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PHYSICAL CONSISTENCY IN UNCERTAINTY BASED DESIGN OPTIMIZATION

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Abstract. A previous work demonstrated that standard probabilistic robust optimization (that takes into account a weighted sum of the expected value and the standard deviation) may become non-consistent from the physical point of view, when too much weight is given to the standard deviation. If this occurs, the optimum designs may have no meaning from the physical/engineering point of view. An alternative probabilistic robust optimization approach was then proposed, that is able to ensure physical consistency of the problem *a priori* (i.e. before the optimization procedure is started). In this work, physical consistency of reliability-based design optimization (RBDO) and worst-case robust optimization is investigated. We show that the RBDO and the worst-case problems are consistent under milder conditions than standard probabilistic robust optimization. This indicates that RBDO and worst-case optimization are inherently more consistent than probabilistic robust optimization, at least from the physical point of view.

Keywords: uncertainty based design optimization, robust optimization, rbdo, worst case optimization, physical consistency

1 Introduction

A recent work demonstrated that probabilistic Robust Optimization problems may give designs that are non-consistent from the physical/engineering point of view [1]. This occurs when the design obtained has features that violate some sort of physical intuition on the phenomenon under study. In the case of compliance-based topology optimization, for example, lack of physical consistency is observed when holes are made in the design even though the volume constraint is dropped. Before proceeding, it is interesting to present a brief overview of the discussions presented by [1].

Consider, for example, the square design domain from Fig. 1 with $a = b = 1$ (see [1] for more details). The structure is supported at both sides and a vertical distributed load is applied along $z = b/2$ (i.e. along the center of the structure). Plane-stress elasticity is considered and compliance minimization is pursued. However, the position of the peak load is a random variable X with uniform distribution on $[0, a]$. Thus, the load may actually be applied anywhere along $z = b/2$. Since the applied loads do not depend on the density field, the compliance is monotone and thus increasing the density anywhere always makes the structure stiffer. Consequently, consistent approaches should remove material of the design domain only if a volume constraint is imposed. Optimum designs were then obtained with three approaches: i) the standard probabilistic robust optimization approach (i.e. linear combination of the expected value and standard deviation), ii) a consistent probabilistic robust optimization approach developed by [1], iii) deterministic optimization.

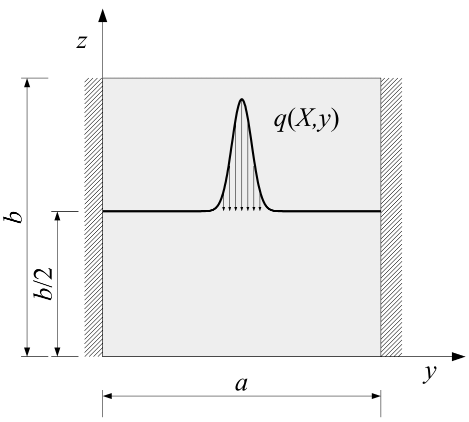


Figure 1. Design domain [1]

The designs from Fig. 2 were obtained without the volume constraint (i.e. by setting the volume fraction to $\alpha = 1.0$), and thus the algorithm is able to fill the entire design domain with material. We observe that even though the volume constraint was dropped, the standard robust approach removed material from the design domain. We thus say that the standard approach is non-consistent in this case, since physical intuition on the problem is violated.

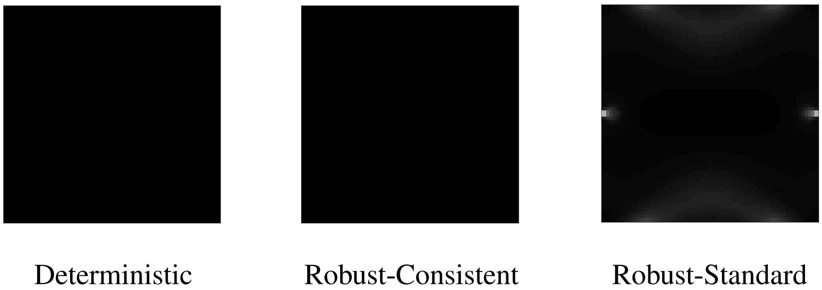
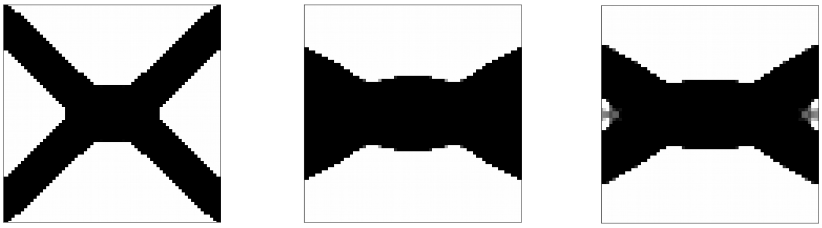


