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A unified framework for local sensitivity analysis of probability of failure

André Jacomel Torii[1], Leandro Fleck Fadel Miguel[2], Rafael Holdorf Lopez[2]

[1]*Programa de Pós-Graduação em Engenharia Civil (PPGECI), Universidade Federal da Integração-Latino Americana (UNILA)*
Parque Tecnológico Itaipu, Av. Tancredo Neves 6731, 85.867-900, Foz do Iguaçu, Brasil
andre.torii@unila.edu.br
[2]*Departamento de Engenharia Civil, Universidade Federal de Santa Catarina (UFSC)*
Rua João Pio Duarte, s/n, 88.040-900 Florianópolis, Brasil
leandro.miguel@ufsc.br, rafael.holdorf@ufsc.br

Abstract. In this work we present a unified framework for sensitivity analysis of probability of failure with respect to design variables (i.e. local sensitivity). Local sensitivity analysis is generally classified in two cases. In first case, the design variables do not affect the probability distribution. In the second case, the design variables do affect the probability distribution (e.g. expected values). Equations for sensitivity analysis for each one of these two cases were developed in the past. However, in this work we demonstrate that the first case (i.e design variables that do not affect the probability distribution) can be viewed as a limit of the second case (i.e. design variables that affect the probability distribution). This unifies the theoretical background of the problem and leads to a single set of equations for both cases. This development is important from the theoretical point of view, since it allows an easier mathematical analysis of the problem. From the computational point of view, it demonstrates how routines developed to address the second case (i.e. design variables that affect the probability distribution) can be adapted to address the first case (i.e design variables that do not affect the probability distribution).

Keywords: probability of failure, sensitivity analysis, unified framework

1 Introduction

In local sensitivity analysis of the probability of failure one is interested in evaluation of the derivative of the probability of failure with respect to design parameters. This problem is important in practice because it can be employed to guide design procedures based on reliability and safety. Local sensitivity analysis can be broadly classified in two cases:

- Case I: the design parameter affects the distribution of the random variables, but does not affect the limit state function;
- Case II: the design parameter affects the limit state function, but does not affect the distribution of the random variables.

Sensitivity in Case I has been extensively studied in the past and can be evaluated with the score function method [1–3]. Case II, on the other hand, was more recently addressed and can be evaluated with a weak derivative approach [4–6].

Since Cases I and II are conceptually different, the resulting methods and expressions employed for each are also different. Consequently, general computer codes for sensitivity analysis require a set of routines for Case I and another set of routines for Case II. However, this leads to some difficulties for the programmer, because a computer code of larger complexity must be maintained.

However, in this work we demonstrate that Case II can be written as an asymptotic situation of Case I. This puts both cases into a unified framework and allows existing routines for Case I to be adapted for Case II. In other words, it is demonstrated that the score function method can be employed to evaluated derivatives for Case II, that are generally treated with the weak derivative approach. This development is also important from the theoretical point of view, since it allows a unified mathematical treatment of the problem.

In the next section we present an overview of local sensitivity analysis of probability of failure. In Section 3 a computational approach is presented. A numerical example is discussed in Section 4. The conclusions of this work are presented in Section 5.

2 Sensitivity of probability of failure

In this section we present an overview of sensitivity analysis of probability of failure. It is comprised by the developments presented in [4], [5], [1], [2] and [6]. Assume that $g(\rho, X) < 0$ indicates failure of the system under study, where $\rho \in \mathbb{R}$ is a design parameter and X is a random variable. The probability of failure can be defined as

$$P_f = \int_{-\infty}^{+\infty} I(g(\rho, x)) f_X(x) dx = E[I(g(\rho, X))], \tag{1}$$

where f_X is the probability density function (PDF) of the random variable X, $E[\cdot]$ represents the expected value and I is the indicator function

$$I(t) = \begin{cases} 0, & t \geq 0 \\ 1, & t < 0 \end{cases}. \tag{2}$$

We then wish to evaluate the sensitivity of P_f according to design parameter ρ, i.e. the derivatives $dP_f/d\rho$. In Case I we have $dg/d\rho = 0$ and thus direct differentiation of Eq. (1) gives

$$\frac{dP_f}{d\rho} = \int_{-\infty}^{+\infty} I(g(\rho, x)) \frac{df_X(x)}{d\rho} dx. \tag{3}$$

The above expression is generally rewritten as

$$\begin{aligned} \frac{dP_f}{d\rho} &= \int_{-\infty}^{+\infty} I(g(\rho, x)) s(\rho, x) f_X(x) dx \\ &= E\left[I(g(\rho, X)) s(\rho, X)\right] \end{aligned} \tag{4}$$

where

$$s(\rho, x) = \frac{1}{f_X(x)} \frac{df_X(x)}{d\rho} \tag{5}$$

is known in literature as score function. This case was extensively studied in the past and is discussed in details by [1–3].

