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USE OF THE ADAMS-BASHFORTH FOURTH-ORDER SCHEME IN THE NUMERICAL SIMULATION OF DYNAMIC SYSTEMS

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Abstract. Generally, the real physical systems are chaotic and /or dynamics, modeling problems in conditions similar to this one will be discussed in this work, testing a method that numerically and appropriately models the problems of this class. Ordinary differential equations are used to model these physical systems, since they present non-linear, deterministic and three-dimensional characteristics. The numerical method of type predictor-corrector of fourth-order Adams-Bashforth was tested for conducting simulations. The use of this method proved to be efficient for these types of systems. The method appropriately simulates the Lorenz Attractor, other models familiar to Lorenz are analyzed, such as the attractor of Lorenz with source term, attractor of Rössler, attractor of Pan-Xu-Zhou, attractor of Chen and the modified Chen attractor which are fundamental systems for the processing and estimation of climatic phenomena, whether natural or controlled forms. The fourth-order Adams-Brashforth method appropriately simulated the dynamic systems mentioned, which has proved to be efficient when compared to other scientific works.

Keywords: Dynamic systems, Numerical simulation, Family of attractors.

1 Introduction

Natural systems, whether biological, physical, chemical or other scientific specialties, are generally chaotic and / or dynamic, in order to have a scientific prediction of the possible evolution of these processes, specific mathematical resources and treatments that describe the object of study are used. Initially, these systems are modeled by differential equations, although in practice most solutions of these types of equations cannot be demonstrated through elementary functions or even been solved analytically.

Problems that have evolutionary characteristics are complex to be demonstrated through basic functions and or do not have analytical results. We can use numerical methods to find approximate solutions. Among many possible numerical schemes, in this work we will test and demonstrate the efficiency of the fourth-order Adams-Bashforth numerical scheme to simulate Lorenz's attractor and some similar ones such as the Lorenz attractor with source term, Pan-Xu-Zhou attractor, Rössler attractor, Chen attractor and the modified Chen attractor.

The Lorenz attractor was published in 1963 by E. N. Lorenz as a thermally induced fluid convection model in the atmosphere. Lorenz's convection mathematical model has three variable states (x, y, z). Being the variable x proportional to the amplitude of the circulating fluid velocity, representing clockwise and counterclockwise when positive and negative respectively. The temperature difference between the fluids is represented by the variable y, the variable z demonstrates the distortion of the vertical temperature linearity. The Lorenz attractor in the discretized form is given by the system of differential equations:

$$\begin{cases} x_{k+1} = x_k + \alpha \left(x_k - y_k\right) \Delta t \\ y_{k+1} = y_k + \left(-x_k z_k + \rho x_k - y_k\right) \Delta t \\ z_{k+1} = z_k + \left(x_k y_k - \beta z_k\right) \Delta t \end{cases} \tag{1}$$

α is a parameter that approximates the momentum diffusivity ratio and the thermal diffusivity of a fluid. ρ describes the relationship between buoyancy and viscosity within a fluid. β is related to the horizontal wave number of the convective motions. α, ρ and are dimensionless parameters.

The Lorenz attractor with source term has variables and parameters similar to the one presented above, as it adds a source term (q), which is an external agent that causes disturbance in the original system. The Lorenz attractor with source term (q) in the discretized form is given by the differential equation system:

$$\begin{cases} x_{k+1} = \left(x_k + \alpha \left(x_k - y_k\right) \Delta t\right) q \\ y_{k+1} = \left(y_k + \left(-x_k z_k + \rho x_k - y_k\right) \Delta t\right) q \\ z_{k+1} = \left(z_k + \left(x_k y_k - \beta z_k\right) \Delta t\right) q \end{cases} \tag{2}$$

The Pan-Xu-Zhou attractor bears similarities to the Lorenz attractor, but topologically they are not equivalent as the Pan attractor may have complex variables, [1]. The Pan-Xu-Zhou attractor in discretized form is given by the system of differential equations:

$$\begin{cases} x_{k+1} = x_k + \left(\alpha \left(y_k - x_k\right)\right) \Delta t \\ y_{k+1} = y_k + \left(\gamma \cdot x_k - x_k \cdot z_k\right) \Delta t \\ z_{k+1} = z_k + \left(x_k \cdot y_k - \beta \cdot z_k\right) \Delta t \end{cases} \tag{3}$$