Figure 2. Designs obtained with volume constraint dropped [1]

The reader may wander if the same issue may happen when the volume constraint is taken into account. For this reason, the designs from Fig. 3 were obtained with a volume fraction $\alpha = 0.4$. The deterministic design is clearly non-robust, since no material is placed in regions where the load may indeed be applied. However, the standard robust approach again removed material near the lateral supports, at regions where the load may be applied. The design obtained with the consistent robust approach, on the other hand, did not remove material from these regions.



Deterministic Robust-Consistent Robust-Standard

Figure 3. Designs obtained with constraint on the volume fraction $\alpha = 0.4$ [1]

These results indicate that lack of physical consistency may lead to designs that are not suitable for practical physical/engineering purposes. Thus, non-consistent Robust Optimization problems can be viewed as ill conditioned problems, at least from the physical point of view. Even worse, these results also indicate that the widely employed standard probabilistic approach (i.e. linear combination of expected value and standard deviation) may fail to be consistent in practical situations. See [1] for more details.

In this work we demonstrate that RBDO problems are always consistent when the design parameters do not affect the random variables. We also demonstrate that Worst Case Robust Optimization (WCRO) is consistent provided the worst case scenario is unique. This indicates that RBDO and WCRO are inherently more consistent than probabilistic Robust Optimization, since probabilistic Robust Optimization problems may fail to be consistent under the same assumptions. A general result concerning RBDO problems were the random variables are affected by design parameters is also provided. Finally, we show that the standard probabilistic Robust Optimization approach is consistent when the performance has Normal distribution, assuming design parameters do not affect the random variables. This last result is obtained from an equivalence condition between RBDO and Robust Optimization problems.

2 Previous results on Robust Optimization

In the work by [1] it was demonstrated that physical consistency of probabilistic Robust Optimization problems is generally difficult to ensure, since evaluation of statistical moments is generally required. However, for a particular choice of parameters, a very simple sufficient condition for physical consistency is obtained. Based on these results, an alternative probabilistic Robust Optimization approach was proposed, that ensures physical consistency *a priori*, i.e. before the optimization procedure is started and statistical moments are evaluated [1].

Suppose $W(\rho, \mathbf{X})$ is a performance function that depends on a deterministic design parameter $\rho \in \mathbb{R}$ and a vector of random variables $\mathbf{X}$ with probability density function (PDF) f_X and support Ω [2–4]. From now on we employ the notation

$$W'(\rho, \mathbf{X}) = \frac{dW(\rho, \mathbf{X})}{d\rho} \tag{1}$$

in some passages of the text to represent derivatives with respect to ρ, in order to simplify the expressions.

In a previous work [1], the following definition of physical consistency was presented:

Definition 1 (Physical Consistency) *F is a consistent function of W on $\mathcal{D} \subset \mathbb{R}$ if*

$$F'W' \geq 0 \tag{2}$$

for $\rho \in \mathcal{D}$. In other words, F is a consistent function of W if F' and W' have the same sign for $\rho \in \mathcal{D}$.

It was also demonstrated that lack of physical consistency may lead to designs with no meaning from the physical point of view and convergence issues of the optimization algorithm [1].

Assuming the expected value of the performance satisfies $\mathrm{E}[W(\rho, \mathbf{X})] \geq 0$, the p-norm robust objective function was then defined as [1]

$$F_p(\rho, \mathbf{X}) = \sqrt[p]{(\mathrm{E}[W(\rho, \mathbf{X})])^p + \eta^p\,(\sigma[W(\rho, \mathbf{X})])^p},\ p \geq 1, \eta \geq 0, \tag{3}$$

where $\mathrm{E}[\cdot]$ and $\sigma[\cdot]$ represent the expected value and the standard deviation, respectively [2–4]. This objective function is actually the p-norm of a vector of minimization goals composed by the expected value and the standard deviation weighted by η.

