METHOD OF APPLICATION OF FINITE ELEMENT SEMI-EMBEDDED IN CONCRETE BEAMS SIMULATION PRESTRESSED

Jordana F. Vieira
Raul D. Durand Farfán
jordanaferreirav@gmail.com
raul.durand@gmail.com
University of Brasília
Campus Universitário Darcy Ribeiro, 70910-900, Brasília/DF, Brasil

Abstract. Computer modeling has become an increasingly integral part of the activities of researchers and civil engineering professionals. To characterize a model that best represents a material, one must consider the knowledge of their mechanical behavior. In this context, this article aims to review the constitutive models that are being used to model prestressed concrete structures and present a representation proposal for this type of structure based on the model defined by Durand (2008). This method stems from an adaptation of two already consolidated methods for this type of structure, the embedded method, and the discrete method and was adopted as a semi-embedded method. It differs from the other methods, the reinforcing steel shall be discretized in front of elements and, such as the conditions of contact cannot be corrected in each node of elements. The final element of the bar is obtained after the reinforcement and finite element discretization of the concrete, being necessary to obtain stress and strain fields in the bars and together with the interface. Thus, from the model, it can be said that the semi-supplied method proposed is more advantageous than other methods of literature, since, from it, an example of analysis for concrete and reinforcement. Moreover, by simulating the element with its reinforcement, it is understood that the simulated concrete structures are approximated in a context closer to the real one.

Keywords: Semi-embedded method, computational modeling, prestressed concrete structures, finite element method.

1 Introduction

Over the years, numerical models have been developed to represent the nonlinear mechanical behavior of different types of structures. Computational methods began to be used frequently due to the great benefits offered in solving mathematical models. With the numerical study, in a preliminary analysis, it is possible to visualize the system and make different graphical interpretations of structure behavior. Thus, the ease of visualizing the problems that will occur in the structure in advance provides an initial rapid modification previously considered the structural system, reducing the costs related to the structure design and design time, since the problems were initially identified and modeled through numerical models.

Coupled with the advancement of technology, specific structural analysis software is becoming more widespread, facilitating the numerical study behavior of different structures. Thus, more refined models of calculation can be implemented in order to numerically visualize the real structures to be built. Considering prestressed and reinforced concrete as study materials, among the methods used to study the mechanical behavior of this type of structure, the Finite Element Method (FEM) is one most used, due to the versatility presented to model complex geometries, heterogeneous materials and nonlinear.

Hartl [1] reports that there are some methods for performing analysis of reinforced concrete by the FEM including reinforcement simulation. Among the methods found are distributed, discrete and embedded. The distributed method presents a lower computational expense and the perfect adhesion between steel and concrete is initially assumed. Thus, the simulation mechanical behavior of the reinforcement is not allowed. In the discrete method, it is possible to know the stresses in the reinforcement by considering two element types, but it restricts the domain discretization. The embedded method, on the other hand, presents ease in considering the reinforcement within the modeling and its free localization. In the embedded method, the implementation is more complex, since it allows the sliding reinforcement in the simulation and with this, there is an increase of the degrees of freedom.

Due to the complexity of modeling composite structures, the interface of various structural elements (such as beam, lattice, plate, membrane, solid element, etc.) must be carefully considered for precision modeling. Due to this limitation presented in the already available methods, Durand [2] proposed a new method based on the combination of discrete and embedded methods with consideration interface between materials and contact present in this type of structure. This method was attributed as semi-embedded and it allows the reinforcement elements to freely cross the elements that represent the concrete, now admitting to knowing the characteristics of mechanical behavior steel bars, independently concrete. The initial application of this method was given to analyze reinforcements in soils, and in recent years, several types of research have been developed using the semi-embedded method applied to reinforced concrete structures. In the works of Del Rio [3], Rosales [4], Silva [5], Faria [6] and Pantoja [7] the use of the semi-embedded method in reinforced concrete structures was applied considering different simulation hypotheses and how As a result of this research, the "FemLab" finite element program was developed for structural modeling and has been improved through research to implement the method in different types of structures.

