MODAL ANALYSIS OF A HUMAN LUMBAR SPINE BY EULER-BERNOULLI BEAM THEORY

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Abstract. Human lumbar spine supports most of loads arising daily activities. Those loads are mainly dynamic or periodic, therefore, an accurate study of free vibration of this structure can lead to a better understanding the behavior of the system. In anatomical point of view, there are five lumbar vertebrae in a human lumbar spine, two adjacent vertebral body are separated from each other by upper end plate, intervertebral disc and lower end plate. Each part of the column has a unique structure. In this paper, an analytical approach to study free transverse vibration of a human lumbar spine as a segmental multi-layer beam by the classical beam Euler-Bernoulli theory is developed. A uniform transverse section of lumbar spine is considered and the material properties are composed of different mechanic parameters. Free vibration is analyzed considering different material properties, in order to simulate the aging process. Continuity and equilibrium conditions are imposed as boundary conditions throughout the junctions of the segments. Those conditions are in terms of displacement, slope, shear force, and bending moments.

Keywords: Free Vibration, Euler-Bernoulli beam, Lumbar spine, Segmental multi-layer beam

1 Introduction

In many studies of mechanical vibration, the main features in a structure subject to a dynamic excitation are natural frequencies and normal modes. Considering free vibration, the first represents the magnitude that the structure vibrates after the force applied on it, while the second refers to the way the vibration occurs, relative to geometric aspect. The human spine present a complex anatomical combination of bones, muscles, tendons, and other joint components that form a biomechanical structure. Studies on dynamic analysis of a numerical model of this type of structure help to understand the human spine mechanics, mainly in performing certain competitive activities, such as some specialized movements based on repetitive actions of the lumbar segment.

Several works developed methods to analyze the influence of vibrations on the human spine. Guo et al. [1] developed a three-dimensional finite element model to extract resonant frequencies and normal modes from the twelfth thoracic vertebra to Pelvis, through harmonic response analysis. In order to obtain the resonant frequencies, Xu et al. [2] analyzed the healthy, pre- and post-surgical scoliotic spines vibration mechanical responses using finite element methods (FEM) and modal vibration analysis. These studies involving resonant frequencies make important contributions to the understanding of whole body vibration (WBV) injury, the ergonomic design and the development of mechanical production to protect the safety of the human spine. Natarajan et al. [3] investigated the biomechanical effect of material properties, geometric variables, and anchoring arrangements in a segmented pedicle screw to affect lumbar spine kinematics and instrumentation stresses through a finite element analysis. Results showed that the differences in properties and variations in screw-rod geometry moderately affected spinal biomechanics. Wang et al. [4] used a lumbar spine finite element model and a dynamic vibration model to investigate the biomechanical effects of the combination of combining head-down tilt(HDT) traction and vibration therapy on the age-related degeneration of the lumbar spine. Results showed that the decrease of intradiscal pressure is more effective when vibration was combined with traction therapy.

The human spine is consist in a set of subdivided and overlapping cylindrical segments. Multi-layer stepped models are important in the study of spine vibration and have been used in many studies. Shahmohammadi et al. [5] investigated an artificial intervertebral disc with reinforced fibers under various loading conditions such as compression, torsion axial and bending moment, through a three-dimensional finite element model of second and third lumbar vertebra where units of 12 vertical fiber were incorporated into annulus fibrosis. Results showed that the addition of reinforced fibers in the disc induce a significant decrease in tension in the core and ring. In order to analyze the behavior of human lumbar spine, Shirazi et al. [6] proposed a free transverse vibration analysis of the human lumbar spine through the Timoshenko and Euler-Bernoulli beam theories by changing the mechanical properties of the lumbar spine components, for analysis the effects damages and defects, which occur over time and aging.

This paper aim to study the free transverse vibration of a human lumbar spine. First, the lumbar spine is modeled as a segmental multi-layer beam by classical Euler-Bernoulli beam theory. Due to the level of complexity of the geometry human spine, the cross section of the lumbar spine is assumed to be uniform. Material properties are different for the vertebral bodies, intervertebral discs and endplate. Finally, free vibration problem is analyzed changing material properties in order to simulate the aging process.

2 Euler-Bernoulli beam model

Lumbar spine can be idealized as a segmental two-layer Euler-Bernoulli beam. It can be seen in Fig. 1, in which the vertebral body and intervertebral discs are each composed of two different layers while the endplate is composed of only one layer. In the formulation present in the next sections, subscripts in and out represent the inner and outer layer, respectively.

