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QUANTIFICATION OF UNCERTAINTIES OF RANDOM DISCRETE VIBRATIONS SYSTEMS USING POLYNOMIAL CHAOS

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Abstract. The quantification of uncertainties consists in exploring how much the uncertainties in initial data can propagate and change the final results. One of the most used methods is the Monte Carlo Simulation Method. Despite it's simplicity and versatility for engineering problems, this method presents high computational costs for complex problems, which might render it inapplicable. The Polynomial Chaos is an efficient alternative for Monte Carlo Simulation Method because of its orthogonal properties and convergence. This work approaches the quantification of uncertainties of discrete dynamic systems using generalized Polynomial Chaos. The discrete system will be a general model of mass-spring-damper and the uncertainties will be related to the initial conditions. This method will be validated by comparison with results obtained by Monte Carlo Simulation.

Keywords: Uncertainty quantification, Vibrations, Polynomial Chaos, Monte Carlo Simulation.

1 Introduction

The cost reduction, work time and tests with prototypes are the main reason of the computational simulations became common in engineering. However, despite the computational advances make more sophisticated and numerically accurate simulations possible with deterministic models, the results don't coincide with experimental data, where not even in identical experiments the results are the same [1]. A lot of these uncertainties are related to material properties, loading and environment, they are inevitable in the engineering. Without considering these uncertainties, the results obtained using deterministic numerical methods may be not reliable [2].

The quantification of uncertainties consists in exploring of how much the uncertainties in initial data can propagate and change the final results. A lot of methods attach the uncertainties to a random variable - these methods are said to be probabilistic [2]. The most famous is the Monte Carlo Simulation (MC) that, however simple and versatile, presents high computational costs in complex problems [2–4].

Polynomial chaos expansion (PC) is one alternative for Monte Carlo simulation. Lots of works are published using this method, like quantification of uncertainties using polynomial chaos and Harmonic Balance method [3, 5]; PC and Gauss Integration [2]; Inverse power methods via PC [4]; The Galerkin method [1, 6] and Askey-Wiener scheme, which consists of obtaining approximate solutions from projections in subspace of finite dimension dense in the space of theoretical solution of the problem. One of these subspace is generated by given polynomials or random variables that belong to a set of orthogonal polynomials known as the Askey-Wiener scheme [7].

This paper approaches a quantification of uncertainties of discrete dynamic systems using PC combined with a reduction of order. The discrete system is a model of mass-spring-damper and its uncertainties will be related to the initial conditions. The results obtained by PC and PC-MC are compared to MC results.

2 Polynomial Chaos Method Applied at Stochastic Discrete Vibrations Systems

The Polynomial Chaos Method is applied into the following stochastic one degree of freedom problem:

$$\begin{cases} \text{Determine } x \in C^2((0,T);L^2(\Omega,\mathscr{F},P)) \text{ such that:} \\ (m\ddot{x}+c\dot{x}+kx)(t,\xi)=F(t)\ \forall(t,0)\in(0,T)\times(\omega,\mathscr{F},P); \\ \text{Subject by:} \\ x(0,\xi)=x_0(\xi); \\ \dot{x}(0,\xi)=v_0(\xi); \end{cases} \tag{1}$$

Where ξ is the random variable, m is the mass, k is the stiffness, c is the damper coefficient and $F(t)$ is the load. The approximate numerical solution is wanted in the following form:

$$x(t,\xi)=\sum_{i=1}^{n} u_i\psi_i(t,\xi); \tag{2}$$

Where u_i are coefficients to determine, ψ_i is the i-th approximation function and n is the size of the stochastic system. Introducing eq. (2) into eq. (1), the residual function $R_n(t,\xi), r_n^{x_0}(\xi), r_n^{v_0}(\xi)$:

$$\begin{cases} R_n(t,\xi)=\sum_{i=1}^{n}(\ddot{u}_i^n+\left(\frac{c}{m}\right)\dot{u}_i^n+\left(\frac{k}{m}\right)u_i^n)(t))\psi_i(\xi)-F(t),\ \forall t\in(0,T) \text{ and } \xi\in(\Omega,\mathscr{F},P); \\ r_n^{x_0}(\xi)=\sum_{i=1}^{n} u_i^n\psi_i(\xi)-x_0(\xi); \\ r_n^{v_0}(\xi)=\sum_{i=1}^{n} \dot{u}_i^n\psi_i(\xi)-v_0(\xi). \end{cases} \tag{3}$$

