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BEMCRACKER2D: A SOFTWARE PACKAGE FOR TWO-DIMENSIONAL FATIGUE CRACK GROWTH ANALYSIS

Gilberto Gomes
Álvaro M Delgado Neto
Thiago A A Oliveira
Danilo C de Moura
ggomes2007@gmail.com
alvaro.martins.bok@gmail.com
eng.thiagoarnaud@gmail.com
danilo.cmoura@hotmail.com
University of Brasilia
Campus Darcy Ribeiro, 70910-900, Brasilia/DF, Brazil

Abstract. The stress analysis in structures with complex geometry, such as in aircraft fuselages, where geometry is continuously altered by crack growth, usually requires the use of numerical methods, since the presence of cracks in the structure raises difficulties related to the modeling and, consequently, to the calculation of the stress intensity factors (SIF). In this aspect, the dual boundary element method (DBEM) has been applied in this type of analysis, taking advantage over the FEM considered, since there is no need for continuous remeshing at each crack increment. In this context, this work proposes the development of software written in C ++ and based on Object Oriented Programming, called BemCracker2D, for two-dimensional analysis by DBEM of fatigue crack propagation problems in the field of linear elastic fracture mechanics. Parameters such as SIF, crack propagation path, number of load cycles and others, will be computed and compared with examples from the open literature in order to attest to the program's efficiency and robustness.

Keywords: crack growth; fatigue life; dual boundary element; bemcracker2d.

1. INTRODUCTION

According to De Lacerda and Wrobel (2001), regardless of the physical mechanism involved, the crack propagation process invariably requires the use of incremental numerical methods in the analysis, especially the finite element (FEM) and boundary element methods (BEM).

The FEM, whose history in fracture mechanics applications is due to Gallagher (1978), was applied in the crack propagation process, for example by Swenson and Ingraffea (1988) and by Kocer and Collins (1997). However, in the most formulations there is a need for continuous remeshing to follow the crack extension, which can make this method rather time-consuming. BEM has also been applied to incremental analysis of crack propagation problems through the use of subregions, or more recently using a dual formulation known as DBEM (Bush, 1999; Ingraffea et al., 1983; Portela et al., 1993-1992). The DBEM has been shown to be more efficient and does not present remeshing difficulties as in the FEM.

On the other hand, Computer Graphics has been playing an important role as a support tool in the development of computational programs in the engineering field, especially in the geometric modeling area, which requires studies and development of efficient algorithms and sophisticated data structures (Gomes, 2012). This requires innovation in both the programming tool used and the way it is programmed. In this respect, a new programming philosophy has been highlighted: Object Oriented Programming (OOP) (Booch, 1994), which seeks to define the objects - entities involved in the system - and their classes and relations, instead of decomposing the program into procedures or functions.

An example of a programming language that fully supports OOP is the well-known C++ language developed by Stroustrup (1990). Gomes (2012) developed a data structure for the treatment of two-dimensional boundary element models based on OOP, which has been used in the development of BEMLAB2D (Delgado Neto, 2017), a Graphical Interface for the design and mesh generation necessary to feed the BemCracker2D program (Gomes et al. 2016, Gomes and Miranda 2018).

This paper aims to present the BemCracker2D program, implemented in C ++ language and based on Object Oriented Programming, as well as the BemLab2D graphical interface, as a computational package for analysis and modeling of fatigue crack propagation problems via DBEM.

The paper is organized as follows: section 2 presents the main concepts of fracture mechanics and fatigue life, as well as the basic theory of the DBEM; The main computational aspects of the BemCracker2D program are presented in section 3; Section 4 presents the computational package applications in data processing, modeling and analysis types; Finally, in section 5, the final considerations.

2. THEORETICAL REFERENTIAL

2.1. Linear Elastic Fracture Mechanics

Griffith (1921) proposed a rupture criterion based on energy, since stresses at the crack tip tend to infinity. According to this criterion, the crack will propagate when the available energy equals the energy required for propagation and then the Energy Release Rate parameter has emerged. This method served as the basis for Elastic Linear Fracture Mechanics (ELFM), where linearity is not required. From then on, several works continued, such as Irwin (1957), with the Potential Energy Release Rate, which represents the energy available to separate two crack surfaces during the propagation; the Stress Intensity Factor, K, by Irwin (1957), serving as a parameter to quantify the stress intensity near the crack tip; and Rice's (1968) formulation of Integral J - which brought the key to relating the Energy Release Rate, G, to the stress and strain field near the crack tip for any elastic material. Thus, the Fracture Mechanics can be used in crack prediction, both explaining its occurrence and its behavior and direction.

2.1.1. Stress Intensity Factors

The Stress Intensity Factor (SIF), K, it is the most important parameter in ELFM and quantifies the stresses field close to a crack, providing fundamental information on how the crack starts and propagates (HE, REN and YANG, 2011). The SIF is associated with crack open mode, where K_I is for mode I; K_{II} for mode II; and K_{III} for mode III. When the K value represents the maximum or critical value referring to the fracture toughness of the material, it is called K_{IC}, for mode I and similarly for the other modes.