Rössler's attractor was introduced in the 1970s. Otto Eberhard Rössler was inspired by three-dimensional flow geometry. For continuous chaos, this system is governed by some questions such as: it has a quadratic term making its nonlinearity minimal and is a chaotic attractor with single lobe [1], by contrast Lorenz's attractor has two lobes, [2]. The Rössler attractor in discretized form is given by the system of differential equations:

$$\begin{cases} x_{k+1} = x_k + (-y_k - z_k)\,\Delta t \\ y_{k+1} = y_k + (x_k + \alpha \cdot y_k)\,\Delta t \\ z_{k+1} = z_k + (\beta + z_k\,(x_k - \gamma))\,\Delta t \end{cases} \tag{4}$$

Chen's attractor is suitable for a climate controlled model. In engineering is a model used for regulation in indoor environments. The initial parameters and conditions are such that they introduce a control input that turns a non-chaotic system into chaotic.[3]. Chen's attractor in discretized form is given by the system of differential equations:

$$\begin{cases} x_{k+1} = x_k + (\alpha\,(y_k - x_k))\,\Delta t \\ y_{k+1} = y_k + (x_k(\gamma - \alpha) - x_k \cdot z_k + \gamma \cdot y_k)\,\Delta t \\ z_{k+1} = z_k + (x_k \cdot y_k - \beta \cdot z_k)\,\Delta t \end{cases} \tag{5}$$

The modified Chen attractor carries information on the parameters and variables similar to Chen attractor, however in the second line of eq. (5). A parameter that changes the behavior of the initial system is introduced.The modified Chen attractor in discretized form is given by the system of differential equations:

$$\begin{cases} x_{k+1} = x_k + (\alpha\,(y_k - x_k))\,\Delta t \\ y_{k+1} = y_k + (x_k(\gamma - \alpha) - x_k \cdot z_k + \gamma \cdot y_k + \mu)\,\Delta t \\ z_{k+1} = z_k + (x_k \cdot y_k - \beta \cdot z_k)\,\Delta t \end{cases} \tag{6}$$

2 Theoretical Foundation

Adams-Bashforth / Adams-Moulton is a multistep (predictor and corrector) numerical method, In the academic field there is a lot of work with this method and its basic idea is to approximate a function f (t, y (t)) on a $P_n(t)$ polynomial of degree n using the n + 1 data that were previously calculated to determine the coefficients of the $P_n(t)$ polynomial. An analysis of the Adams-Bashforth method and the characteristics of multistep methods will be performed.

2.1 Numerical method for ordinary differential equations - "Multistep Scheme"

The multistep numerical method is obtained as follows: integrating the solution over the interval $[t_i, t_{i+1}]$ we obtain:

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} y'(t)dt = y_i + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt \tag{7}$$

[1]convex part of a meander.

Approximating $f(t, y(t))$ by a $P_n(t)$ polynomial the approximate solution will be given by:

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} P_n\,(t)dt \tag{8}$$

Consequently the multistep method is characterized in that it is necessary to know only the result of each previous step in order to obtain the numerical solution value of the next step. In this way, we can obtain two values y_1; y_2; y_3; ...; y_n, which approximate value of each ordinate $y(x_1)$; $y(x_2)$; $y(x_3)$; ...; $y(x_n)$ of the solution curve. Once you have identified the approximate values of the solution at some points of x_0, you can use some of this information to calculate the value of the next step. When using information other than that obtained in the last step to gain value for the next step, these methods are known as multistep.

2.2 Adams-Bashforth/Adams-Moulton Scheme

As P_1 is the grade 1 polynomial that interpolates the points $(t_{(i-1)}, f_{(i-1)})$ and (t_i, f_i), written as $P_1 = mt + d$, we have:

$$\begin{cases} P_1(t_i) = f(t_i, y_i) = mt_i + d \\ P_1(t_{i-1}) = f(t_{i-1}, y_{i-1}) = mt_{i-1} + d \end{cases} \tag{9}$$

Solving the system 9, we find the following expressions:

$m = \frac{f(t_i,y_i)-f(t_{i-1},y_{i-1})}{\Delta t}$ e $d = \frac{f(t_{i-1},y_{i-1})t_i - f(t_i,y_i)t_{i-1}}{\Delta t}$.