As it is a method that allows simulating the interface between steel and concrete in reinforced concrete structures, as continuity of Durand's research line [2], the present work intends to study the method in the behavior of prestressed concrete structures via FEM Thus, the work seeks to perform a literature review of the main works that show the constitutive models used to model prestressed concrete structures and presents the proposed method, defending its possibility of use in the simulation of prestressed beams.

2 State of Art

According to Pfeil [8], prestressing is the introduction of an initial stress state in the structure, to improve the resistance or behavior element under various load situations. Compared to the conventional reinforced concrete structure, a prestressed concrete structure has advantages and disadvantages. Among the advantages, we highlight the greater rigidity structure, produced by the previous state of stress that totally or partially limits the cracking part. Also, with the use of prestressed concrete, slender and large-spanned structures can be designed. However, unlike reinforced concrete structures, prestressed concrete structures require more specialized equipment and skilled labor, as minimum prestressing cable must be positioned as projected.

The materials that make up prestressed concrete structures are mainly active reinforcing steel and concrete. Complementary materials present in this type of system are sheaths in case of post-bent elements, cement creams, hydraulic jacks, and tie bar anchor elements.

In this article, a literature study was carried out, searching the main and most relevant works for the modeling of prestressed concrete structures. Were chosen for the research, databases Web of Science and Scopus to be the basis of consolidated data and recognized quality. To outline the development theme, two types of research were carried out, the first without any temporal delimitation and the second using a refinement last 5 years.

In 1989, Martins [9] proposes a computational method, implementing in the CARPE program , to calculate isostatic beams with mixed prestressing. CARPE is a beam analysis program capable of analyzing the behavior until the rupture of beams with internal, external or mixed prestressing. This method considers the contribution of tensioned concrete in the stiffness of the pieces ("tension stiffening") even when there is a variation of adhesion between steel and concrete. Following the studies by Martins [9], Désir [10] implements the proposal of a hyperstatic model of beams with external prestressing.

Koni g & Qian [11] implemented a finite element model for the static and dynamic structural analysis of reinforced and prestressed beams, considering material nonlinearity, shear effects and concrete strength between successive cracks. Another numerical model present in the literature is the NAPCCB, proposed by Kodur & Campbell [12], which is based on a formulation using a macroscopic finite element model to analyze partially or fully prestressed reinforced concrete beams. In the same year, other proposals by other authors were developed. After approximately 10 years, Rabczuk & Eibl [13] used finite elements to simulate the rupture of prestressed beams, where the problem is treated linearly by Garlekin's method.

Lou and Xiang [14] presented a numerical modeling of the flexural behavior of continuous prestressed concrete beams, including load-deflection characteristics, increased stress on external wires, moment redistribution, secondary moments and the effect of long-time term in the nonlinear analysis. The concrete was modeled from an adaptation of the model presented in Hognestad [15] and the prestressing steel by a model proposed by Menegotto and Pinto [16]. Subsequently, the authors Lou et al. [17] also applied their numerical formulation to beams with non-adherent internal prestressing, considering a finite plane gantry model and from the discretization of the cable in straight sections, in each concrete element. Lou et al. [18] also published their numerical formulation for FRP tendon prestressed beams, prestressed concrete pillars (LOU et al. [19]) and externally prestressed steel composite beams (LOU et al. [20]).

Yapar et al [21] developed a nonlinear finite element model for prestressed concrete beams. The work states that it was the first time that a prestressed concrete beam was successfully modeled by nonlinear finite elements, allowing the plasticity, the behavior of concrete damage and its failures. As a goal, the work accurately predicted the behavior of a prestressed concrete beam for all load stages and the performance of a beam subjected to bonded composite patch repair. The plasticity-damage model for the concrete and the bond slip model for the steel-concrete interface are discussed. The modeling process faithfully reflected all loading conditions, starting with wire pre-tensioning, voltage transfer and subsequent loading to failure. The proposed scheme was modeled by Abaqus software.

Xu and Sun [22] numerically analyzed nonlinear finite elements of prestressed steel reinforced concrete beams, simulating the complete failure procedure of prestressed steel reinforced concrete beam using the three-dimensional nonlinear finite element method. For the modeling of materials, this work used the plasticity model to simulate steel rebar, which met Von Mises productivity criteria and simulated the elastoplastic properties of metallic materials. To simulate the bond between stress and strain of prestressed steel and concrete, a linear elastic model was used and for the plastic properties, the Ramberg-Osgood curve was used. The elastoplastic damage model provided by Abaqus was used to simulate concrete, which assumed that the concrete material presents mainly tensile cracks, compression fracture, and damage when subjected to high loads.