Generally, the K value is shown as:

$$K = \sigma\sqrt{\pi a} * Y\left(\frac{a}{W}\right) \tag{1}$$

where $Y\left(\frac{a}{W}\right)$ is a dimensionless function that depends on geometry. The value of the applied load, σ, and the crack size, a, can be directly related to SIF (SANFORD, 2002).

These intensity factors are generally obtained by employing extrapolation or stress extrapolation techniques. In this paper, the J-Integral (Rice and Tracey, 1973) technique for obtaining the SIFs is employed, since its elastic field inside can be exactly determined along the contour path by the BEM, whether the exact variation of its elastic field is built into the fundamental solution.

2.1.2. Crack Direction

According to Carvalho (1998), there are several criteria for use with numerical methods in determining the growth direction of crack propagation in the elastic linear regime, the main ones being summarized below.

Maximum Circumferential Stress (MCS)

According to this criterion, the crack will propagate normal to the plane in which the circumferential tension is maximum, which occurs when the shear stress is zero. However, the crack only propagates if the circumferential stress is greater than the corresponding stress for K_IC (without influence of fatigue). Thus, according to this criterion, the direction of the local crack growth θt is determined by the nullity condition of the shear stress, as,

$$K_I \sin\theta_t + K_{II}(3\cos\theta_t - 1) = 0 \tag{2}$$

where, *KI* e *KII* are the SIFs, respectively, opening displacement modes (Mode I) and shear (Mode II) at the crack tip. θ_t is an angular coordinate centered on the crack tip and measured from the crack axis in front of the tip.

Maximum Potential Energy Release Rate (MPERR)

This criterion is based on Griffith's energy release rate methods, G, and consists of measuring the potential energy that is released during fracturing. When studying linear elastic fractures that do not change direction, G is easily related to K, where the total potential energy for fracturing is equivalent to the sum of the potential energies of each mode - G_I and G_II. However, in mixed mode fracturing, the crack changes direction and, according to Hussain (1974), crack propagation will occur in the direction in which there is the highest energy release rate for fracturing, with the total energy release rate, given by

$$\boldsymbol{\mathcal{G}(\theta)_I} = \frac{K_I^2(\theta)}{E'} \tag{3}$$

$$\boldsymbol{\mathcal{G}(\theta)_{II}} = \frac{K_{II}^2(\theta)}{E'} \tag{4}$$

$$\boldsymbol{\mathcal{G}(\theta)} = \boldsymbol{\mathcal{G}}(\theta)_I + \boldsymbol{\mathcal{G}}(\theta)_{II} \tag{5}$$

Therefore, cracking will occur in the direction θ in which there is greater energy release and when this maximum released energy rate is greater than the critical energy rate required for cracking, $\mathcal{G}_C$.

Minimum Deformation Energy Density (MDED)

This criterion, proposed by Sih (1974), is based on the magnitude of the strain energy density, S, as a parameter to evaluate the direction of crack propagation. Thus, by this criterion, the deformation energy per area element is given by,

$$\frac{dW}{dA} = \frac{1}{r}\left(a_{11}{K_I}^2 + 2a_{12}K_I K_{II} + a_{22}{K_{II}}^2\right) \tag{6}$$

In the Equation (6) the term in parentheses that multiplies 1/r is called the factor S, and represents the intensity of dW/dA within an infinitesimal element, as a function of angle θ. In general it can be written as:

$$S(\theta) = a_{11}{K_I}^2 + 2a_{12}K_I K_{II} + a_{22}{K_{II}}^2 \tag{7}$$

The value of S is not valid when the value of r is too small and is limited by a critical r, r_0. In addition, crack propagation will occur if the value of S (θ) is equal to a critical strain energy density factor, S_{cr} and occurs in the direction of minimum strain energy density (SIH, 1974).

2.2. Fatigue Crack Propagation

The fatigue crack propagation analysis presented here comes from the results extracted from the incremental crack growth analysis, in which the SIFs are calculated at each increment. Applying the Paris Law (1962), we can calculate the crack propagation rate as,

$$\frac{da}{dN} = C(\Delta K_{Ieq})^m \tag{8}$$

where a is the crack length, N is the number of load cycles, C and m are the empirical constants of the material.

Equation (8) translates what we call fatigue life and means the variation in the number of load cycles required to propagate the crack, or how far its length is advanced with each increment.

2.3. Dual Boundary Element Method (DBEM)

Problems involving cracks in general cannot be achieved in a single region analysis with direct application of the conventional boundary element method because the coincidence of crack contours produces a poorly posed problem. This occurs for a pair of source points, coincident in the crack contour, where the algebraic equations for one of the points are identical to those of the opposite point, since the same integral displacement equation is applied at both source points, with same integration path and along the entire contour of the problem.

To overcome this difficulty, the dual boundary element method (DBEM) (Portela 1992; Aliabadi 2002) considers two independent equations - the displacement and traction integral equations, with the same integration path for each pair of coincident source points, but with distinct boundary integral equations which, therefore, give rise to independent algebraic equations.