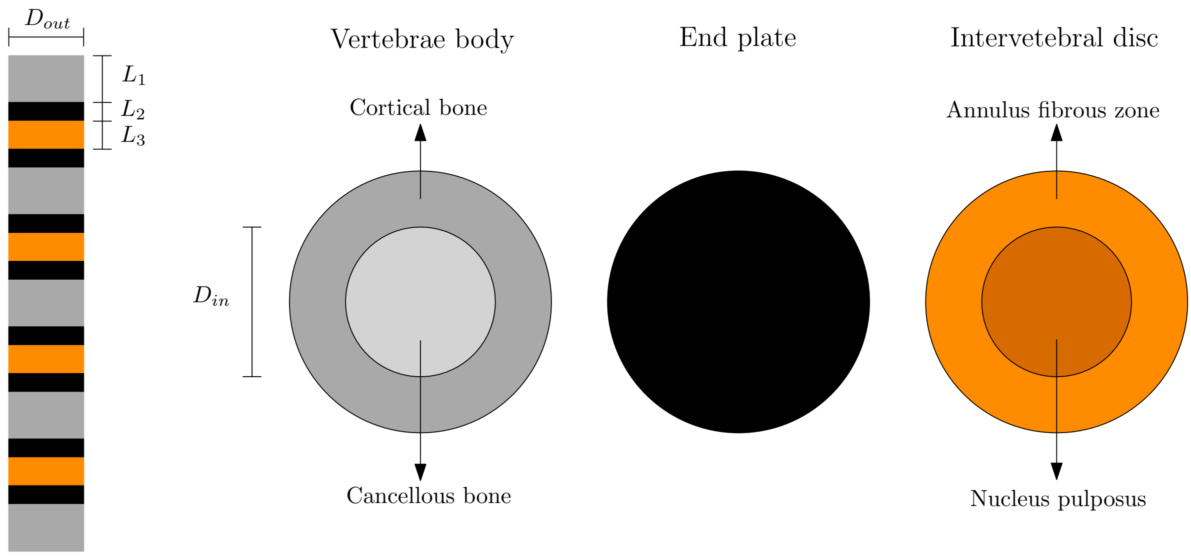

Figure 1. Lumbar spine scheme.

2.1 Two-layer beam

The energy of an Euler-Bernoulli beam are respectively derived from the bending deformation and the translational inertia. Hence, the potential and kinetic energy relative to translation inertia are given respectively by:

$$U = \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx}\, dV \quad \text{and} \quad T = \frac{1}{2}\int_0^L (\rho_{in}A_{in} + \rho_{out}A_{out})\left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx, \tag{1}$$

where L is the length of beam, ρ the mass per unit volume, A the cross-sectional area, $w(x,t)$ the transverse deflection, σ_{xx} and ε_{xx} are respectively the stress and strain components in direction x, which can be written as (Shirazi et al. [6])

$$\sigma_{xx} = \begin{cases} E_{in}\varepsilon_{xx}, & 0 \leq r_1 \leq r_{in}, \\ E_{out}\varepsilon_{xx}, & r_{in} \leq r_2 \leq r_{out} \end{cases} \tag{2}$$

and

$$\varepsilon_{xx} = -z\frac{\partial^2 w(x,t)}{\partial x^2}, \tag{3}$$

in which E is the Young modulus, r_1 and r_2 are respectively the radius of the cross-section areas of the inner and outer layer of the beam and z is the distance of any fiber to the neutral axis of the beam. Substituing the Eqs. 2 and 3 on the Eq. 1, we obtain the potential energy in terms of the transverse deflection:

$$U = \frac{1}{2}\int_0^L (E_{in}I_{in} + E_{out}I_{out})\left(\frac{\partial^2 w(x,t)}{\partial x^2}\right)^2 dx. \tag{4}$$

The equation of motion can be obtained using Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T-U)\,dt + \int_{t_1}^{t_2} \delta W_{nc}\,dt = 0, \tag{5}$$

where δW_{nc} is the virtual work done by non conservative forces, t_1 and t_2 are times at which the configuration of the system is known and $\delta(\ )$ is the symbol denoting virtual change, in the quantity in parentheses. Considering a free transverse vibration analysis, $\delta W_{nc} = 0$, using the potential and kinetic energy on the Hamilton's principle and, after some manipulations, the governing equation of motion can be expressed as (Shirazi et al. [6])

$$(E_{in}I_{in} + E_{out}I_{out})\frac{\partial^4 w(x,t)}{\partial x^4} + (\rho_{in}A_{in} + \rho_{out}A_{out})\frac{\partial^2 w(x,t)}{\partial t^2} = 0. \tag{6}$$

2.2 Segmental multi-layer beam

Consider that the segmental multi-layer beam shown in Fig. 2 is excited harmonically with an angular frequency ω and

$$w(x,t) = X(x)e^{jwt}, \tag{7}$$

in which $j = \sqrt{-1}$ and $X(x)$ is the normal function of w.