Also, Thoma [23] used nonlinear finite element analysis to perform numerical analysis of full-scale plates and beams, directly and indirectly, tested experimentally. Numerical analysis was performed from an ANSYS Mechanical APDL implementation for a bent beam, a directly supported reinforced concrete beam, a directly supported prestressed concrete beam, and an indirectly supported prestressed concrete beam. In this work, prestressed concrete beam analyzes are performed from finite element models considering the geometry prestressed cable as an open polygon consisting of continuous members. Anchorage forces and friction due to prestressing are applied as external loads. The stress-strain relationship prestressing steel includes the tensile stiffening effect and the cables can only transfer tensile forces. Finally, the tension rope model was defined to model the bonding tension between prestressing steel and concrete.

Moreira [24] proposed a model that considers that the tension in the prestressing cable is constant along the length, that is, there is no friction between the plastic sheath and the prestressing steel. The prestress cable in this work was modeled as a lattice element, with the lattice element coordinates associated with the flat gantry element. The reinforced concrete beam is modeled by nonlinear flat structure elements based on the Euler-Bernoulli- Navier theory and the total Lagrangean approach.

Given the chronology of the recently published works on numerical modeling of prestressed concrete beams, it is noted that the recent work follows the same line of research, however, with the proposal to use new constitutive models for finite elements. Thus, the work fits into a relevant theme for the area in question, showing the great motivation of this study. Moreover, the work is contained within the research line of Professor Raul Darío Durand Farfán, at the Graduate Program in Structures and Civil Construction, University of Brasilia, where since his master's degree and through the works of Velez [25], Gaitán [26] and Rosales [4] have been studying the insertion of the semi-embedded method to several types of structures, however, the first visualization of the prestressed concrete method application will be in this work.

3 Finite element methods on prestressed structures

The Finite Element Method (FEM) is an indispensable numerical method in the modeling and simulation of advanced engineering systems in various fields. FEM is a computational technology designed to obtain approximate solutions to the partial differential equations that arise in scientific and engineering applications. In many situations, it is a suitable model for approaching the solution, using a finite number of well-defined points, which can be called discrete models. However, in other problems, the subdivision is continued indefinitely, and the problem can only be defined using infinitesimal mathematical fiction. Thus, these systems are continuous because they result in differential equations or equivalent statements implying an infinite number of elements (Zienkiewicz, [27]).

According to Brebbia & Ferrante [28], in the mid-1940s, the Courant proposed a special method derived from the Rayleigh-Ritz variational method to approximate solutions based on the use of polynomial functions to formulate elastic problems in triangular subregions. However, according to Carey & Oden [29], the method was discovered from the work of Hrennikof and McHenry, who documented a two-dimensional problem domain in a set of one-dimensional bars and beams. Finally, Turner, Clough, Martins, and Topp [30] presented the FEM as it is currently used, applying the method to the analysis of aeronautical structures, considering the concept of discretization and shape functions.

Given any system, it is possible to differentiate the domain and boundary conditions that represent the known variables and which condition the change of the system to be solved, such as loads, displacements, temperatures, among others. After defining the boundary and unknown conditions, system variables can be found from the approximate solution for each finite element. The FEM uses a variational problem that involves a differential integral equation over the problem domain, as opposed to the finite difference method that approximates partial differential equations by finite differences. This domain is divided into several subdomains called finite elements and the solution of the partial differential equation is approximate to the polynomial functions of each element. The polynomials of each finite element are grouped so that the approximate solution has an appropriate degree to represent the content domain (Flaherty, [31]).

As technology evolves, discrete problems can provide exact solutions regardless of the number of elements. But by identifying the capacity of the computer as finite, continuous problems can be solved only by mathematical manipulation. Mathematical techniques for solving exact equations limit the possibilities of simplified functions. That is, for more complex problems, it is necessary to introduce the discretization methods that have been proposed by scientific and mathematical engineers over the years. Discretization divides the problem into subdomains and the solution is given from the approximate solution that approximates the true continuous solution as the number of discrete variables increases (Zienkiewicz [27]).