3. THE BEMCRACKER2D PROGRAM

The automation process took place through the implementation of two programs: the BEMLAB2D graphical interface for pre- and postprocessing, written in MATLAB, and BemCracker2D. The latter was implemented in C ++ and based on OOP concepts (BOOCH, 1994) for analysis of two-dimensional elastostatic problems via the boundary element method, which involves 3 processing modules, namely:

- Standard BEM (module I);
- DBEM With no Propagation (module II);
- DBEM With Propagation (module III)
    - Stress Analysis with BEM;
    - SIF Evaluation (J-Integral);
    - Assessment of the direction/correction of crack growth by the MCS, MPERR and MDED criteria;
    - Fatigue Life Assessment (Paris Law);
    - Cracked Linkup Analysis.

The automation scheme is represented by the flow chart of figure 1.

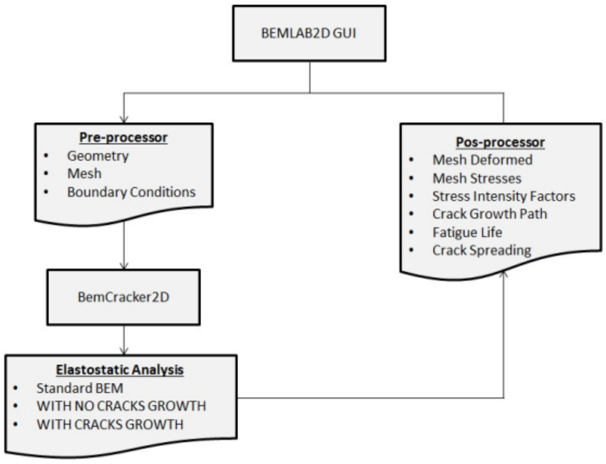


Figure 1 - Crack propagation automation scheme.

3.1. The BEMLAB2D Graphical Interface

BEMLAB2D is a GUI-type graphical interface for pre- and post-processing, written in MATLAB, aimed at two-dimensional generation and visualization of different mesh types. BEMLAB2D is based on user-defined actions using the buttons, mouse and dialog tools (Fig. 2b), pre- and post-processing modules as illustrated in Fig. 2a.

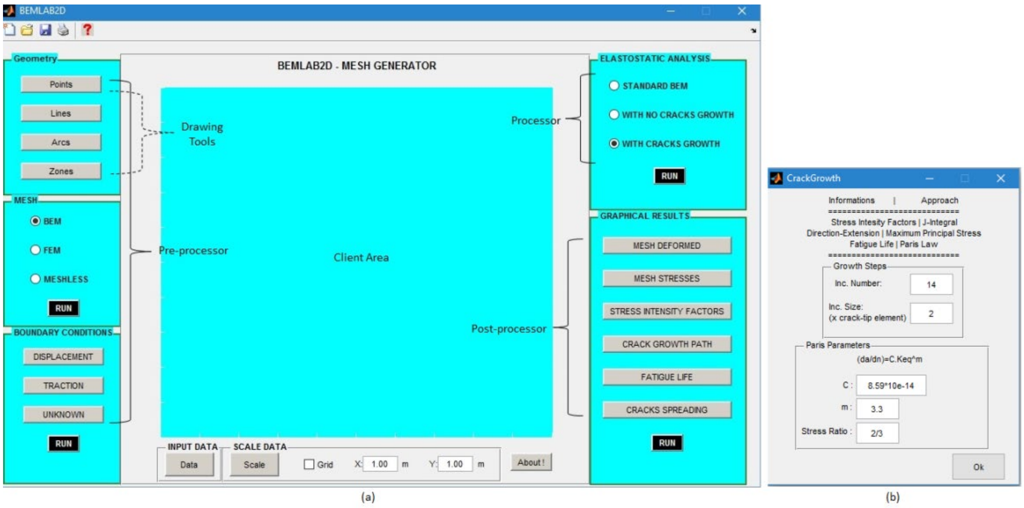


Figure 2. (a) BEMLAB2D GUI Lay-out; (b) Dialog for propagation analysis.

4. MODELING AND ANALYSIS WITH THE BEMCRACKER2D

In this section, we present three application examples of the bemcracker2d program, whose purpose is to show its functionality and robustness in the ELFM field. All examples are taken from work that performed analysis and/or implementations using bemcracker2d. For the sake of simplicity, only in the first example will we bring details of preprocessing prints with BEMLAB2D.

4.1. Example 1

Here, an example will be presented based on the study by Leite (2017) with analytical solution taken from the literature. After stress analysis performed via BEM, the SIFs are determined by J-Integral. The propagation direction and fatigue life results depend on the accuracy of the SIF values to model the fatigue crack propagation satisfactorily. Therefore, the aim is to compare the results of the fracture mode I and mode II with the available analytical solutions, evaluating the accuracy of the numerical SIF results obtained by the BemCracker2D program. Figure 3 shows the geometry of the Diagonally Loaded Square Plate (DLSP) specimen used in this example (AYATOLLAHI and ALIHA, 2009).