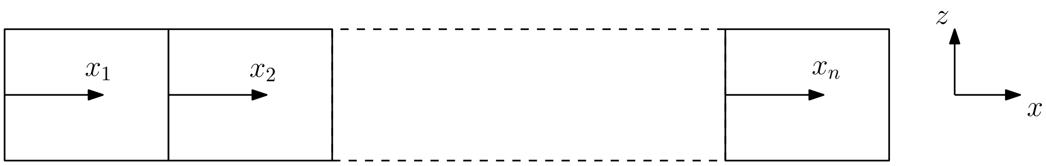

Figure 2. Segmental beam model.

The solution of Eq. 6 can be expressed in trigonometric and hyperbolic functions as

$$X_i(x_i) = B_{b1}sin(\beta_i x_i) + B_{b2}cos(\beta_i x_i) + B_{b3}sinh(\beta_i x_i) + B_{b4}cosh(\beta_i x_i), \tag{8}$$

where

$$0 \leq x_i \leq L_i \quad \text{and} \quad \beta_i = \left[\frac{(\rho_{in}A_{in} + \rho_{out}A_{out})\omega^2}{(E_{in}I_{in} + E_{out}I_{out})}\right]^{\frac{1}{4}}, \tag{9}$$

$i = 1, 2, ..., n$, in which n is the number of segments. For each segment there is a value of the four constants B_b which are obtained by the boundary conditions. Continuity and equilibrium conditions are imposed as boundary conditions throughout the junctions between adjacent segments. The displacement, slope, bending and shear conditions are respectively given by (Vaz [7]):

$$X_{p-1}(x_{p-1})\Big|_{x_{p-1}=L_{p-1}} = X_p(x_p)\Big|_{x_p=0}, \tag{10}$$

$$\left.\frac{dX_{p-1}(x_{p-1})}{dx_{p-1}}\right|_{x_{p-1}=L_{p-1}} = \left.\frac{dX_p(x_p)}{dx_p}\right|_{x_p=0}, \tag{11}$$

$$(E_{in}I_{in} + E_{out}I_{out})_{p-1}\left.\frac{d^2X_{p-1}(x_{p-1})}{dx_{p-1}^2}\right|_{x_{p-1}=L_{p-1}} = (E_{in}I_{in} + E_{out}I_{out})_p\left.\frac{d^2X_p(x_p)}{dx_p^2}\right|_{x_p=0}, \tag{12}$$

$$(E_{in}I_{in} + E_{out}I_{out})_{p-1}\left.\frac{d^3X_{p-1}(x_{p-1})}{dx_{p-1}^3}\right|_{x_{p-1}=L_{p-1}} = (E_{in}I_{in} + E_{out}I_{out})_p\left.\frac{d^3X_p(x_p)}{dx_p^3}\right|_{x_p=0}, \tag{13}$$

where p is location of the segment analyzed. Substituting the Eq. 8 and its derivatives in the continuity and equilibrium conditions yields the linear system of homogeneous equations which can be expressed as:

$$[\mathbf{H}]\{\mathbf{b}\} = \{0\}, \tag{14}$$

in which $[\mathbf{H}]$ is the coefficients matrix and $\{\mathbf{b}\}$ the vector of unknowns. In case of a segmented beam in n parts, the coefficient matrix is of the order $4n \times 4n$. Note that solving the linear system by the non-trivial solution, it is possible to evaluate the natural frequencies through the characteristic equation.

3 Numerical results

3.1 Method validation

An numerical example for an uniform free-free segmental two-layer Euler-Bernoulli beam of circular cross section with $L = 321.222\ mm$, $d = 20.0022\ mm$, $E = 68.732\ GPa$ is considered. This example was presented by Vaz [7]. He considered that all segments of the free-free stepped segmental beam are made of the same material and have only one layer and obtained experimental results for natural frequencies which are presented in Table 1.

Table 1. Experimental values of natural frequencies for uniform beam

$f(Hz)$	Natural frequencies
1ª	875
2ª	2375
3ª	4500

In order to validate our method, a convergence study was developed considering the values of E_{in} and $_{in}$ as ratios of E_{out} and $_{out}$, starting at ratio zero, representing a beam with hollow circular cross section, up to ratio one. Results are listed on Table 2.