The discretization of continuous problems has been consolidated through different approaches by engineers and mathematicians. According to Cook et al. [33], mathematicians developed general techniques applied directly to the differential equations that govern the problem, such as finite-difference approximations and weighted residual techniques. The engineer, on the other hand, usually approaches the problem more intuitively, creating an analogy with discrete real elements and finite parts of a continuous domain.

Considering the application of FEM in prestressing, Hartl [1] defines that in a prestressing element, the tensile stress of the prestressed cable is increased to arbitrary stress, which is lower than the strength of the steel. The increased tension is converted to residual forces. These forces are reapplied to the element matrix, causing compression stresses in the matrix. Given this, it can be assumed that the prestressed can be incorporated by modifying the cable tension. According to Lin [34] and Leonhard [35], prestressing is explained by the use of the concept of load balancing, characterized by treating concrete cables and prestressing as free bodies. According to Hofstetter [36], Roca / Mari ([37], [38]), this concept presented can be applied to the finite element method.

4 Semi-embedded method

As in reinforced concrete structures, prestressed concrete structures are constituted of materials of different properties. As presented by Baetu & Ciongradi [39], three models capable of simulating the interface between steel and concrete through the finite element method are highlighted. They are the distributed model, the discrete model, and the embedded model.

In the distributed model, according to Ožbolt & Sharma [40], steel is represented as a continuous strip along with the concrete element with structure orientation axes. Thus, the constitutive relationship is given as a homogeneous steel-concrete model, while achieving a perfect bond between the two materials. In this one, we initially have the characteristic values for steel and reinforcement and then are integrated to represent the distributed model. Modeling the reinforcement in a distributed form within the solid element thickness for the concrete has perfect adhesion between the two components.

For the discrete model, according to Hartl [1], the reinforcements are positioned over the edges finite elements that represent the concrete matrix, that is, they are modeled as discrete lattice elements whose nodes coincide with those of the concrete elements. One of the disadvantages of this method is that the reinforcement positioning in this method depends in advance on the finite element mesh and with each reinforcement configuration, a new mesh model must be generated so that the nodes remain coincident with the element nodes. In this model, in general, a perfect adhesion between the concrete and the reinforcement is assumed.

In the third model, the embedded model, the reinforcement bars are incorporated into the solid elements so that the displacements between the two elements are shared. Unlike the discrete model, the placement of armor does not depend on the initial mesh, so the mesh does not need to be modified for the insertion of steel. The model is characterized as the most real of the three presented because it is possible to assign different rigidity to each material. Structures with more complex reinforcement distributions are better analyzed by this method. For Yamaguchi & Ohta [41], the formulation is said to be advantageous in concrete structures where the reinforcement has a complex distribution, where the embedded representation analysis are more efficient than those discretely represented.

The models presented are seen as the main models to represent the interface between concrete and steel through the Finite Element Method. In order to improve the representation of the bond between steel and concrete, Durand [2] proposed the semi-embedded method based on the discrete and embedded methods. In the semi-embedded method, the boundary conditions can be applied directly to the inclusions and the reinforcement is discretized in bar elements. In the discretization process, interface elements are generated which connect the mesh (concrete) to the bar elements (reinforcement). The interface elements do not necessarily have their knots coinciding with the solid knots and from varying the stiffness properties of these elements, various levels of adhesion between steel and concrete can be simulated.

The finite bar element obtained after the discretization of the reinforcement is independent of the finite element of the concrete and with this it is possible to apply the boundary conditions in each node, resulting later in obtaining stress and deformation fields in the bars and along with the contact.

