# Prediction of dynamic response in a cantilever beam using a smoothed version of the experimental modal matrix

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**Abstract.** It is not economically viable, nor technically interesting the installation of accelerometers in all locations of interest in a given machine or structure. To estimate vibration responses in non-instrumented locations, a smoothed and expanded version of the experimental modal matrix can be employed. For this, a finite element model and operational modal analysis can be used. In this work, we used a smoothed version of the experimental modal matrix of a cantilever rectangular aluminum beam, to estimate vibration responses. Impact test and the EFDD method were used for modal parameter identification. The SEREP reduction technique was used to match numerical DOFs with the measured ones. Then, the local correspondence principle (LC) was used to smooth the measured DOFs in the experimental modal matrix. Finally, the smoothed version of the experimental modal matrix was used to estimate the acceleration signal in DOF no.5 and results showed high accuracy between the measured and the estimated signals.

**Keywords:** Operational Modal Analysis, Cantilever beam, Modal Expansion, Response Prediction

## 1 Introduction

Continuous monitoring can be a difficult and expensive task as with the technology available the number of sensors for vibration measurement is limited or the complexity of the system sometimes makes it impossible to always measure vibration in locations with higher risk of failure, particularly when these locations have poor accessibility.

One way to estimate vibration levels in the structure regions of interest is by using finite element method (FEM) models, Qu [1] and Chen [2]. However, these analytical models are based upon assumptions as to the actual structural characteristics and, as such, need to be validated by experimental results. These experimental models are critical to the success of any structural dynamic analysis and contain elements that cannot be obtained analytically (Avitabile [3]).

As noted by Avitabile [3], the concept of model reduction and expansion play a significant role in this aspect of modeling especially in the efficient comparison of the large analytical set of DOF (Degree-Of-Freedom) to the relatively small set of experimental DOF. In addition, these reduction and expansion processes play a significant role in the correlation and updating of analytical models.

In this work, we used a expanded experimental modal model of a cantilever beam, to obtain estimates of vibration amplitudes in a non-instrumented location. The experimental modal matrix was obtained using impact tests and the EFDD (Enhanced Frequency Domain Decomposition) method for the identification of modal parameters.The SEREP technique was used to reduce the beam FE model, thus ensuring the compatibility between the degrees of freedom of the numerical and experimental modal models. The estimation of the unmeasured (smoothed) DOFs was carried out using the Local Correspondence Principle (LC), as defined by Brincker et al. [4]. The expanded modal model was used to estimate acceleration signals in a beam location that were not instrumented.

Results showed high accuracy between the measured acceleration signals and those estimated by the expanded experimental modal model.

# 2 Theoretical Framework

On the following sections a theoretical description of the steps undertaken in this work is presented. First the DOFs are compatibilized between the experimental and numerical models by means of a SEREP reduction, then, the experimental results are smoothed through the local correspondence (LC) principle. The quality assurance criteria used are also described and finally the virtual sensing technique used to estimate the response in unmeasured positions is characterized.

## 2.1 Dynamic Response Prediction

Prediction of the dynamic response at unmeasured locations can be developed by making use of the modal decomposition approach. The acceleration vector, $\ddot{X}(t)$, can be written as a linear combination of the mode shape vectors $\Phi_p$, as shown in Eq. 1 (Iliopoulos et al. [5]).

$$\ddot{X}(t) = \Phi_p \ddot{q}_p(t) \tag{1}$$

Where $\ddot{q}_i(t)$ is the vector of the acceleration modal coordinates for each time instance *t*.