From that, the minimization of the residual projection at the polynomial chaos subspace:

$$\begin{cases} \langle R_n^t,\psi_j\rangle_{L^2(\Omega,\mathscr{F},P))}=0,\ \forall\psi_j\in span[\{\psi_j\}_{i=1}^n]; \\ \langle r_n^{x_0},\psi_j\rangle_{L^2(\Omega,\mathscr{F},P))}=0; \\ \langle r_n^{v_0},\psi_j\rangle_{L^2(\Omega,\mathscr{F},P))}=0; \end{cases} \tag{4}$$

Where $\langle .,. \rangle$ is the inner product. For $n=2$, you have the following Initial Problem Value:

$$\begin{cases} \text{Determine } (u_1^{(2)},u_2^{(2)})\in C^{(2)}((0,T;\Re^{(2)}) \text{ such that:} \\ \left(\ddot{u}_1^{(2)}+\frac{c}{m}\dot{u}_1^{(2)}+\frac{k}{m}u_1^{(2)}\right)(t)=F(t); \\ \left(\ddot{u}_2^{(2)}+\frac{c}{m}\dot{u}_2^{(2)}+\frac{k}{m}u_2^{(2)}\right)(t)=0\ \forall t\in(0,T); \\ \text{Subject by:} \\ u_1^{(2)}(0)=\langle x_0,\psi_1\rangle_{L^2}; \\ u_2^{(2)}(0)=\langle x_0,\psi_2\rangle_{L^2}; \\ \dot{u}_1^{(2)}(0)=\langle v_0,\psi_1\rangle_{L^2}; \\ \dot{u}_2^{(2)}(0)=\langle v_0,\psi_2\rangle_{L^2}. \end{cases} \tag{5}$$

Is defined a order reduction like it follows:

$$\begin{cases} u_1^{(2)}(t)=u_1(t); \\ u_2^{(2)}(t)=u_2(t); \\ \dot{u}_1^{(2)}(t)=\dot{u}_1(t)=u_3(t); \\ \dot{u}_2^{(2)}(t)=\dot{u}_2(t)=u_4(t); \\ \ddot{u}_1^{(2)}(t)=\dot{u}_3(t); \\ \ddot{u}_2^{(2)}(t)=\dot{u}_4(t). \end{cases} \tag{6}$$

With the relations above and setting $F(t)=0$, the eq. (5) can be written as:

$$\begin{Bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{u}_3(t) \\ \dot{u}_4(t) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m} & 0 & -\frac{c}{m} & 0 \\ 0 & -\frac{k}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{Bmatrix}; \tag{7}$$

Or in a vector-matrix form:

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t). \tag{8}$$

The solution of eq. (8) is given by:

$$\mathbf{u}(t) = e^{\mathbf{A}t}\mathbf{u_0}; \tag{9}$$

Where:

$$\mathbf{u_0} = \begin{Bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ u_4(0) \end{Bmatrix} = \begin{Bmatrix} \langle x_0, \psi_1 \rangle_{L^2} \\ \langle x_0, \psi_2 \rangle_{L^2} \\ \langle v_0, \psi_1 \rangle_{L^2} \\ \langle v_0, \psi_2 \rangle_{L^2}. \end{Bmatrix}. \tag{10}$$

The mean μ and variance σ can be determined by:

$$\begin{cases} \mu = \mathbb{E}\big[u^{(2)}(t)\big]; \\ \sigma^2 = \mathbb{E}\big[(u^{(2)} - \mu)^2(t)\big]; \end{cases} \tag{11}$$