Using the following notation $f_n \equiv f\,(t_n, y_n)$, then:

$\int_{t_i}^{t_{i+1}} P_1\,(t)\,dt = \int_{t_i}^{t_{i+1}} \left[\frac{f_i - f_{i-1}}{\Delta t} t + \frac{f_{i-1}t_i - f_i t_{i-1}}{\Delta t}\right] dt =$

$= \frac{f_i - f_{i-1}}{2\Delta t}\left(t_{i+1}^2 - t_i^2\right) + \frac{f_{i-1}t_i - f_i t_{i-1}}{\Delta t}\left(t_{i+1} - t_i\right) =$

$= \frac{f_i t_{i+1}^2 - f_{i-1}t_{i+1}^2 - f_i t_i^2 + f_{i-1}t_i^2}{2\Delta t} + \frac{f_{i-1}t_i t_{i+1} - f_{i-1}t_i^2 - f_i t_{i-1}t_{i+1} + f_i t_i t_{i-1}}{\Delta t} =$

$= \frac{f_i t_{i+1}^2 - f_{i-1}t_{i+1}^2 - f_i t_i^2 + f_{i-1}t_i^2 + 2f_{i-1}t_i t_{i+1} - 2f_{i-1}t_i^2 - 2f_i t_{i-1}t_{i+1} + 2f_i t_i t_{i-1}}{2\Delta t} =$

$= \frac{f_i t_{i+1}^2 - f_{i-1}t_{i+1}^2 - f_i t_i^2 - f_{i-1}t_i^2 + 2f_{i-1}t_i t_{i+1} - 2f_i t_{i-1}t_{i+1} + 2f_i t_i t_{i-1}}{2\Delta t}$.

If $t_{i-1} = t_i - \Delta t$, then:

$\int_{t_i}^{t_{i+1}} p\,(t)\,dt = \frac{1}{2\Delta t}\left[\frac{f_i t_i^2 + 2\Delta t f_i t_i + f_i \Delta t^2 - f_{i-1}t_i^2 - 2\Delta t f_{i-1}t_i - f_{i-1}\Delta t^2 - f_i t_i^2 - f_{i-1}t_i^2}{\Delta t}\right.$

$\left. + 2f_{i-1}t_i^2 + 2\Delta t f_{i-1}t_i - 2f_i t_i^2 - 2\Delta t^2 f_i + 2f_i t_i^2 - 2\Delta t f_i t_i\right] = \frac{3f_i - f_{i-1}}{2}\Delta t.$

So the 2^a order Adams numeric scheme is given by:

$$y_{i+1} = y_i + \frac{\Delta t}{2}\left[3f\,(t_i, y_i) - f\,(t_{i-1}, y_{i-1})\right]. \tag{10}$$

Proceeding analogously for a degree 3 polynomial, we obtain the 4th order Adams method, as shown by equation 11.

$$y_{i+1} = y_i + \frac{\Delta t}{24}\left[55f\left(t_i, y_i\right) - 59f\left(t_{i-1}, y_{i-1}\right) + 37f\left(t_{i-2}, y_{i-2}\right) - 9f\left(t_{i-3}, y_{i-3}\right)\right]. \quad (11)$$

In this paper, we will test the ref 4.3 equation for the Lorenz equation, source-term Lorenz attractor, Pan-Xu-Zhou attractor, Rössler attractor, Chen attractor, and modified Chen attractor. The Adams-Moulton formula will be used for the corrector step, as pointed out in the equation. 12.

$$y_{i+1} = y_i + \frac{\Delta t}{24}\left(9f(t_{i+1}, y_{i+1,}) + 19f(t_i, y_i) - 5f(t_{i-1}, y_{i-1}) + 1f(t_{i-2}, y_{i-2})\right). \quad (12)$$

Equation ref 4.3 requires knowledge of the values of $y_0; y_1; y_2; y_3$ so we can obtain y_4. The value of y_0 is the given initial condition itself and the values of $y_1; y_2; y_3$, can be found by the Euler's method 13 [4] .