The modal coordinates are calculated by using the pseudo inverse as indicated in Eq. (2), assuming that the number of modes $p$ is less then the number of active DOF's (Iliopoulos et al.[5]):

$$\ddot{\hat{q}}(t) = \left(\Phi_p^T \Phi_p\right)^{-1} \Phi_p^T \ddot{X}_p^{meas}(t) = \Phi_p^{\dagger} \ddot{X}_p^{meas}(t) \tag{2}$$

The acceleration prediction at unmeasured locations,$\ddot{X}_p^{pred}(t)$ can be obtained by including the wanted degrees of freedom in the smoothed version of the experimental modal matrix $\Phi_p$:

$$\ddot{X}_p^{pred}(t) = \Phi_p \ddot{\hat{q}}(t) = \Phi_p \Phi_p^{\dagger} \ddot{X}_p^{meas}(t) \tag{3}$$

## 2.2 Assessment Criteria

The MAC, criteria presented below will be used, for comparison between modal matrices. TRAC, FRAC, MAE and RMSE criteria will be used to evaluate the prediction accuracy of dynamic responses in a given degree of freedom, in time and frequency domains.

### Modal Assurance Criteria (MAC)

It is a simple way to correlate two mode shapes, check its linear dependence, verifying the modal assurance between modes, it is calculated as (Allemang [6]):

$$MAC\left(\Phi_A, \Phi_B\right) = \frac{\left|\Phi_A^T \Phi_B\right|^2}{\left(\Phi_A^T \Phi_A\right)\left(\Phi_B^T \Phi_B\right)} \tag{4}$$

Where $\Phi_A$ e $\Phi_B$ are the mode shapes A e B, respectively. A MAC value close to 1 suggests that the two modes are well correlated, and values close to 0 suggest bad correlated modes. According to Qu [1] the MAC correlation is the first step in the correlation process.

### Time Response Assurance Criterion (TRAC)

The time response assurance criterion (TRAC) is a correlation for one DOF over the time of the predicted signal $\ddot{X}_p^{pred}(t)$ with the measured signal in the time domain $\ddot{X}_p^{meas}(t)$ (Iliopoulos et al. [5]).

$$TRAC = \frac{\left[\ddot{X}_p^{meas}(t)^T \ddot{X}_p^{pred}(t)\right]^2}{\left[\ddot{X}_p^{meas}(t)^T \ddot{X}_p^{meas}(t)\right]\left[\ddot{X}_p^{pred}(t)^T \ddot{X}_p^{pred}(t)\right]} \tag{5}$$

### Frequency Response Assurance Criterion (FRAC)

The frequency response assurance criterion (FRAC) is a correlation for one DOF over all frequencies of the predicted complex frequency domain $\ddot{X}_p^{pred}(f)$ with the measured complex frequency domain signal $\ddot{X}_p^{meas}(f)$ (Iliopoulos et al. [5]).

$$FRAC = \frac{\left[\ddot{X}_p^{meas}(f)^H \ddot{X}_p^{pred}(f)\right]^2}{\left[\ddot{X}_p^{meas}(f)^H \ddot{X}_p^{meas}(f)\right]\left[\ddot{X}_p^{pred}(f)^H \ddot{X}_p^{pred}(f)\right]} \tag{6}$$

**Mean Absolute Error in the time domain ($MAE_{TD}$)**

Is defined as the mean absolute error over $n_s$ data samples in the time domain (Iliopoulos et al. [5]).

$$MAE_{TD} = \frac{1}{n_s}\sum_t \left|\ddot{X}_p^{meas}(t) - \ddot{X}_p^{pred}(t)\right| \tag{7}$$

**Mean Absolute Error in the frequency domain ($MAE_{FD}$)**

Is defined as the mean absolute error over $n_f$ data samples in the frequency domain (Iliopoulos et al. [5]).

$$MAE_{FD} = \frac{1}{n_f}\sum_f \left|\ddot{X}_p^{meas}(f) - \ddot{X}_p^{pred}(f)\right| \tag{8}$$

**Root Mean Squared error($RMSE$)**

The root-mean-square error is a mean of calculating the deviation between two samples, the predicted and measured. It is defines as the square root of the second sample moment of the deviations between the observed and predicted values.