It was then demonstrated that F_p is consistent with respect to W if the weight given to the standard deviation satisfies [1]

$$\eta^p \leq \left(\frac{\mathrm{E}[W]}{\sigma[W]}\right)^{p-2}. \tag{4}$$

From this result we conclude that checking for physical consistency of F_p requires, in general, evaluation of statistical moments. Besides, this result also indicates that robust objective functions commonly employed in practice (e.g. standard probabilistic robust optimization with $p = 1$) may fail to be consistent, depending on the value taken for the weight η.

Also note that for general values of p it is generally impossible to set η as to ensure physical consistency *a priori* (i.e. before the optimization procedure is started), since statistical moments are required for this task. However, for $p = 2$ the above sufficient condition is simplified, giving the following result:

Corollary 1 *F_p is a consistent function of $W(\rho, \mathbf{X}) \geq 0$ for $p = 2$, $\eta \leq 1$.*

Thus, this choice of parameters can be employed to ensure physical consistency of the Robust Optimization problem *a priori*, i.e. before the optimization procedure is started and statistical moments are evaluated. This makes statement of the problem much easier and has significant practical implications, as demonstrated by [1].

3 Consistency of the Probability of Failure

We now discuss physical consistency of RBDO problems, where the probability of failure plays a fundamental role. Suppose then that

$$g(\rho, \mathbf{X}) = \omega - W(\rho, \mathbf{X}) \leq 0 \tag{5}$$

indicates failure of the system under analysis, where $\omega \in \mathbb{R}$ is a given failure threshold and g is the resulting limit state function. In this case, the system is safe when $W < \omega$. The probability of failure is then defined as [5, 6]

$$P_f(\rho, \mathbf{X}) = P(g(\rho, \mathbf{X}) \leq 0) = \int_\Omega I(g(\rho, \mathbf{X}))f_X(\rho, \mathbf{X})), \tag{6}$$

where $P(\cdot)$ indicates the probability of occurrence of some event and I is the Indicator function

$$I(t) = \begin{cases} 0, & t > 0 \\ 1, & t \leq 0 \end{cases} . \tag{7}$$

We then have the following result[1]:

Theorem 1 *Suppose that $W(\rho, \mathbf{X})$ is monotone almost surely on Ω and satisfies the Dominated Convergence Theorem. Then the probability of failure $P_f(\rho, \mathbf{X})$ is a consistent function of $W(\rho, \mathbf{X})$ if*

$$E[W'] \int_\Omega I(g) \frac{df_X}{d\rho} \geq 0. \tag{8}$$

Proof 1 *The derivative of P_f with respect to ρ is given by [7–12]*

$$\frac{dP_f}{d\rho} = \int_\Omega I(g) \frac{df_X}{d\rho} - \int_\Omega \delta(g) \frac{dg}{d\rho} f_X. \tag{9}$$

where $\delta(t)$ is Dirac's Delta. From Eq. (5) we have

$$\frac{dP_f}{d\rho} = \int_\Omega I(g) \frac{df_X}{d\rho} + \int_\Omega \delta(g) W' f_X. \tag{10}$$

Since W is monotone almost surely on Ω, the last term has the same sign as W'. We then conclude that a sufficient condition for consistency of P_f is

$$E[W'] \int_\Omega I(g) \frac{df_X}{d\rho} \geq 0, \tag{11}$$

i.e. the first term on the right hand side should have the same sign as W'. This concludes the proof.

Note that multiplication by $\mathrm{E}[W']$ in Eq. (8) is just a sign switching condition. In other words, for $W' \leq 0$ almost surely the condition gives

$$\int_\Omega I(g) \frac{df_X}{d\rho} \leq 0. \tag{12}$$

In this case, the condition from Eq. (8) indicates that f_X should not be increased in the failure domain, in a general sense, when ρ is increased. We thus conclude that for $W' \leq 0$ almost surely, P_f is a consistent function of W provided the design parameter does not increase the probability of occurrences in the failure domain.