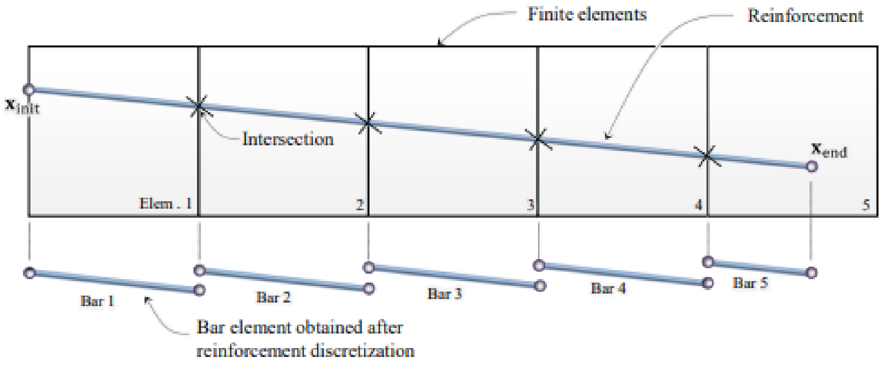

Figure 1. Reinforcement discretizacion

The formulation for modeling the interface was presented by Durand [2] as “point modeling of the interface”. Subsequently, in response to the previously proposed modeling improvement, Durand & Farias [42] defines it as “continuous interface modeling”, bringing a more realistic simulation than the previous one. For point interface modeling, the joint elements are made up of a set of springs placed on the bar nodes. In continuous interface modeling, the springs were replaced by a special single joint element. The special single joint element connects the bar element to the corresponding pierced element without adding extra nodes to the system. In this case, these elements represent all contact points at the interface and their stiffness is determined by integration using Gauss quadrature. Figure 1 shows the shape of the special joint elements.

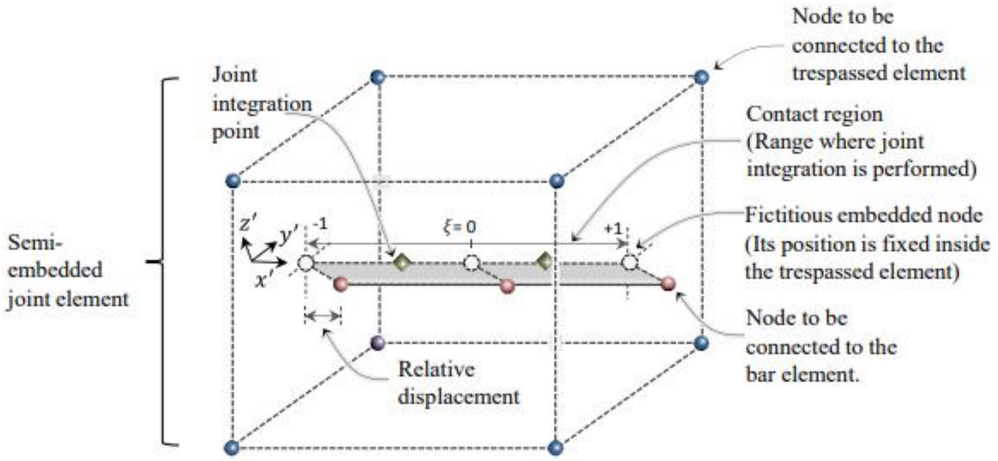

Figure 2. Special joint element

In Fig. 2, the possible displacements along the contact region are represented by the shaded area. Black nodal points are those that connect to solid elements, and white nodes are nodes that connect to bar elements. The white nodes are control points and their position are fixed within the traversed element and on these, the interpolation offsets of the nodes connected to the solid element are calculated.

Considering the displacements of the nodes in the solid elements and bars as $\mathbf{u} = \left[\mathbf{u}^{s} + \boldsymbol{u}^{b}\right]^{\boldsymbol{T}}$ where, $\boldsymbol{u}^{s}$ are the displacements of the nodes of the solid element and $\boldsymbol{u}^{b}$ are the displacements of the bar nodes (interface floating), the relative offsets between the bar and solid elements are calculated from the transformation matrix **B** as follows:

$$\mathbf{u}^{\mathbf{r}} = \mathbf{B}\mathbf{u} \tag{8}$$

Where **B** is the matrix that transforms the interface element nodal displacements into relative displacement along the contact area and can be found from the relative displacement expressed by:

$$\mathbf{u}^{\mathbf{r}} = \mathbf{R}[\mathbf{N} \quad -\boldsymbol{N}]\begin{Bmatrix}\mathbf{u}^{*} \\ \boldsymbol{u}^{b}\end{Bmatrix} \tag{9}$$

The displacements **u*** and $\mathbf{u}^{\mathbf{b}}$ are displacements of dummy and floating interface nodes, respectively, **R** is the matrix of cosine directors for the bar contact area and **N** is the matrix containg the interface interpolation functions, defined by:

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_n \end{bmatrix} \tag{10}$$

The term **u*** can be calculated by interpolating the nodal displacements of the traversed solid element $\mathbf{u}^{\mathbf{s}}$, given:

$$\mathbf{u}^{*} = \mathbf{M}\mathbf{u}^{\mathbf{s}} \tag{11}$$

Eq. (X) can be rewritten by:

$$\begin{Bmatrix} u_{x1}^{*} \\ \boldsymbol{u_{y1}^{*}} \\ \boldsymbol{u_{z1}^{*}} \\ u_{x2}^{*} \\ \boldsymbol{u_{y2}^{*}} \\ \vdots \\ \boldsymbol{u_{xn}^{*}} \\ \boldsymbol{u_{yn}^{*}} \\ \boldsymbol{u_{zn}^{*}} \end{Bmatrix} = \begin{bmatrix} M_{1,1} & 0 & 0 & M_{2,1} & 0 & \vdots & M_{m,1} & 0 & 0 \\ 0 & M_{1,1} & 0 & 0 & M_{2,1} & \vdots & 0 & M_{m,1} & 0 \\ 0 & 0 & M_{1,1} & 0 & 0 & \vdots & 0 & 0 & M_{m,1} \\ M_{1,2} & 0 & 0 & M_{2,2} & 0 & \vdots & M_{m,2} & 0 & 0 \\ 0 & M_{1,2} & 0 & 0 & M_{2,2} & \vdots & 0 & M_{m,2} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & \ldots \\ M_{1,n} & 0 & 0 & M_{2,n} & 0 & \vdots & M_{m,n} & 0 & 0 \\ 0 & M_{1,n} & 0 & 0 & M_{2,n} & \vdots & 0 & M_{m,n} & 0 \\ 0 & 0 & M_{1,n} & 0 & 0 & \vdots & 0 & 0 & M_{m,n} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \boldsymbol{u_{y1}} \\ \boldsymbol{u_{z1}} \\ u_{x2} \\ \boldsymbol{u_{y2}} \\ \vdots \\ \boldsymbol{u_{xn}^{*}} \\ \boldsymbol{u_{yn}^{*}} \\ \boldsymbol{u_{zn}^{*}} \end{Bmatrix} \tag{9}$$

5 Conclusion

Modeling a prestressed concrete structure is not a trivial task. Due to the heterogeneity material, numerical models need to be previously evaluated and studied so that the results are satisfactory and consistent with the experimentally acquired results.

In this paper, a proposal semi-embedded finite element method for the simulation of prestressed concrete beams was defended. Other existing proposals have been submitted. For the distributed model, steel is represented as a continuous strip along with the concrete element with structure orientation axes, ie the actual reinforcement simulation is not performed. Considering the discrete model, the reinforcement is modeled as discrete lattice elements whose nodes coincide with one concrete element.

The third model available in the literature is the embedded method. Until then, the model is characterized as the most real three presented. In the inline model, you can assign different stiffness to each material. Another point considered is that it can be characterized to simulate more complex situations. Therefore, to improve the application of a method that makes numerical simulation in prestressed concrete structures more realistic, the semi-embedded method was presented.

The use semi-embedded model in the representation of prestressed concrete beams will allow a better representation of the element, since the consideration of prestressing makes the formulation a little more complex, as the cable element contributes to the stiffness matrix and forces at a point of equilibrium must be indicated correctly. As in the semi-embedded method, the bar is represented independently of the matrix, one positive point for the simulation of prestressed reinforcement is that the consideration initial stresses present is palpable. In the aforementioned works, it is noted that the method has already been applied to reinforced concrete structures and due to the accuracy presented, the method is defended for prestressed structures. In this sense, as a continuation of this line of research, the purpose of this work was defended and it is intended to implement, in a future job, the prestressing of concrete beams for numerical modeling using the semi-embedded finite element method.

Acknowledgements

CNPq for financial support throughout this research.