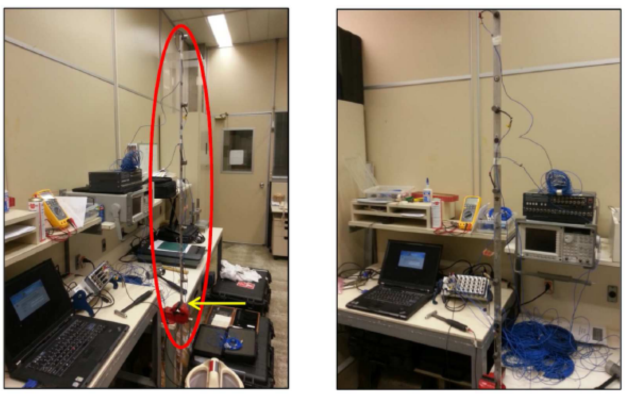
It can be calculated for the signal in the time domain as in Eq. 9 or in the frequency domain as in Eq. 10.

$$\mathrm{RMSE_{TD}} = \sqrt{\frac{\sum_t \left(\ddot{X}_p^{meas}(t) - \ddot{X}_p^{pred}(t)\right)^2}{n_t}} \tag{9}$$

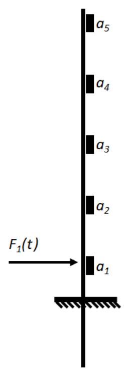
$$\mathrm{RMSE_{FD}} = \sqrt{\frac{\sum_f \left(\ddot{X}_p^{meas}(f) - \ddot{X}_p^{pred}(f)\right)^2}{n_f}} \tag{10}$$

## 3 Experimental Setup

A cantilever aluminum beam, instrumented with accelerometers and used for modal tests. It has a full length of 2,145 m, however, the length of the segment which the accelerometers were positioned is 1,31 m from the clamped point to the top. The beam is 25.42mm wide and has a thickness of 6.17mm.



(a) Rectangular beam in cantilever boundary condition and instrumented with 05 accelerometers.




(b) Rectangular beam configuration($F_i$: force, $a_i$: accelerometer).

Figure 1. Experimental Setup

These 05 accelerometers were positioned so that the distances between two consecutive sensors were equivalent. As the free length of the beam is 1310 mm, the distances between the sensors is 300 mm, being the first accelerometer installed 110mm from the fixation point. A multiple reference impact testing was performed using an instrumented hammer with a steel tip mounted. Fifteen (15) impacts were applied near the clamped point, as shown in Fig. 1a.

# 4 Results and Discussion

Firstly, the output-only modal analysis results will be presented. Next, results for model correlation using the LC method is presented, as well as the improvement in the correlation, using MAC as a criteria. Finally, results of virtual sensing the vibration response at the DOF no. 5, which is at the end of the beam, will be presented.

## 4.1 Modal Parameters Identification

Only vibration responses were used in the Enhanced Frequency Domain Decomposition (EFDD) algorithm. The idea was to simulate practical situations where it is not always possible to measure the excitation forces. Details of this analysis can be found in Busson et al. [7]. Table 1 shows the first five (5) natural frequencies. The corresponding experimental mode shapes can be seen in Fig. 2.

Table 1. Natural Frequencies estimated using EFDD

| Modes | Frequency[Hz] |
|---|---|
| 1 | 2.39 |
| 2 | 14.82 |
| 3 | 41.22 |
| 4 | 81.19 |
| 5 | 136.96 |

## 4.2 Smoothing of the Experimental Modal Matrix

Smoothing of the experimental modes with numerical data will cause the differences between the numerical and experimental DOFs amplitudes to be minimized. This will tend to increase MAC values.

The LC is based on the selection of an optimum base of numerical modal vectors. In this process we have a increasing MAC number for the fitting DOFs as the number of modes increases used are greater, while for the observation DOFs, the MAC number reaches a maximum and then decreases indicating that any number of modes above the one where the maximum occurred yields a correlation that can be considered over fitting. This maximum number of modes is the optimal choice for the correlation.