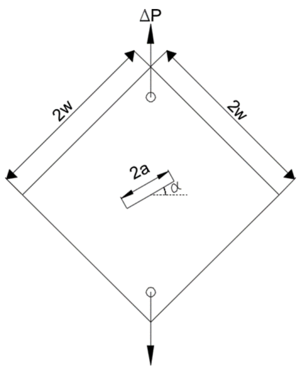
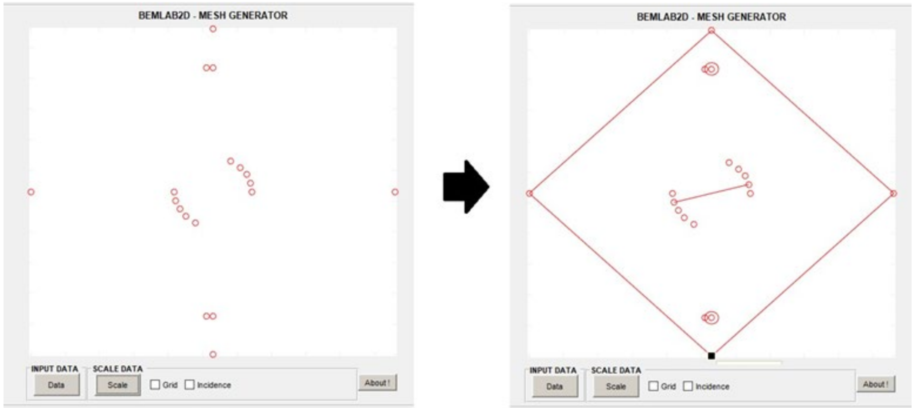


Figure 31 - DLSP specimen geometry and loading position.

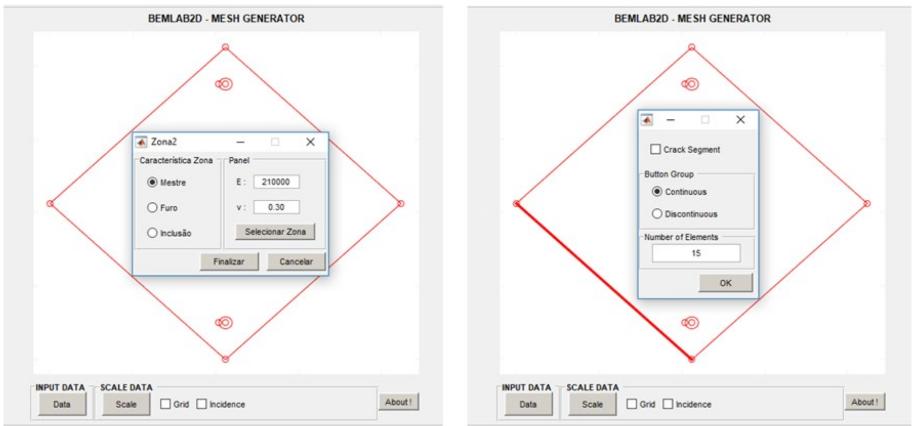
Figure 4 presents the considered points and the model geometry finished in BemLab2D.



(a) (b)

Figure 4 - (a) Points to construct the segments; (b) Full geometric model in BemLab2D

Figure 5 (a) illustrates the BemLab2D zone definitions window, and Figure 5 (b) illustrates the element definitions window. The last preprocessing step is the definition of the displacement and traction boundary conditions,



(a) (b)

Figure 5 - (a) Window to define the zone type; (b) Window to define the type and quantity of elements.

Figures 6a and 6b illustrate the windows for defining the displacement and traction boundary conditions in BemLab2D, respectively. In Figure 7 we have the final model for angle α equal to 15 ° that will be sent to BemCracker2D to perform the processing step. We have considered 86 boundary elements in the model, six of which were to model the internal crack of the problem in a ratio of 0.5, 0.3, 0.2 (three elements for each crack face).

At the beginning of the processing step performed by BemCracker2D, the processing module is defined as module II (DBEM without propagation) in order to obtain the SIFs at the crack tip. The values obtained by BemCracker2D are presented in Table 1 compared to the analytical results, with the differences presented in percentage. The largest difference found was 3% for KI in the 45 ° model, a considerably small difference.

