown in Eq. (70). As such equation depends on the natural frequency of vibration and mode, the prior knowledge of both is necessary.

In order to obtain a simplified formulation, Eq. (72) has found that from the first vibration frequency of the element (w_n), the axial load at the top and the critical load of the studied element, it is possible to obtain the natural frequency of vibration related to the load level P. The first vibration frequency (w_n) of the element without considering the axial load at the top of the element is given according to Paz (1997) by:

$$W_n = (\alpha_n L)^2 \sqrt{\frac{EI}{\bar{m}L^4}}, \tag{73}$$

$$W_1 = \frac{1{,}875^2}{L^2} \sqrt{\frac{EI}{\bar{m}}}. \tag{74}$$

The vibration modes are obtained from Eq. (26) and evaluated in L = 3m. It is observed that λ_1 is obtained from Eq. (9) and λ_2 is by Eq. (27) and both depend on the relations given in Eq. (9) and (8). It is also necessary previously to normalize the vibration modes from Eq. (28).

Thus, to exemplify the behavior of the free-fixed column shown in Fig. 13, it is calculated to the amplification M2/M1 as a function of the increase of axial load (P) at the top of the element to the condition of an element with the same properties shown in Table 4, but from the simplification shown in Eq. (72), with F equal to 150 N, disregarding the distributed load (p) and moment at the top (M).

Looking at Fig. 13 and considering as external forces only the point force F and the axial load P, we have:

$$\frac{M2}{M1} = X_w = \frac{FL + Pu_1}{FL}. \tag{75}$$

The development to obtain amplification M2 / M1 as a function of the increase of axial load (P) at the top from natural frequency and vibration mode is shown in Fig. 16.

$$w_n = w_1 = \frac{1.875^2}{L^2}\sqrt{\frac{EI}{\bar{m}}};\quad Pcrit = \frac{\pi^2 EI}{(2L)^2}$$

→

$$w_p = w_n\left(1 - \frac{P}{Pcrit}\right)^{1/2.091}$$

→

$$k^2 = \frac{P}{EI};\quad \beta^4 = \frac{{w_p}^2\bar{m}}{EI}$$

↓

$$\lambda_1 = \sqrt{-\frac{k^2}{2}\sqrt{\frac{k^4}{4}+\beta^4}};\quad \lambda_2 = \sqrt{k^2 + {\lambda_1}^2}$$

↓

$$\emptyset(\boldsymbol{L}) = [cosh(\lambda_1 L) - cos(\lambda_2 L)] + B\left[senh(\lambda_1 L) - \frac{\lambda_1}{\lambda_2}sen(\lambda_2 L)\right]$$

$$B = \frac{-{\lambda_2}^2 cos(\lambda_2 L) - {\lambda_1}^2 \cos h(\lambda_1 L)}{\lambda_1\lambda_2 sen(\lambda_2 L) + {\lambda_1}^2 senh(\lambda_1 L)}$$

$$\emptyset_{\boldsymbol{n}}(\boldsymbol{L}) = \frac{\emptyset(L)}{\sqrt{\bar{m}\int_0^L \emptyset(x)^2 dx}}$$

↑

$$u_1 = u(L) = \frac{\emptyset_{11}^2 f_1(t)}{w_{n1}^2} = \frac{\emptyset_{\boldsymbol{n}}(\boldsymbol{L})^2 F}{\boldsymbol{w_p^2}}$$

←

Majority of moment.

$$\frac{M2}{M1} = X_w = \frac{FL + Pu_1}{FL}$$

Figure 16: Organization of the procedures to obtain the amplification M2/M1 as a function of the increase of axial load (P) at the top from natural frequency and vibration mode

Fig. 16 shows the relationship of the amplification solutions M2/M1 as a function of the axial load increase (P) in an exact but complex and difficult way already demonstrated from Eq. (41) and the simplified form is shown in Eq. (75). It is observed that the simplification is in obtaining the displacement from the dynamic characteristics of the free-fixed column, that is, Eq. (72) and Eq. (26).

It is possible to note that the simplified equation of modal analysis as a method of estimating second-order effects on a free-fixed column had a good approximation of the traditional exact method.