$$y_{i+1} = y_i + \Delta t f(x_i, y_i). \quad (13)$$

2.3 Consistency of Adams-Bashforth/Adams-Moulton Scheme

The 4th order Adams method is predictor and corrective. These types of numeric schemes are known as the Multistep method. The most general form of a multistep $(n+1)$ method is given by the equation:

$y_{i+1} = \sum_{j=0}^{n} \alpha_j y_{i-j} + \Delta t \sum_{j=-1}^{n} \beta_j f_{i-j}, \qquad \forall i \geq n.$

For $\beta_{-1} \neq 0$ we get an implicit schema and for $\beta_{-1} = 0$ an explicit schema. The truncation error is given by:

$E_T = y_E\left(t_{i+1}\right) - y_{i+1} = y_E\left(t_{i+1}\right) - \left[\sum_{j=0}^{n} \alpha_j y_{i-j} + \Delta t \sum_{j=-1}^{n} \beta_j f_{i-j}\right] =$

$= y_E\left(t_{i+1}\right) - \left[\sum_{j=0}^{n} \alpha_j y_{i-j} + \Delta t \sum_{j=-1}^{n} \beta_j y'_{i-j}\right].$

Where y_E is the exact solution. Substituting the approximate solution for the exact solution and expanding on Taylor Series, we have the following expression for the truncation error:

$E_T \approx \sum_{r=0}^{m} y_E^r\left(t_{i+1}\right) \frac{\Delta t^r}{r!} - \left[\sum_{j=0}^{n} \alpha_j \sum_{r=0}^{m} y_E^r\left(t_{i+1}\right) \frac{(-j\Delta t)^r}{r!} + \right.$

$\left. + \Delta t \sum_{j=-1}^{n} \beta_j \sum_{r=0}^{m-1} y_E^{r+1}\left(t_{i+1}\right) \frac{(-j\Delta t)^r}{r!}\right].$

That is:

$E_T \approx \left[1 - \sum_{j=0}^{n} \alpha_j\right] y_E + \left[1 + \left(\sum_{j=0}^{n} j\alpha_j + \sum_{j=-1}^{n} \beta_j\right)\right] \Delta t y'_E +$

$+ \sum_{r=2}^{m} \left[1 - \sum_{j=0}^{n} \alpha_j (-j)^r - r \sum_{j=-1}^{n} \beta_j (-j)^{r-1}\right] \Delta t y'_E.$

From the above result and the condition below for a numerical method to be consistent, then the following equalities must be verified:

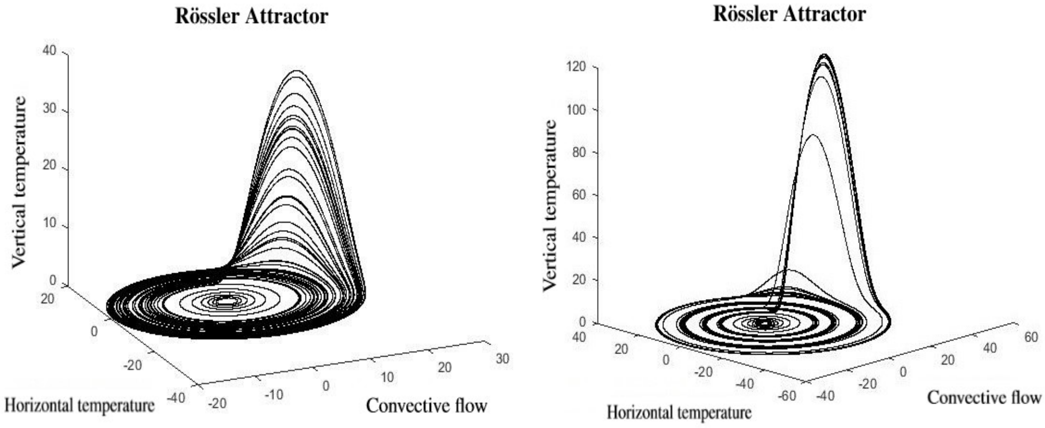
$$lim \Delta t \to 0, \; \frac{E_T}{\Delta t} = 0 \Rightarrow \sum_{j=0}^{n} \alpha_j = 1 \wedge \sum_{j=-1}^{n} \beta_j - \sum_{j=0}^{n} j\alpha_j = 1 \quad (14)$$

Stability and convergence analysis of the 4th order Adams Bashforth method can be seen in detail in the work of [5].