From eq. (11), the mean for displacement μ_x and velocity μ_v are given by:

$$\begin{cases} \mu_x = (e^{At})_{11}\langle x_0, \psi_1\rangle_{L^2} + (e^{At})_{12}\langle x_0, \psi_2,\rangle_{L^2} + (e^{At})_{13}\langle v_0, \psi_1\rangle_{L^2} + (e^{At})_{14}\langle v_0, \psi_2\rangle_{L^2} \\ \mu_v = (e^{At})_{31}\langle x_0, \psi_1\rangle_{L^2} + (e^{At})_{32}\langle x_0, \psi_2\rangle_{L^2} + (e^{At})_{33}\langle v_0, \psi_1\rangle_{L^2} + (e^{At})_{34}\langle v_0, \psi_2\rangle_{L^2}, \end{cases} \tag{12}$$

and the variance for displacement σ_x^2 and velocity σ_v^2 are given by:

$$\begin{cases} \sigma_x^2 = \big[(e^{At})_{21}\langle x_0, \psi_1\rangle_{L^2} + (e^{At})_{22}\langle x_0, \psi_2\rangle_{L^2} + (e^{At})_{23}\langle v_0, \psi_1\rangle_{L^2} + (e^{At})_{24}\langle v_0, \psi_2\rangle_{L^2}\big]^2, \\ \sigma_v^2 = \big[(e^{At})_{41}\langle x_0, \psi_1\rangle_{L^2} + (e^{At})_{42}\langle x_0, \psi_2\rangle_{L^2} + (e^{At})_{43}\langle v_0, \psi_1\rangle_{L^2} + (e^{At})_{44}\langle v_0, \psi_2\rangle_{L^2}\big]^2. \end{cases} \tag{13}$$

3 Methodology Validation

The validation of the application of Polynomial Chaos is made by comparison with results obtained by Monte Carlo Simulation. The case study is a one degree of freedom mass-spring-damper system, with no loads and uncertainties at initial conditions $x_0(\xi)$ and $v_0(\xi)$. The main objective is to calculate the mean and variance of displacement and velocity, where the PC is implemented by two approaches. The first one is using eq. (12) and eq. (13). The second one is combined PC with MC, where samples are generated using eq. (9) and then the mean and variance are determined. The relative errors between MC/PC and MC/PC-MC are determined, as the computational times for MC/PC-MC. Since the random variable ξ follows up a uniform distribution with $\xi \in [-1,1]$, the optimal polynomial type is the Legendre polynomials [8]. The simulations were made in Matlab R2018a, running into a computer with operational system Windows X Professional 64 bits, stored into a Solid State Drive (SSD); processor Intel i3 2.0 GHz and 12 Gb of RAM.

4 Results

Setting $m = 300\,kg$; $c = 2064.86\,Ns/m$; $k = 9869.6\,N/m$; $x_0 = 0.1\,m$; $v_0 = 0.05\,m/s$; $t \in [0,2]\,s$; the computational times for MC for 5000 and 10000 samples are, respectively, 891.62 s and 1806.4 s. The PC-MC was slower with 941.85 s for 5000 samples and 1962.1 s for 10000 samples. The reason for these results might be the fact the model solved by MC (that also uses the order reduction and the exponential solution) has only the variables $x_1(t,\xi)$ and $x_2(t,\xi)$ - displacement and velocity, respectively- to be determined. PC-MC has $u_1(t,\xi)$, $u_2(t,\xi)$, $u_3(t,\xi)$ and $u_4(t,\xi)$ to be solved and provide approximate solutions for the displacement and velocity.

The relative errors between MC and PC-MC for mean and variance, in a 10000 samples, are shown in Fig. 1 and Fig. 2.