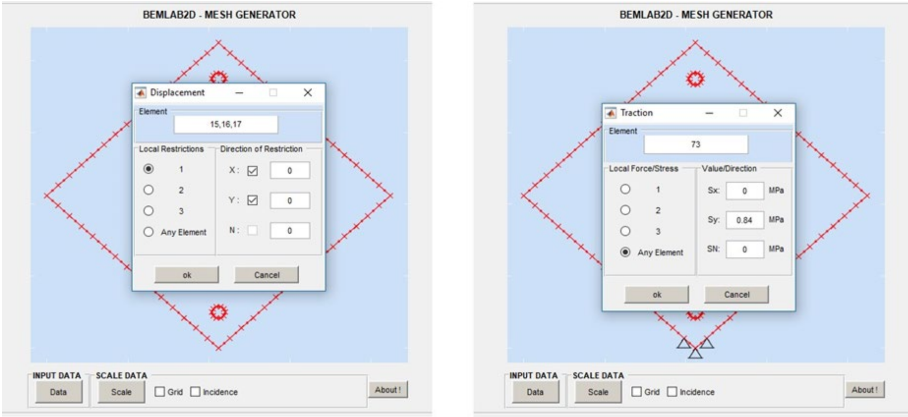


Figure 6 – Dialog for boundary condition: (a) Displacement; (b) Traction

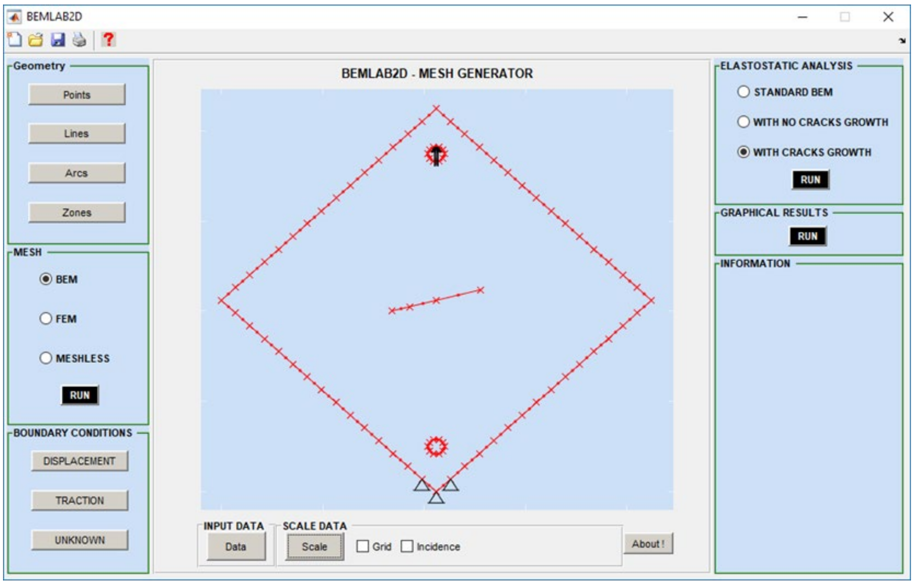


Figure 7 - BEM mesh for the 15º crack slope model with displacement and traction boundary conditions.

Table 1 - Comparison between analytical and numerical results of the DLSP specimen

α	ANALYTICAL			BEMCRACKER2D			DIFFERENCE	
(°)	KI (MPa√m)	KII (MPa√m)	Keq (MPa√m)	KI (MPa√m)	KII (MPa√m)	Keq (MPa√m)	KI	KII
0	1.480	0.000	1.480	1.480	-0.001	1.480	0.0%	0.0%
15	1.450	0.469	1.480	1.452	0.460	1.480	0.1%	1.9%
30	1.082	0.809	1.480	1.096	0.804	1.479	1.3%	0.6%
45	0.541	0.876	1.480	0.557	0.875	1.480	2.9%	0.0%
63	0.000	0.880	1.480	0.030	0.880	1.480	0.0%	0.1%

4.2. Example 2

This example is due to Rodrigues (2018) and illustrated in Figure 8a. It represents a physical model widely used in the literature for generalized damage analysis MSD. The purpose of this model is to analyze crack coalescence according to the three plasticization criteria: Irwin, Dugdale and von Mises, but here one of the criteria will be shown. It consists of a plate subjected to MSD fatigue in a row of holes in the panel, with arrangement of five cracks and three holes. The contour of the plate and the holes presented 66 continuous quadratic elements. A mesh with 8 quadratic elements and discontinuous were modeled in each crack, whose ratio was 0.4, 0.3, 0.2 and 0.1, in that order, making a total of 106 elements and 212 nodes. The discrete model is presented in Figure 8b.

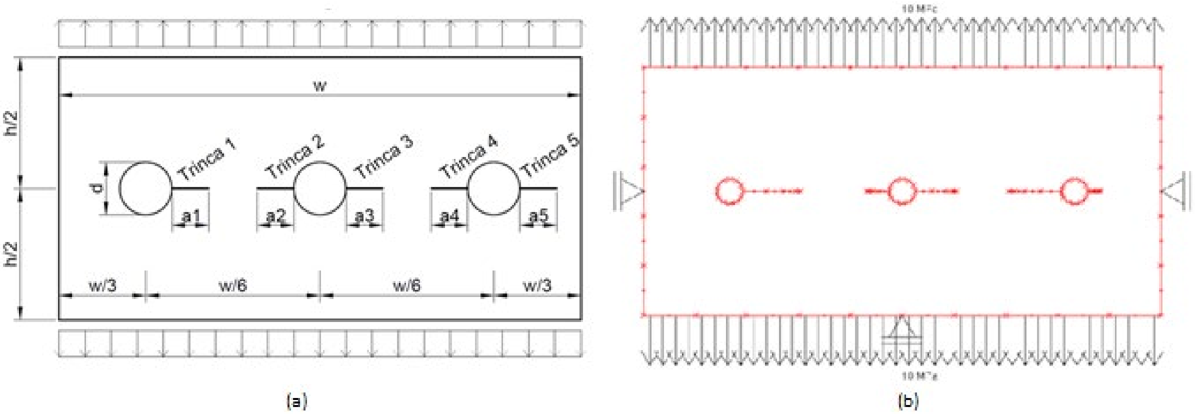


Figure 8 - a) Physical model; b) Discrete model in BEMLAB2D.