3 Results

The 4th order Adams-Bashforth method was used to simulate the attractors of Rössler, Lorenz, Lorenz with source term (q), Pan-Xu-Zhou, Chen and modified Chen's.

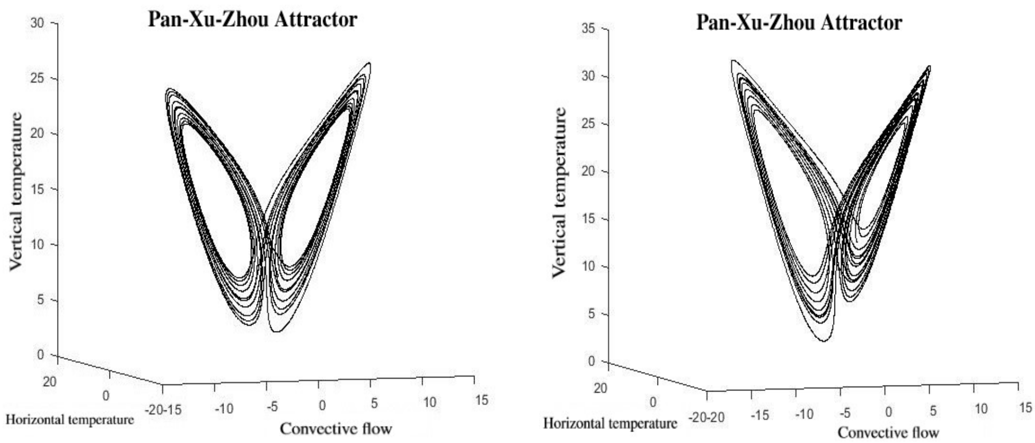
Starting with the Rössler attractor that was modeled by the Adans method and in these simulations the initial values were, $Tf = 300$, $dt = 0.001$, $n = 4$, $\sigma = 10$, $\alpha = 35$, $\beta = 3$ and γ has been varied.



(a) Rössler Attractor with $\gamma = 14$. (b) Rössler Attractor with $\gamma = 28$.

Figure 1. Rössler Attractor with variation in γ.

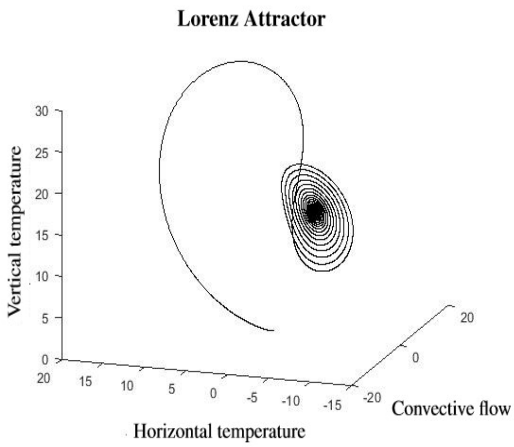
The Pan-Xu-Zhou attractor was modeled by the Adans method and in these simulations the initial values were, $Tf = 20$, $dt = 0.001$, $n = 4$, $\sigma = 10$, $\alpha = 10$, $\beta = 8/3$ while γ has been varied.



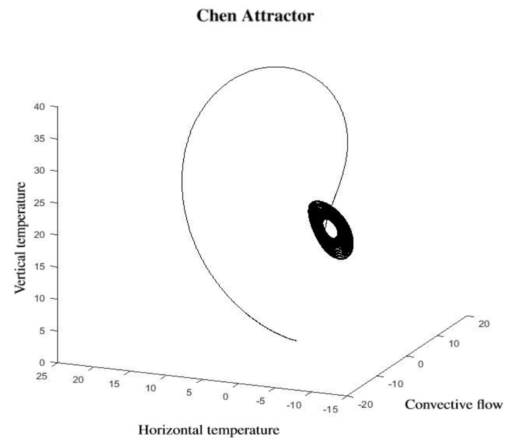
(a) Pan-Xu-Zhou Attractor with $\gamma = 16$. (b) Pan-Xu-Zhou Attractor with $\gamma = 20$.