The fitted (calibrated or smoothed) mode shapes can be seen in Fig.2, plotted along with the non-fitted (numerical) and the experimental mode shapes for comparison purposes. MAC values appearing in the plots are between the fitted and the experimental mode shapes, and, in general, a clear improvement can be seen.

The improvements achieved with the LC methodology in comparison with the results obtained with the non-fitted reduced procedure can be observed in Fig. 3. The MAC results show a clear improvement on the correlation of the LC fitted model in comparison with the non-fitted reduced model.

## 4.3 Dynamic Responses Prediction

As mentioned in chapter 4.1, the beam's modal parameters were identified considering the impacts near the clamped point. Thus, it was possible to identify the first five bending mode shapes, in the range of 0.1 to 140 Hz. For the virtual sensing procedure, the response signals measured due to the impact on DOF 1 was used, as shown schematically in Fig.1b. In addition, dynamic response predictions will be evaluated at DOF 5 and the acceleration signal measured in this DOF will be used as a reference to assess the accuracy of the predictions. Two situations were analyzed: (i)the possibility of obtaining good results if only one accelerometer is used. And, (ii)the influence to the prediction errors if more sensors and mode shapes are used.

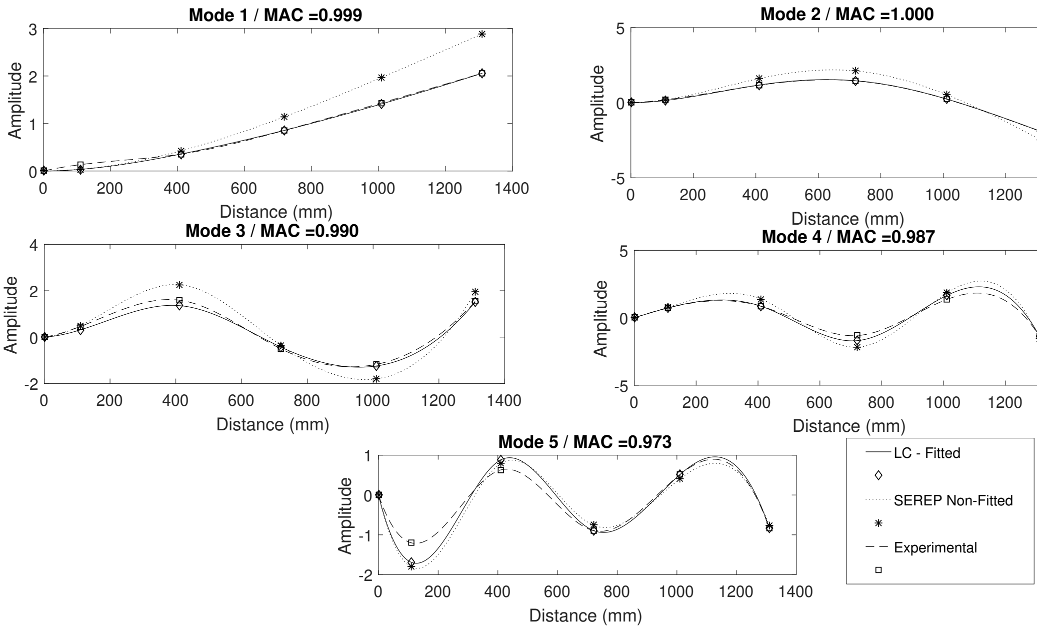



Figure 2. Modal plots for calibrated, non-fitted and experimental mode shapes.

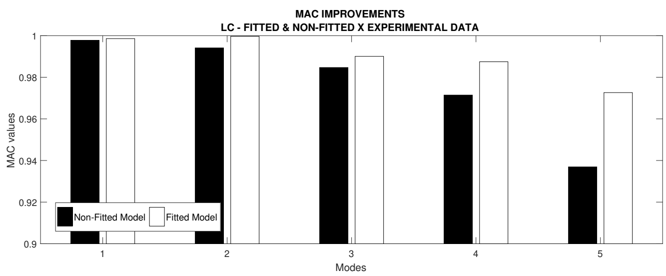



Figure 3. MAC improvements from LC correlation procedure.