Initially, without Linkup's consideration, the stress intensity factors along the growth increments of cracks 1 and 5 are shown in Figures 9a and 9b, respectively (see Figure 8a) of the Dugdale criterion.

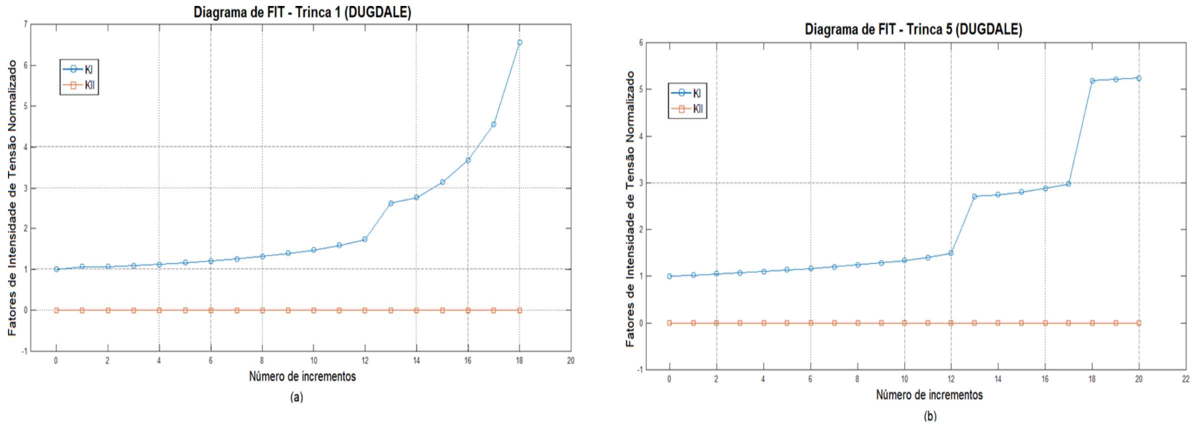


Figure 9 - SIFs by number of increments: a) Trinca 1; b) Trinca 5.

In Figure 10, we have the deformed configuration for the first and second linkup, respectively. Figures 11a and 11b show the data for the first and second linkup occurring according to the method of calculating the plastic zone proposed by Dugdale.

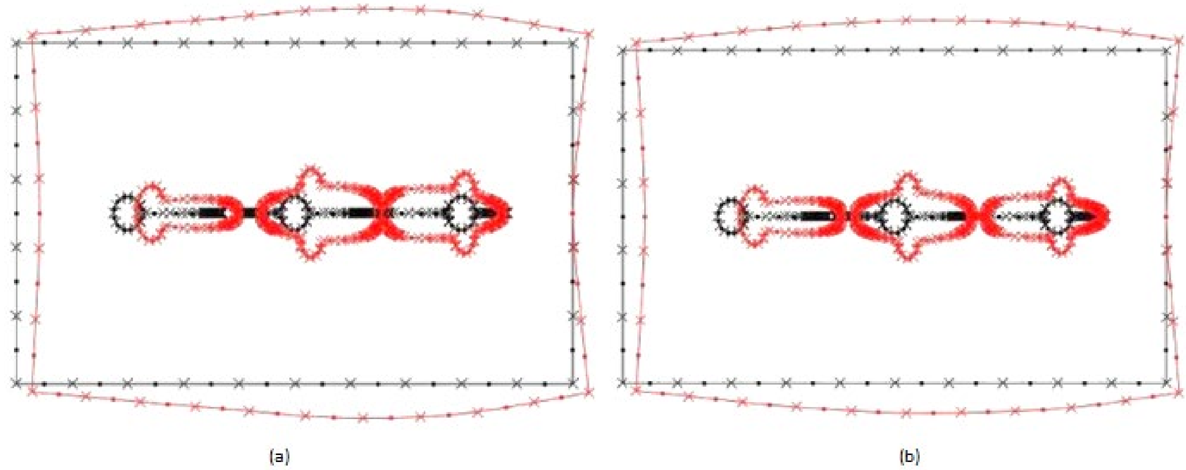


Figure 10 - Deformed Plate Configuration (a) First Linkup (b) Second Linkup.