Figure 2. Rössler Attractor with variation in γ.

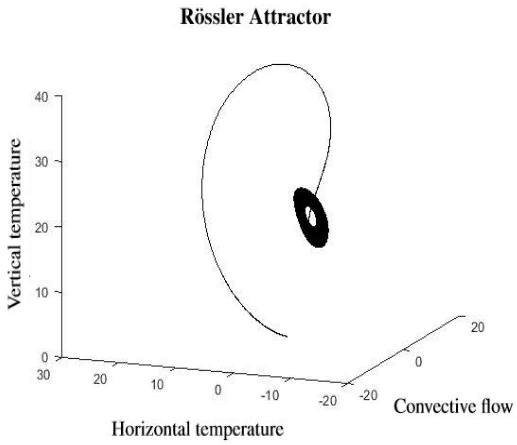
In the following image we have the lorenz attractor simulation, in these simulations the initial values were $T_f = 20$, dt = 0.01, n = 4, $\alpha = 10$, $\beta = 8/3$ and ρ was being varied. It is possible to notice that after $\rho = 28$ the system starts to diverge.



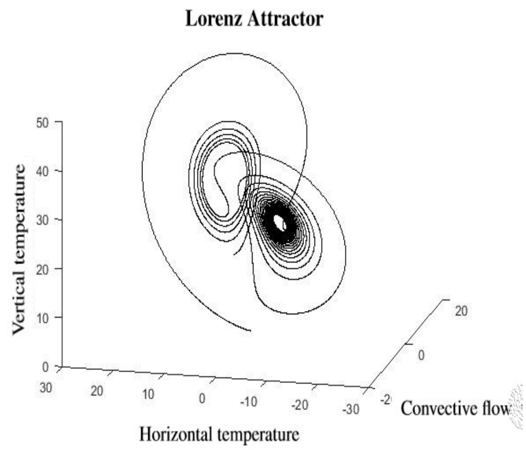
(a) Lorenz attractor with $\rho = 18$.



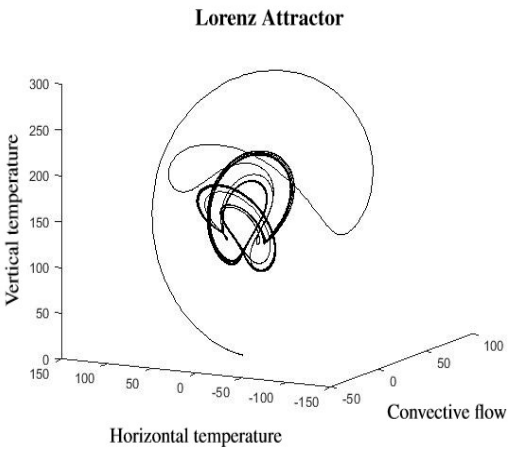
(b) Lorenz attractor with $\rho = 22.30$.



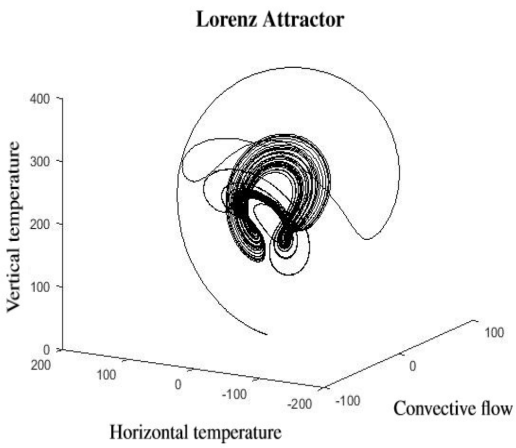
(c) Lorenz attractor with $\rho = 22.40$.



(d) Lorenz attractor with $\rho = 28$.



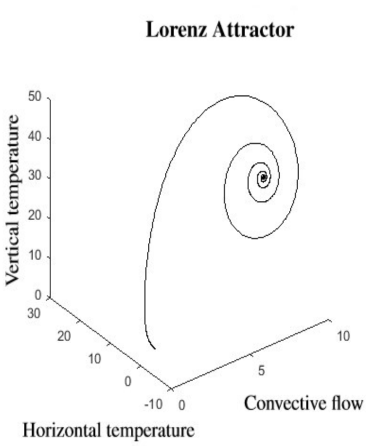
(e) Lorenz attractor with $\rho = 150$.