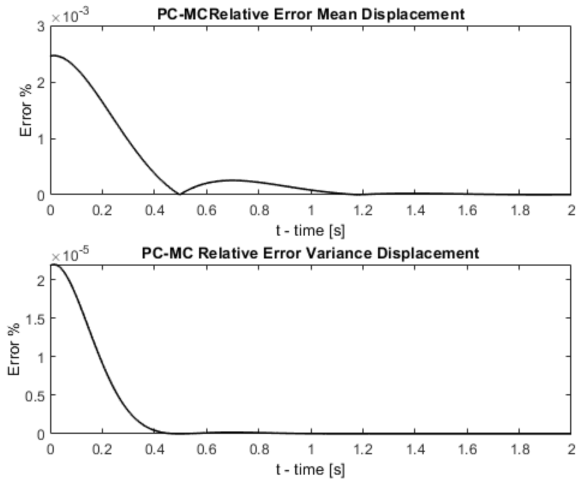

Figure 1. Displacement relative errors between MC and PC-MC

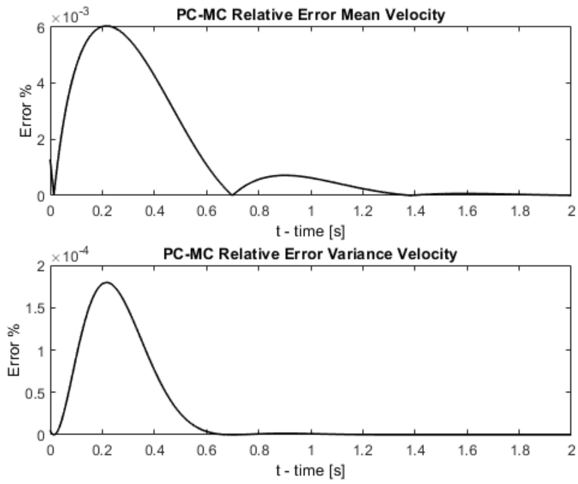

Figure 2. Velocity relative errors between MC and PC-MC

Despite the slower computational times, the relatives errors between PC-MC and MC are less than 1%. The relative errors for PC also are less than 1% as shown in Fig. 3 and Fig. 4.

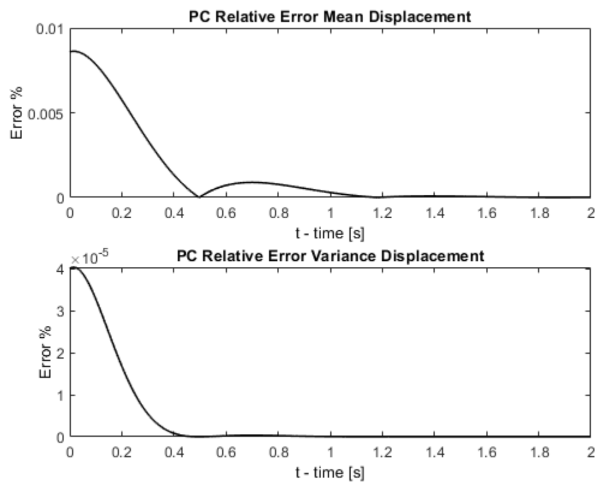

Figure 3. Displacement relative errors between MC and PC

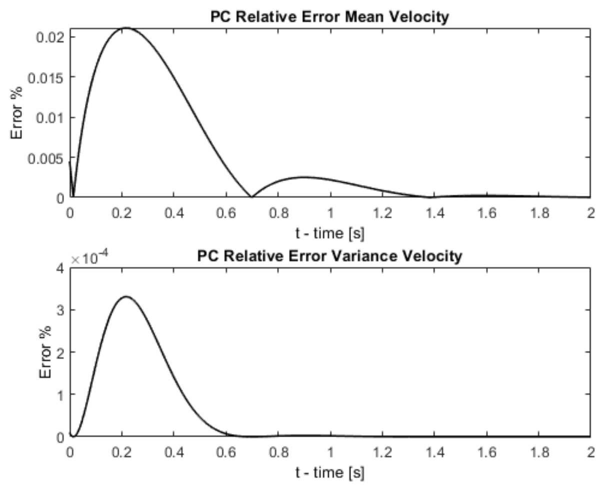

Figure 4. Velocity relative errors between MC and PC

5 Conclusions

This paper is a part of a research project about applications of the Faedo-Galerkin method and the Askey-Wiener scheme for quantification of uncertainties for vibration in structural systems. In this particular work, a mathematical modeling is proposed for the quantification of uncertainties of a discrete dynamics system using Polynomial Chaos Method (PC) and a reduction order to a one degree of freedom mass-spring-damper system. The Monte Carlo Simulation (MC) was used as benchmark for the mean and variance obtained by PC and PC-MC. The results showed that despite being a good approximation, the PC-MC was slower than MC. Additionally, the direct determination of the mean and variance by PC may not be as simple and versatile for certain problems as MC is, but is shown to be generally a good alternative, since it doesn't require a sample generation.

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