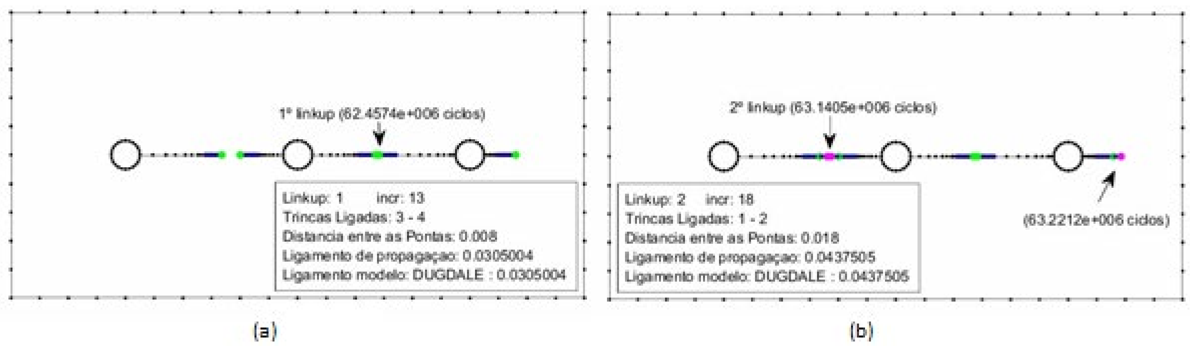


Figure 11 - Dugdale crack coalescence (a) First linkup (b) Second linkup.

4.3. Example 3

Example 3 is based on the work of Oliveira (2019) and refers to an aircraft fuselage subjected to fatigue with normal load P and shear load Q, where different stress plate fields follow the continuum mechanics analysis due to Johnson (1985). The stress field plot was restricted to the 20 x 40 cm region of the workpiece in order to predict the maximum stress field locations, as shown in Figure 12.

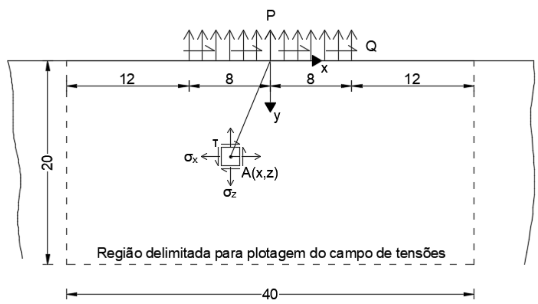


Figure 12 - Multiscale analysis model (cm).

With the stress field obtained in the macro model, the micro model analysis is performed in BemCracker2D, shown below. Micro analysis was performed at 3 different positions of the macro analysis. Position 1 considers the micro element in the center of the plate with coordinates (0,10); position 2 considers this element at the origin of the system axis (0,0); and position 3, at the limit of external request application (8,0), such positions are illustrated in Figure 13a, as well as the discrete model in Figure 13b.

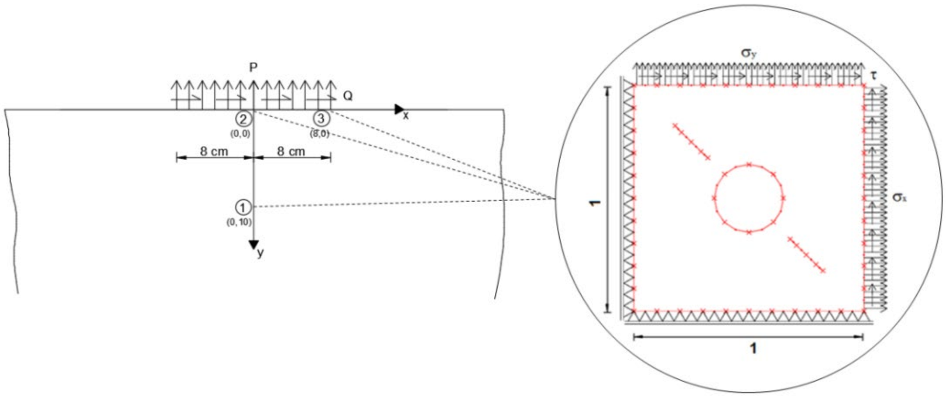
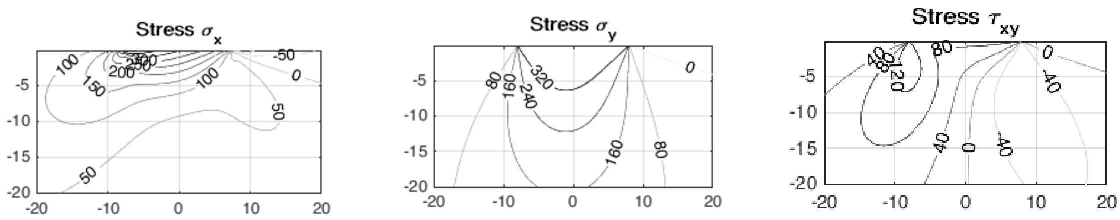


Figure 13 - Positions for initial defect analysis: a) micro analysis (cm); b) discrete model in BEMLAB2D.

Figure 14 shows the stress field of the delimited region for analysis of the macro model (Figure 12), whose data are represented in Table 2:



(a) σ_x (b) σ_y (c) τ

Figura 24 - Stresses Field (MPa).

Table 2 – Input data and its values

P (MPa)	360.47
Q (MPa)	92.78
C	7.20e-11
m	3.52
r (cm)	0.093
L1 (cm)	0.086
L2 (cm)	0.093

where P and Q are the normal and shear stress, respectively, C and m are the Paris constants, r the radius of the central hole, L1 and L2 the size of the upper and lower cracks, respectively. The following are the micro element analysis for position 1 only. Table 3 shows the resulting stress field, considering the micro element in position 1 of Figure 13.