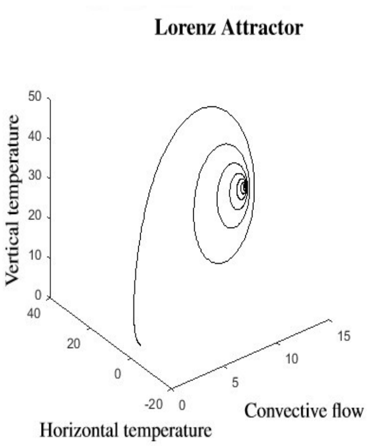
(f) Lorenz attractor with $\rho = 200$.

Figure 3. Lorenz attractor with variation in ρ

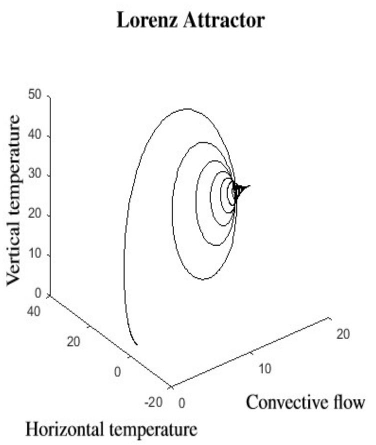
In the following image we have simulations of the Lorenz attractor with source term q, in these simulations the initial values were $T_f = 40$, dt = 0.005, n = 4, $\alpha = 1$, $\beta = 8/3$, $\rho = 28$ the source term (q) has been varied and the convergence point could be verified.



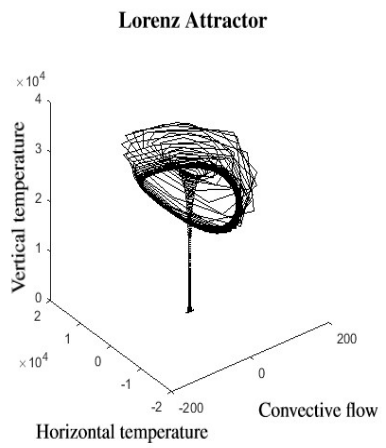
(a) lorenz with term external source $q = 1.001$.



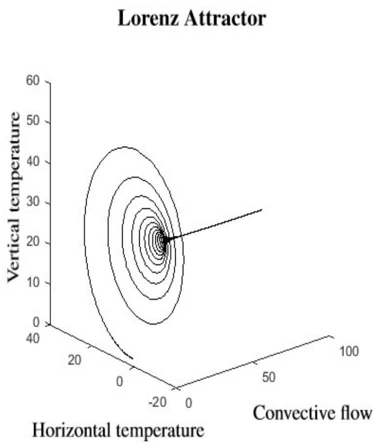
(b) lorenz with term external source $q = 1.0030$.



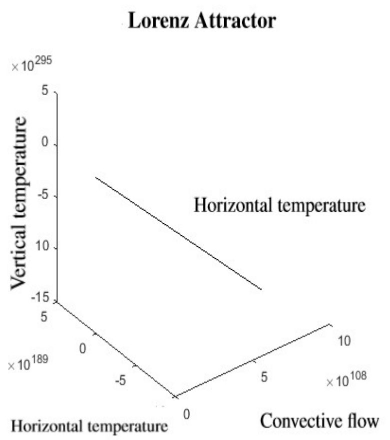
(c) lorenz with term external source $q = 1.0040$.



(d) lorenz with term external source $q = 1.0050$.



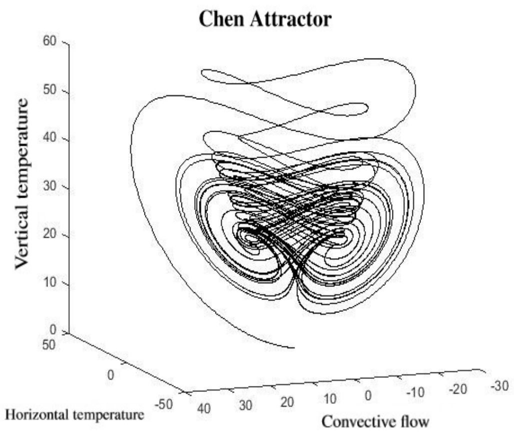
(e) lorenz with term external source $q = 1.0051$.