When the design parameter does not affect the random variables we have $df_X/d\rho = 0$ and the condition from Eq. (8) is always satisfied. This leads to the following result:

Corollary 2 *Suppose ρ does not affect the vector of random variables $\mathbf{X}$ and that W satisfies the Dominated Convergence Theorem. Then the probability of failure $P_f(\rho, \mathbf{X})$ is a consistent function of $W(\rho, \mathbf{X})$.*

These results prove that the probability of failure is a consistent function of the performance W under conditions frequently encountered in practice. Consequently RBDO problems, that are based on the probability of failure, are consistent under very mild conditions.

[1]Here we say that W is monotone almost surely if either $W' \geq 0$ almost surely or $W' \leq 0$ almost surely.

4 Consistency of Worst Case Robust Optimization

Consider now the worst case performance defined by

$$W^* = W(\rho, \mathbf{x}^*) = \max_{\mathbf{x}\in\mathcal{S}} W(\rho, \mathbf{x}), \tag{13}$$

where $\mathcal{S} \subseteq \Omega$ is a given set of events and $\mathbf{x}^*$ is the worst case scenario. Note that the worst case scenario $\mathbf{x}^*$ is actually the realization of $\mathbf{X}$ that maximizes the performance on $\mathcal{S}$. We then have the following result.

Theorem 2 *Suppose the worst case scenario $\mathbf{x}^*$ is unique. Then the worst case performance W^* is a consistent function of W.*

Proof 2 *Assuming $\mathbf{x}^*$ is unique we have, from Eq. (13),*

$$\frac{dW^*}{d\rho} = W'(\rho, \mathbf{x}^*) \tag{14}$$

We thus conclude that the derivative $dW^*/d\rho$ has the same sign as W' if W is monotone on Ω.

This result proves that WCRO is consistent under somewhat mild conditions. This basically occurs because the worst case derivative is actually the derivative of some realization of $\mathbf{X}$, and then the sign of the derivative is preserved when W is monotone.

5 Consistency of Robust Optimization with Normal Performance

In this section we consider only the case where W has Normal distribution. This is an important case both from the conceptual and practical points of view. We then have the following result:

Corollary 3 *Suppose $W(\rho, \mathbf{X})$ has Normal distribution and that the design parameter ρ does not affect the vector of random variables $\mathbf{X}$. Suppose additionally that $E[W] > 0$. Then F_p is a consistent function of $W(\rho, \mathbf{X})$.*

Proof 3 *Since W has Normal distribution, the probability of failure with respect to some failure threshold $\omega \in \mathbb{R}$ can be written as*

$$P_f = 1 - \Phi(\beta) \tag{15}$$

where

$$\beta = \frac{\omega - E[W]}{\sigma[W]} \tag{16}$$

is the reliability index [5, 6] and Φ is the standard Normal cumulative distribution function (CDF)[2–4]. By the chain rule we then have

$$\frac{dP_f}{d\rho} = -\phi(\beta)\frac{d\beta}{d\rho}, \tag{17}$$

where ϕ is the standard Normal PDF. The derivative of the reliability index gives

$$\frac{d\beta}{d\rho} = -\frac{1}{\sigma[W]}\left(\frac{dE[W]}{d\rho} + \beta\frac{d\sigma[W]}{d\rho}\right). \tag{18}$$

The derivative of F_p gives

$$\frac{dF_p}{d\rho} = (F_p)^{1-p}\left((E[W])^{p-1}\frac{dE[W]}{d\rho} + \eta^p(\sigma[W])^{p-1}\frac{d\sigma[W]}{d\rho}\right), \tag{19}$$

that can be rearranged as

$$-\frac{1}{\sigma[W]}\left(\frac{F_p}{E[W]}\right)^{p-1}\frac{dF_p}{d\rho} = -\frac{1}{\sigma[W]}\left(\frac{dE[W]}{d\rho} + \eta^p\left(\frac{\sigma[W]}{E[W]}\right)^{p-1}\frac{d\sigma[W]}{d\rho}\right). \tag{20}$$