Table 3 – Stress Field (MPa) for position 1

$\boldsymbol{\sigma_x}$	42,90
$\boldsymbol{\sigma_y}$	266,78
$\boldsymbol{\tau}$	11,04

Figure 15a illustrates the crack propagation for the element at position 1, as well as the deformed mesh after all propagated increments in figure 15b.

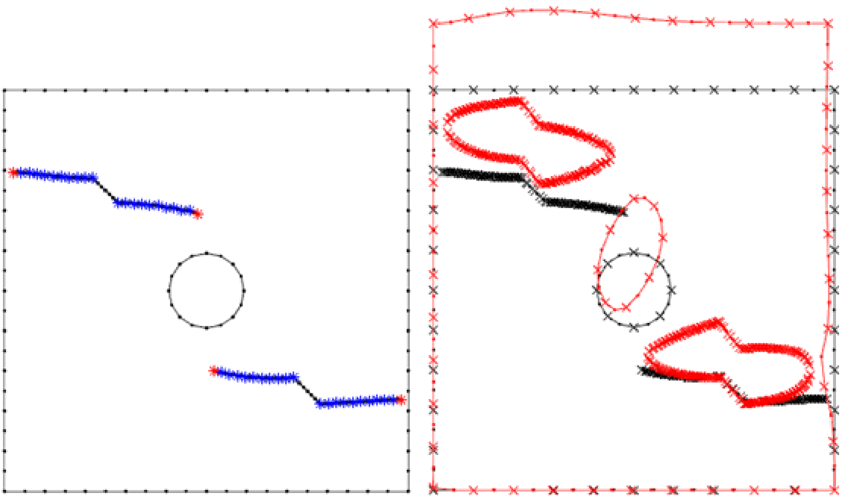
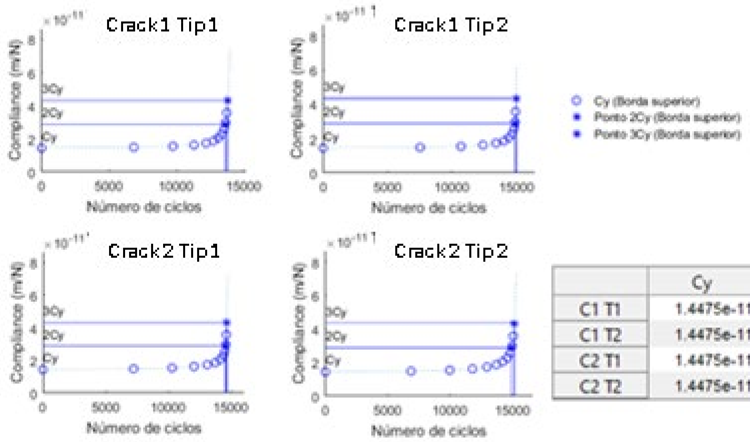


Figure 35 - a) Crack Propagation; b) Deformed Mesh.

Each increment has the points for the construction of the fatigue life curve (N) x mean edge compliance (Figure 16). As a result, the smallest number of cycles to reach 2Cy and 3Cy are respectively N (2C) = 1.3635e+04 and N (3C) = 1.3811e+04.



	Cy	2Cy	N (2Cy)	3Cy	N (3Cy)
C1 T1	1.4475e-11	2.8950e-11	1.3635e+04	4.3424e-11	1.3811e+04
C1 T2	1.4475e-11	2.8950e-11	1.4885e+04	4.3424e-11	1.4956e+04
C2 T1	1.4475e-11	2.8950e-11	1.4567e+04	4.3424e-11	1.4629e+04
C2 T2	1.4475e-11	2.8950e-11	1.4954e+04	4.3424e-11	1.5145e+04

Figure 16 - Number of cycles x edge compliance for element in position 1.

5. CONCLUSIONS

In this work, the BemCracker2D program, as well as its BemLab2D graphical interface, was presented as a computational package for analysis and modeling of fatigue crack propagation problems via MECD.

The modeling process and incremental analysis were automated by the interaction between the BEMLAB2D and BemCracker2D programs, respectively. The first one is responsible for the crack surface modeling using the two integral boundary equations (DBEM) and discontinuous quadratic elements, the domain contour using continuous quadratic elements, the mesh generation and the visualization of the propagation crack path. The second is invoked by the BEMLAB2D to perform a stress analysis of the structure with the BEM and, at each increment, after calculating the SIFs, the crack propagation direction is calculated by three direction criteria (MCS, MPERR and MDED). Fatigue life is performed with Paris low and coalescence analysis is possible with three plastic radius criteria.

Finally, the three examples presented illustrate the functionality of the graphical interface and the results of the crack propagation path, FIT calculations, number of cycles and residual resistance are obtained by the BemCracker2D program, showing that the use of the two boundary equations integrals greatly simplify the modeling process as well as attest to the robustness of the program.

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