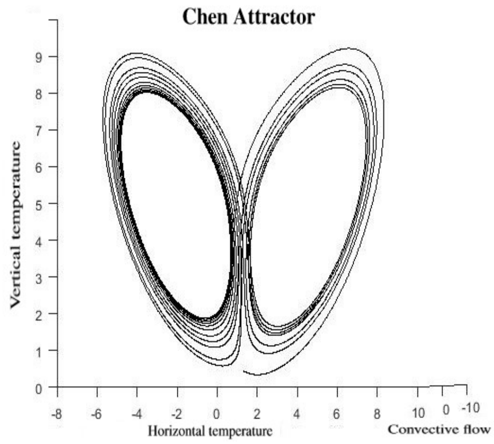
(f) lorenz with term external source $q = 1.0053$.

Figure 4. Lorenz Attractor with source term (q).

Chen's attractor was modeled by the Adans method and in these simulations the initial values were, $Tf = 20$, $dt = 0.001$, $n = 4$, $\sigma = 10$, $\alpha = 10$, $\beta = 3$ while γ has been varied.



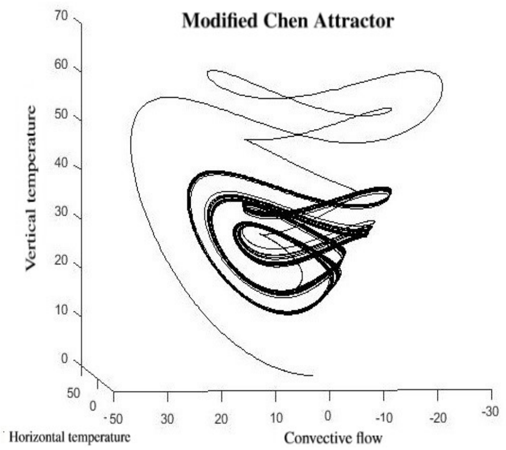
(a) Chen Attractor with $\gamma = 28$.



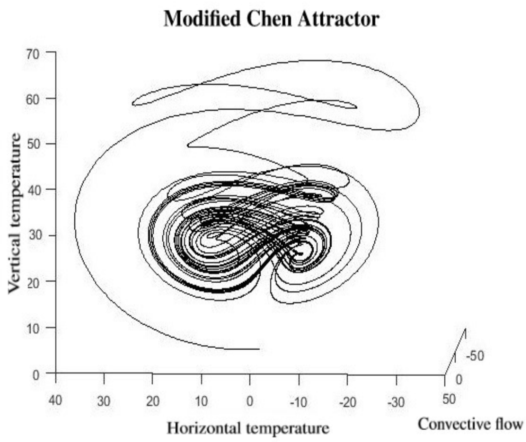
(b) Chen Attractor with $\gamma = 28$.

Figure 5. Chen Attractor with variation in γ.

The Chen attractor modified when it was modeled by the Adans method and in these simulations the initial values were, $Tf = 20$, $dt = 0.001$, $n = 4$, $\sigma = 10$, $\alpha = 35$, $\beta = 3$, $\gamma = 28$ and μ has been varied.



(a) Modified Chen Attractor with $\mu = 23.1$.



(b) Modified Chen Attractor with $\mu = 10$.

Figure 6. Modified Chen Attractor with variation in μ.

4 Conclusion

In the Lorenz Attractor and Lorenz simulations with the source term (q), (ρ) and (q) were varied respectively so that it was possible to analyze the divergence point of the method used, and when compared to the work of cite song the method of Adans behaves appropriately the previous author used the numerical method of Runge-Kutta (see [4]) to obtain the results.

The Rössler, Pan-Xu-Zhou, Chem and modified Chem models were simulated using the 4th order Adams-Brashforth method. The results can be compared respectively with [1], [2], [3] and [6]. However in these simulations the parameters of the models were not analyzed, having as main objective the validation of the behavior of the Adams Method in the mentioned systems. It is noteworthy that the Adams-Brashforth Method behaved properly when compared to the aforementioned works.

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