### Response Prediction in DOF 5 using the Acceleration Measured in DOF 4

Figure 4 shows the measured and predicted signals in DOF 5, in the time domain. Figure 5 shows the same signals, now in the frequency domain. In this case, only the acceleration measured in DOF 4 ($a_4$) was used for the prediction, and the second to fifth mode shapes ($2^{nd}$,$3^{rd}$,$4^{th}$ and $5^{th}$) were used. Note that both TRAC and FRAC have values above 0.95, and the mean absolute errors (MAE) and the RMSE errors are small, which shows the high accuracy in the prediction.

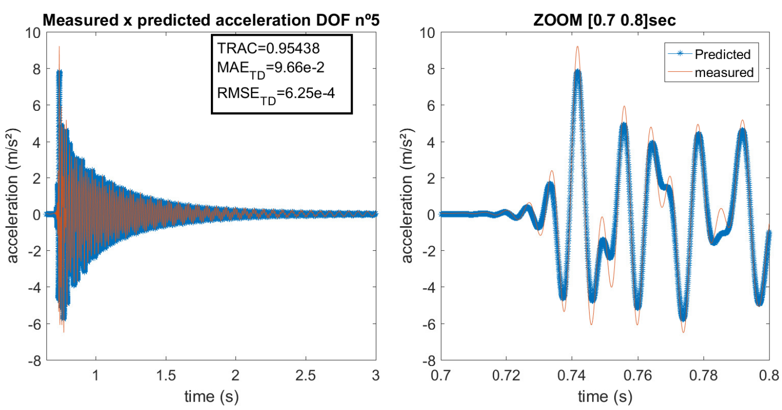



Figure 4. Time domain comparison of predicted and measured accelerations at the fifth DOF.

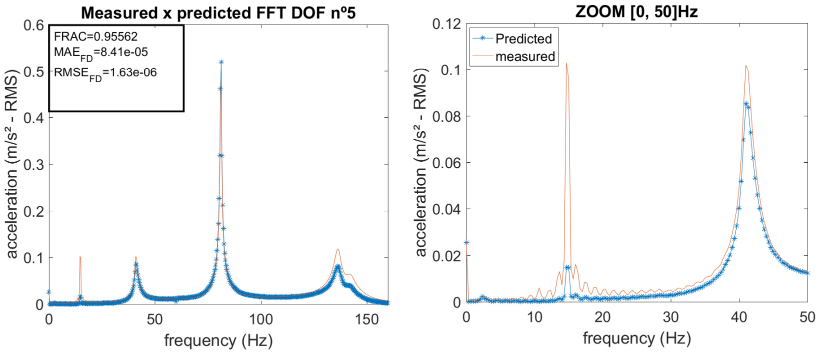



Figure 5. Frequency domain comparison of predicted and measured accelerations at the fifth DOF.

## Response Prediction in DOF 5 using Multiple Acceleration Measurements

Figures 6 and 7 show the measured and predicted signals for DOF 5, in the time and frequency domains. In this case, the method used to choose the number of accelerometers and modes in the prediction was to calculate the correlation for the prediction signals with the measured in all combinations possible with the available accelerometers and modes. Then, the assessment criteria were optimized one by one and the RMSE in the time domain yielded the best result. The combination of accelerometers and modes that minimized the $RMSE_{TD}$ was using 2 accelerometers ($a_1$ and $a_4$) and 2 ($3^{rd}$ and $5^{th}$) mode shapes. Note that the TRAC and FRAC are above 0.98, indicating that both signals have the same shape and, from the amplitude viewpoint, mean absolute (MAE) and the RMSE errors are low, which shows the high accuracy of the predicted signal.