References

[1] Hartl , H., “Development of a continuum-mechanics-based tool for 3D finite element analysis of reinforced concrete structures and application to soil-structure interaction problems,” PhD Thesis, Graz University of Technology, Graz, Austria, 250 p, 2002.
[2] Durand, R. (2008). Three-dimensional analysis of geotechnical structures submitted to reinforcement and drainage. Doctoral thesis. University of Brasilia, Brasilia - DF, 153 p.
[3] Del Rio, JD; Application of the Semi-Embedded Finite Element Method in the Simulation of Reinforced Concrete Beams. Master Thesis, Publication E.DM-006A / 15, Department of Civil and Environmental Engineering. University of Brasilia. Brasilia, DF, 100p.
[4] Rosales, YM (2016). “Numerical - experimental analysis of the steel-concrete adhesion behavior”. Diss. Master's degree. University of Brasilia, p. 115
[5] Silva, F.H.B.T. (2017). Numerical Analysis of Concrete Mode I and Mixed Mode Fracture Using Joint Element. Master Thesis in Structures and Construction, Publication E.DM-08A / 17, Department of Civil and Environmental Engineering, University of Brasilia, Brasilia, DF, 106p.
[6] Faria, T. (2017). Numerical Study on Steel Fiber Reinforced Concrete Using the Finite Element Method and Constitutive Models. Masters dissertation. University of Brasilia - PECC.
[7] Pantoja R, B.G. (2018). Dynamic Analysis of Reinforced Concrete Structures via Finite elements. Master Thesis in Structures and Construction, Publication E.DM - 01A / 18, Department of Civil and Environmental Engineering, University of Brasilia, Brasilia, DF, 167p.
[8] Pfeil , W. (1984). Prestressed concrete. Rio de Janeiro: LTC - Technical and Scientific Books Editora SA, 1985. ______. PROTENDED CONCRETE, 1: introduction. Rio de Janeiro: LTC - Technical and Scientific Books Editora SA
[9] Martins, PCR & Regis, PA (1998) - Mixed prestressing in Hyperstatic Beams - Proceedings of the 13 th FIP Congress Amsterdam.
[10] Désir, JM (1993) Nonlinear analysis of the behavior until rupture of external prestressed concrete beams. Doctoral Thesis, COPPE-UFRJ, Rio de Janeiro.
[11] Kônig, G. & Qian, L. (1993) - FEMAS: A Structural Non-Linear Analysis Program. Workshop on Behavior of External Prestressing in Structures, Saint- Rémy - Iès - Chevreuse, France.
[12] Kodur, VK R & Campbell, TI (1993) - A Computer Program for the Nonlinear Analysis of Prestressed Concrete Continuous Beams. Workshop on Behavior of External Prestressing in Structures, Saint- Rémy - Iès - Chevreuse, France.
[13] Rabczuk, T & Eibl , J. (2004) Numerical Analyzes of Prestressed Concrete Beams using a Coupled Element Free Garlekin / Finite Element Aprouch , International Journal of Solids and Structures, V.41, pp. 1061-1080.
[14] Lou T, Xiang Y. (2006) Finite element modeling of concrete beams prestressed with external tendons. Eng Struct ;28(14):1919–26. http://dx.doi.org/10.1016/j. engstruct.2006.03.020.