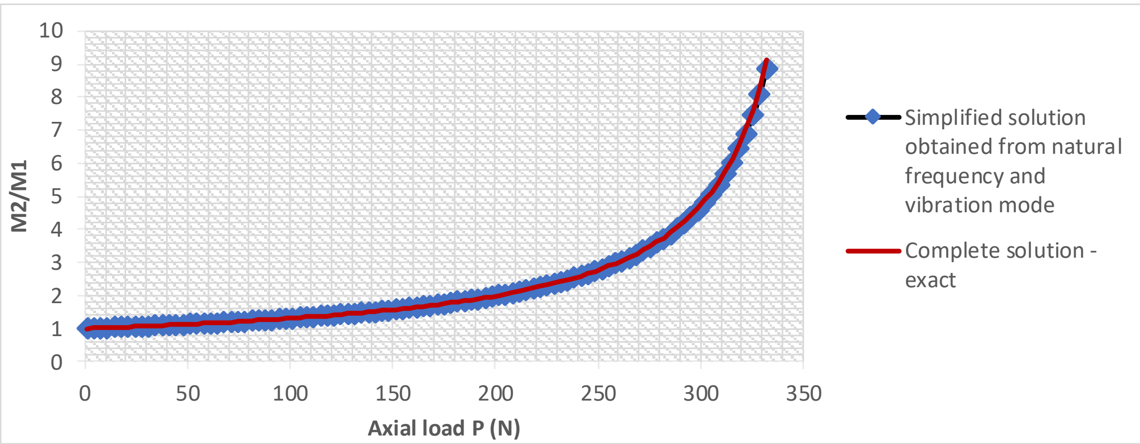


Figure 17: Comparison of the M2/M1 Amplification as a function of the increase in load accurately and with the use of modal analysis

Also, to demonstrate the error ratio of the exact solution obtained from Eq. (41) in relation to the solution obtained in a simplified way by Eq. (75), Figure 18 is shown. Note that with increasing load there is an error, in which it approaches 3% when the axial load (P) approaches the critical load (Pcrit) of the free-fixed column.

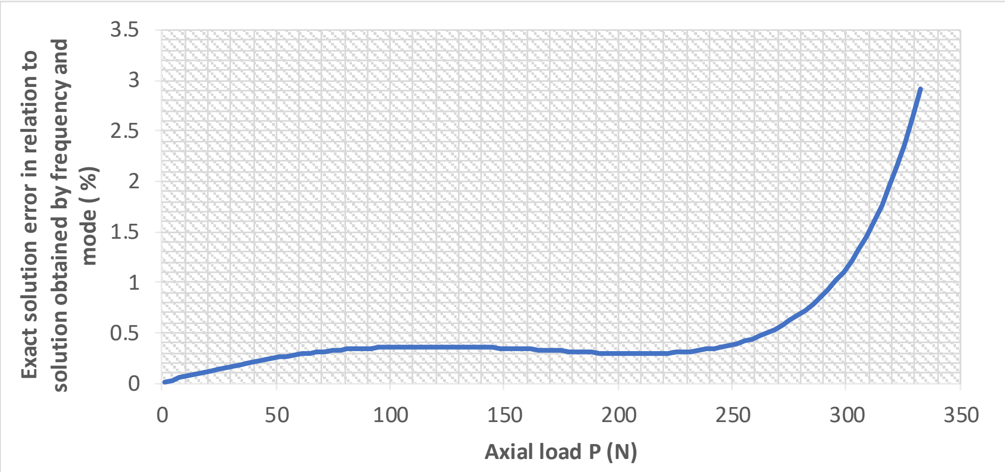


Figure 18: Exact solution error in relation to the solution obtained by natural frequency and vibration mode

5 Final considerations

This work aims to study the dependence between the dynamic characteristics (natural frequencies and modes of vibration) and the loads in the structures of a free-fixed column. With such relation find an alternative parameter based on the modal analysis that can indicate the susceptibility of the structure a suffers second-order effects.

The analytical and numerical analyses were performed on a free-fixed column without axial load P at the top of the element and with an axial load P at the top equal to half the critical element load - 182N. These comparisons provided the verification that in both cases there is a good approximation in the first natural frequency of the vibration from the numerical to the analytical and from the second frequency the numerical model has errors and is decreased with the discretization method.

It has also been found that the natural frequency is decreased by an axial load having a considerable change in the first natural frequency of vibration and little change to the second frequency of vibration having little change in the analytical results over the numerical ones. These analyzes allowed to obtain by means of the graphs a simplified approximation equation for the natural vibration frequency of a free-fixed column.

It was verified that both the analytical and numerical solutions present good results in relation to the vibration modes and the natural frequencies of vibration. It has been found that the first mode of vibration differs when subjected to different types of axial load P.

In this work, the solution of moment amplification by the second-order effect in a free-fixed column was also presented. It was observed that the amplification ratio M2/M1 as a function of the increase of axial load (P) on the top of the element independent of the lateral load (F), the distributed load (p) and the moment at the top (M). However, it is possible to verify a change in M2/M1 amplification when the section type and section type geometry is modified.

With the use of the dynamic characteristics, it was possible to obtain the displacement at the top of the free-fixed column and then the calculation of the amplification M2/M1 as a function of the increase of axial load (P) at the top of the element. It is important to note that a simplified equation was obtained to obtain the natural frequency of vibration considering the axial load at the top of the column.

With this study is expected to assist future research on the use of modal analysis as a method of estimation of second-order effects, as well as the poorly disclosed development of the analytical formulation.

Acknowledgments

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