Table 2. Natural frequencies obtained by convergence of E_{in} and ρ_{in}

$E_{in} - \rho_{in}$	f_1	f_2	f_3
$0 - 0$	971.5785	2678.1938	5250.3306
$0.1E_{out} - 0.1\rho_{out}$	958.9601	2643.4105	5182.1416
$0.2E_{out} - 0.2\rho_{out}$	946.9776	2610.3804	5117.3893
$0.3E_{out} - 0.3\rho_{out}$	935.5816	2578.9667	5055.8060
$0.4E_{out} - 0.4\rho_{out}$	924.7276	2549.0473	4997.1519
$0.5E_{out} - 0.5\rho_{out}$	914.3758	2520.5123	4941.2119
$0.6E_{out} - 0.6\rho_{out}$	904.4904	2493.2627	4887.7918
$0.7E_{out} - 0.7\rho_{out}$	895.0388	2467.2091	4836.7163
$0.8E_{out} - 0.8\rho_{out}$	885.9917	2442.2703	4787.8264
$0.9E_{out} - 0.9\rho_{out}$	877.3223	2418.3728	4740.9777
$E_{out} - \rho_{out}$	869.0063	2395.4493	4696.0385

Notice that the last row shows the natural frequency values for the beam having the inner layer with the same properties as the outer layer material. Comparing these results with those presented in Table 1, we have deviations of 0.689%, 0.853% and 4.174% for the values of the first, second and third natural frequencies. The results obtained showed good accuracy with those reported by Vaz [7].

3.2 Analysis of the human lumbar spine

The free vibration characteristics of lumbar spine were investigated considering it as a clamped-clamped beam given that it is fixed between the thorax and the sacrum region. Each column component was considered as a segmental two-layer beam with distinct mechanical properties.

As shown in Fig. 1, the lumbar spine model has five vertebrae, four intervertebral discs and eight endplates. Geometric parameters $D_{out} = 35\ mm$, $D_{in} = 32\ mm$, $L_1 = 27\ mm$, $L_2 = 0.5\ mm$, $L_3 = 12\ mm$ are considered. Table 3 presents the material properties of each lumbar spine component used in this work according to the literature.

Table 3. Mechanical properties of lumbar spine components. Source: Adapted of Shirazi et al. [6]

Spinal Components	Density(kg/m^3)	Normal case	Caso 1	Caso 2	Caso 3	Caso 4
		Young modulus(MPa)				
Cortical bone	1830	5000	5000	5000	5000	5000
Cancellous bone	1000	500	500	500	500	500
Endplate	1830	100	100	100	1000	5000
Nucleus pulposus	1360	10	50	100	10	10
Annulus fibrous zone	1200	175	175	175	175	175

In order to investigate the influence of the reduced flexibility of the endplates and nucleus pulposus caused by aging, natural frequencies were calculated for five distinct cases that simulate the mechanical properties of lumbar spine components, as shown in Table 4. In normal case there is no change in the properties of column components. Cases 1 and 2 represent a reduction in the flexibility of the nucleus pulposus and cases 3 and 4 the stiffening of the endplates.

Table 4. Natural frequencies for each case studied

Case	$f_1(Hz)$	$f_2(Hz)$	$f_3(Hz)$
Normal	479.6534	1365.0191	3399.9447
1	554.5017	1569.2323	3581.9283
2	621.8529	1748.4519	3748.8705
3	488.7381	1390.0971	3440.4404
4	489.5708	1392.3934	3444.0871

In case 1, increase of the first three frequencies represents an addition of 15.60%, 14.95% and 5.35% in relation to the normal case, respectively, and in case 2 an addition of 29.65%, 28.09% and 10.26%. In case 3, the first three natural frequencies presents an increase of 1.89%, 1.83% and 1.19%, respectively, compared to the normal case, and case 4 an increase of 2.06%, 2.00% and 1.30%.

4 Conclusion

From an analytical approach, a analysis of the free transverse vibration of a human lumbar spine as a segmental multi-layer beam by classical beam Euler-Bernoulli theory was developed.

Two analysis were performed. First, a case with reduction of the nucleus pulposus flexibility (cases 1 and 2) in relation to the normal case. Finally, a case with stiffening of the endplates (cases 3 and 4).

After investigating the effects of cases 1 to 4, it was concluded that the reduced flexibility due to aging causes an increase in the natural frequencies of the lumbar spine. Furthermore, it has been noted that the effect of stiffening of the nucleus pulposus is much more significant than that of the endplates with respect to the increase in natural frequencies.

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