[15] Hognestad, E. N. W. Hanson, and D. McHenry. Concrete Distribution in Ultimate Strength Design. ACI Journal 52 (December, 1955): 455-479
16] Menegotto M, Pinto PE. (1973). Method of analysis for cyclically loaded reinforced concrete plane frames, including changes in geometry and non-elastic behavior of elements under combined normal force and bending. In: Preliminary report for symposium on resistance and ultimate deformability of structures acted on well-defined repeated loads, vol. 13 of IABSE, 1973. p. 15–22.
[17] Lou T, Lopes S, Lopes A. (2013) Nonlinear and time-dependent analysis of continuous unbonded prestressed concrete beams. Comp Struct 2013; 119: 166 -76. http: // dx . doi.org/10.1016/j.compstruc.2012.12.014.
[18] Lou T, Lopes S, Lopes A. (2015). A comparative study of continuous beams prestressed with bonded frp and steel tendons. Compos Struct 2015; 124: 100 –10. http://dx.doi.org/ 10.1016 / j.compstruct.2015.01.009.
[19] Lou T, Lopes S, Lopes A. (2016) Numerical modeling of externally prestressed steel-concrete composite beams. J Constr Steel Res 2016; 121: 229 –36. http://dx.doi.org/10.1016/j.jcsr.2016.02.008 .
[20] Lou T, Xiang Y. (2010) Numerical analysis of second-order effects of externally prestressed concrete beams. Struct Eng Mech 2010; 35 (5): 631–43.
[21] Yapar , O., Basu , P., & Nordendale , N. (2015). Accurate finite element modeling of pretensioned prestressed concrete beams. Engineering Structures, 101, 163-178.
[22] Xu, Jie & Sun, Xiaobo. (2018). Nonlinear finite element parametric analysis of prestressed steel reinforced concrete beams. IOP Conference Series: Materials Science and Engineering. 439, 042050. 10,1088 / 1757-899X / 439/4/042050.
[23] Thoma , K. (2018). Finite element analysis of experimentally tested RC and PC beams using the craked membrane model. Engineering Structures, Vol. 167, pp. 592-607. ISSN 0141-0296. https://doi.org/10.1016/j.engstruct.2018.04.010.
[24] Moreira, L. S. (2016). Nonlinear analysis via finite elements of concrete beams with non-sticking strain. 2016. 135 f. Dissertation (Master in Civil Engineering: Structures and Civil Construction) - Technology Center, Federal University of Ceará, Fortaleza.
[25] Velez, DAY (2015). Optimization of lattice structures using genetic algorithms. Masters dissertation. University of Brasilia, p. 97
[26] Gaitán , JDDR (2015). Application of the semi-embedded finite element method in the simulation of reinforced concrete beams. Diss. Master's degree. University of Brasilia, p. 96
[27] Zienkiewicz , O C .; Taylor, RL (1989). The finite element method. McGraw-Hill, 4th Ed .. SAS. (2013). ANSYS 14.5 Mechanical APDL. SAS IP, Inc.
[28] Brebbia , CA and Ferrante , AJ (1975). The Finite Element Technique: An Introduction for Engineers. Porto Alegre: Ed. UFRGS, 410p.
[29] Oden, JT Carey, GF (1984). Finite Element Methods: An Introduction. Macmillian Book Clubs, Inc.
[30] Turner, MJ, Clough, RW, Martin HC and Topp , LJ (1956). Stiffness and Deflection Analysis of Complex Structures. Journal of the Aeronautical Sciences, Vol. 23 No. 9, pp. 805-823.
[31] E. Flaherty, Joseph & Eaton Professor, Amos. (2000). Finite Element Analysis - Lecture Notes: Spring 2000.
[33] Cook, RD, Malkus , DS, Plesha , ME, & Witt, RJ (2001). Concepts and Applications of
Finite Element Analysis. John Wiley & Sons, New York, p. 784.
[34] Lin, T. Y & Burns, NH (1982). Design of Prestressed Concrete Structures. 3rd ed. New York: John Wiley & Sons.
[35] Leonhardt F., “ Vorlesungen über Massivbau, ” vol. 1-6, Springer, Berlin, 1980
[36] Hofstetter
[37] Roca P., Mari AR, “Numerical treatment of prestressing tendons in
the nonlinear analysis of prestressed concrete structures,” Computers & Structures,
46 , 1993, pp. 905-916
[38] Roca P., Mari AR, "Nonlinear geometric and material analysis of
prestressed concrete general shell structures", Computers & Structures, 46 , 1993,
p. 917-927
[39] Baetu , S., Ciongradi , I. (2011). Nonlinear analytical models of the reinforced concrete structural walls.

[40] Ožbolt , J., & Sharma, A. (2011). Numerical simulation of reinforced concrete beams with different shear reinforcements under dynamic impact loads. International Journal of Impact Engineering, v. 38, no. 12, pp. 940–950. doi: 10.1016 / j.ijimpeng .2011.08.003.

[41] Yamaguchi, E., & Ohta , T. (1993). Accurate and Efficient Method for Analysis of Reinforced Concrete Structures. Journal of Structural Engineering . v. 119, pp. 2017-2035. doi: 10.1061 / (ASCE) 0733-9445 (1993) 119: 7 (2017).

[42] Durand , R., & Farias, M. (2012). Nonlinear joint element for the analysis of reinforcement. Sao Paulo: World Congress on Computational Mechanics.