An advantage of the modal expansion method, in predicting responses, is that it only uses the modal matrix and does not depend on natural frequencies and damping ratios, as can be seen in the Eqs. 2 and 3. Other methods, such as the Kalman Filter, need those informations (Maes et al. [8]).

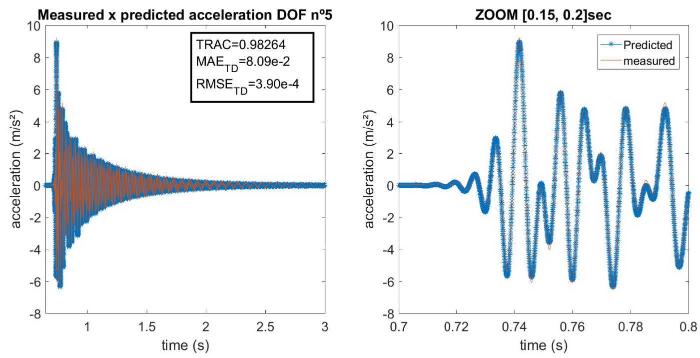



Figure 6. Time domain comparison of predicted and measured accelerations at the fifth DOF.

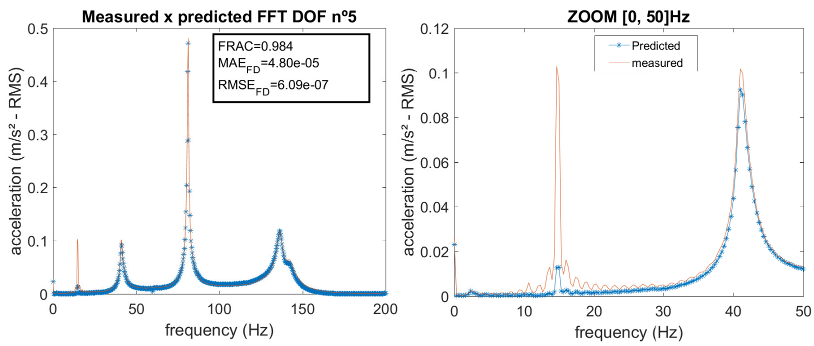



Figure 7. Frequency domain comparison of predicted and measured accelerations at the fifth DOF.

## 5 Conclusions and Recommendations

In this paper, one used a calibrated or smoothed expanded modal model of a cantilever aluminum beam, to estimate acceleration amplitudes in DOF 5.

The numerical modal model, obtained through the FE method, was necessary (i) to serve as a basis for smoothing the experimental modal matrix; (ii) for obtaining a mass matrix to M-orthonormalize the experimental modal matrix (this step is not necessary for response prediction, but it is crucial when predicting forces) and (iii) to serve as a basis to correct the spatial aliasing of higher experimental mode shapes.

The smoothing process, based on the LC principle, showed to be dependent on the choice of fitting DOFs. Therefore, it is considered as part of future work the optimization of the fitting DOFs, for each mode shape to be calibrated.

An advantage of the modal expansion method, in predicting responses, is that it only uses the modal matrix and does not depend on natural frequencies and damping ratios. This method is fast, easy to implement and effective. The virtual sensing, both in the time and frequency domains, showed good agreement with the measured acceleration signals.

The optimization process used to select the accelerometers and modes for the signal prediction resulted in a selection of accelerometers with signals that don't represent all amplitude response, thus resulting in predicted signal that for some frequencies are below the real measurement amplitude, as can be seen in Fig. 7. For future works this could be improved by using a more robust optimization technique to select the attributes for the prediciton process, for example, by using other optimization algorithms or machine learning in the process.

The next step of this research is the extension of this method to predict strains and stresses to evaluate fatigue failure, in locations that are difficult to measure, of structural elements and machines, for example, the ones used in offshore structures, especially those that need to have high reliability.

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