We then have

$$\frac{dF_p}{d\rho} = -\sigma[W]\left(\frac{F_p}{E[W]}\right)^{1-p}\frac{d\beta}{d\rho} \tag{21}$$

for

$$\beta = \eta^p\left(\frac{\sigma[W]}{E[W]}\right)^{p-1} \tag{22}$$

and, finally,

$$\frac{dF_p}{d\rho} = \frac{\sigma[W]}{\phi(\beta)}\left(\frac{F_p}{E[W]}\right)^{1-p}\frac{dP_f}{d\rho}. \tag{23}$$

Since $E[W], \sigma[W], F_p, \phi(\beta) > 0$, we conclude that F'_p and P'_f have the same sign. From Corollary 2 we know that P_f is consistent under the assumptions, and consequently F_p is also consistent. This concludes the proof.

This result shows that probabilistic Robust Optimization approaches based on Eq. (3) are always consistent if the performance has Normal distribution and design parameters do not affect the random variables. Thus, physical consistency is not an issue in these cases.

A similar result for the standard probabilistic Robust Optimization approach (i.e. F_p with $p = 1$) reads:

Corollary 4 *Suppose $W(\rho, \mathbf{X})$ has Normal distribution and that the design parameter ρ does not affect the vector of random variables $\mathbf{X}$. Then*

$$F(\rho, \mathbf{X}) = E[W(\rho, \mathbf{X})] + \eta\sigma[W(\rho, \mathbf{X})],\ \eta \geq 0 \tag{24}$$

is a consistent function of $W(\rho, \mathbf{X})$.

Proof 4 *Take $p = 1$ in the proof of Corollary 3. The condition $E[W] > 0$ can be dropped since $E[W]$ does not appear in Eq. (23) for $p = 1$.*

Corollary 4 shows that the requirement $\mathrm{E}[W] \geq 0$ may be dropped in the standard probabilistic Robust Optimization approach. The important conclusion from Corollaries 3 and 4 is that the probabilistic Robust Optimization problem is consistent under mild conditions if W has Normal distribution.

6 Summary of Results

Several theoretical results were presented in this work. We now carefully summarize them in the following remarks:

Remark 1 *Corollary 2 proves that the probability of failure is a consistent function if the design parameters do not affect the random variables. The general robust objective function F_p, on the other hand, may fail to be consistent even under this assumption (see Section 2). This proves that RBDO is inherently more consistent than probabilistic Robust Optimization from the physical point of view, i.e. it is consistent under milder conditions.*

Remark 2 *Theorem 2 proves that the worst case performance is consistent if the worst case scenario is unique. This also leads to the conclusion that WCRO is inherently more consistent than probabilistic Robust Optimization, at least from the physical point of view.*

Remark 3 *Theorem 3 and Corollary 4 show that the probabilistic robust objective function is consistent under very mild conditions when the performance has Normal distribution. This is an important case from both the conceptual and practical points of view. It also indicates that physical consistency should not be an issue when the distribution of the performance is almost Normal.*

Remark 4 *Theorem 3 is proved by showing that the probabilistic robust objective function is equivalent to the probability of failure when the performance has Normal distribution. In other words, we can check if some probabilistic Robust Optimization approach is consistent by checking if it is equivalent to some RBDO approach, assuming the design parameter does not affect the random variables. This approach may be easier to employ in some cases.*

7 Concluding Remarks

In this brief work we demonstrated that the probability of failure is a consistent function of the performance if the design parameter does not affect the random variables. We also show that the worst case performance is consistent when the worst case scenario is unique. Consequently, RBDO and WCRO are inherently more consistent than probabilistic Robust Optimization from the physical point of view. A general result for RBDO cases where the random variables are affected by the design parameter was also presented.

The idea behind Remark 4 was then employed to discuss consistency of the probabilistic Robust Optimization problem when the performance has Normal distribution. It was demonstrated that the standard probabilistic Robust Optimization approach is consistent provided the design parameter does not affect the random variables. This is an important conclusions from the conceptual and practical points of view.

We emphasize that the implications of employing non-consistent approaches are discussed in details in the work by [1]. It was demonstrated that non-consistent approach may lead to designs with no meaning from the physical point of view. This indicates that ensuring consistency of uncertainty based design optimization approach is important from the practical point of view. In this context, we hope the results presented in this work will contribute to a better understanding of the concept of physical consistency in uncertainty based design optimization.

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