In Case II, the main difficulty is that I is discontinuous, and thus its derivative must be evaluated in the sense of distributions. One approach to obtain the desired result is to approximate it by a continuous function that, in the limit, converges to I [4, 5]. Here we take the approximation

$$I_h(t) = \begin{cases} \frac{1}{2} + \frac{t}{2h}, & -h \leq t \leq h \\ 0, & t < -h, t > h \end{cases}. \tag{6}$$

Note that the approximation I_h converges to the indicator function I from Eq. (2) for $h \to 0$, in the sense of distributions. Also note that other approximations for the indicator function can be employed without changing the results that follow. In this work we employ the above approximation because it has a simple mathematical structure. The failure probability can then be written as

$$\begin{aligned} P_f &= \lim_{h \to 0} \int_{-\infty}^{+\infty} I_h(g(\rho, x)) f_X(x) dx \\ &= \lim_{h \to 0} E[I_h(g(\rho, X))]. \end{aligned} \tag{7}$$

Assuming $df_X/d\rho = 0$ (i.e. the design parameter does not affect the distribution), application of the chain rule to the above expression gives

$$\frac{dP_f}{d\rho} = -\lim_{h \to 0} \int_{-\infty}^{+\infty} \psi_h(g(\rho, x)) \frac{dg(\rho, x)}{d\rho} f_X(x) dx, \tag{8}$$

where $\psi_h(t)$ is the weak derivative of $I_h(t)$, given by

$$\psi_h(t) = \begin{cases} \frac{1}{2h}, & -h \leq t \leq h \\ 0, & t < -h, t > h \end{cases}. \tag{9}$$

Note that Dirac's Delta $\delta(t)$ satisfies

$$\int_{-\infty}^{\infty} \delta(t)dt = \lim_{h \to 0} \int_{-\infty}^{\infty} \psi_h(t)dt. \tag{10}$$

We thus have

$$\begin{aligned} \frac{dP_f}{d\rho} &= -\int_{-\infty}^{+\infty} \delta(g(\rho, x)) \frac{dg(\rho, x)}{d\rho} f_X(x)dx \\ &= -E\left[\delta(g(\rho, X)) \frac{dg(\rho, X)}{d\rho}\right]. \end{aligned} \tag{11}$$

This case was more recently addressed and it is discussed in details by [4–6].

Since Dirac's Delta cannot be reproduced numerically, the above expression is generally approximated by

$$\frac{dP_f}{d\rho} \approx -E\left[\psi_h(g(\rho, X)) \frac{dg(\rho, X)}{d\rho}\right], \tag{12}$$

where a small value for h is taken.

In practice, local sensitivity analysis of probability of failure generally involves evaluation of Eq. (4) or Eq. (12) depending on how the design parameters ρ affects the problem. However, this leads to computational difficulties, since two different sets of computational routines must be implemented in order to address Case I and Case II. In the next section we demonstrate that the two cases can be written in the same unified framework, based on the score function method.

3 A unified framework

In this section we demonstrate that Eq. (11) can be written as a limit case of Eq. (4). In other words, Case II can be treated as an asymptotic situation of Case I. This unifies both cases into a unique framework based on the score function method (i.e. Case I).

Suppose that ρ is a deterministic design parameter that do not affect the distribution f_X (i.e. Case II). This design parameter can be approximated by a random variable Y with very small variance. We then write the augmented random vector

$$\overline{X} = \{Y, X\}, \tag{13}$$

with joint PDF

$$f_{\overline{X}}(\overline{x}) = f_X(x)\varphi(y, \rho, h), \tag{14}$$

where $f_X(x)$ is the PDF of the original random vector and $\varphi(y, \rho, h)$ is the PDF of the random variable Y, employed to approximate ρ. For now we take Y with Uniform distribution and

$$\varphi(y, \rho, h) = \begin{cases} \frac{1}{2h}, & \rho - h \leq y \leq \rho + h \\ 0, & y < \rho - h, y > \rho + h \end{cases}, \tag{15}$$

that is the uniform Probability Density Function (PDF) with mean ρ and support $[\rho - h, \rho + h]$.

The probability of failure can then be approximated by

$$\overline{P}_f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(g(y, x)) f_X(x)\varphi(y, \rho, h)dxdy. \tag{16}$$

The derivative of Eq. (16) with respect to ρ gives

$$\frac{d\overline{P}_f}{d\rho} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(g(y, x)) f_X(x) \frac{d\varphi(y, \rho, h)}{d\rho} dxdy. \tag{17}$$

The derivative of the uniform PDF from Eq. (15) with respect to ρ, in the sense of distributions, is given by

$$\frac{d\varphi(y,\rho,h)}{d\rho} = \frac{1}{2h}\left[\delta(y-(\rho+h)) - \delta(y-(\rho-h))\right], \tag{18}$$

that represents singular pulses of size $1/2h$ at the bounds of the distribution. Substitution of this result into Eq. (17) gives

$$\frac{d\overline{P}_f}{d\rho} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(g(y,x))f_X(x)\frac{1}{2h}\left[\delta(y-(\rho+h)) - \delta(y-(\rho-h))\right]dxdy. \tag{19}$$

By integrating with respect to y and invoking the properties of Dirac's Delta we get

$$\frac{d\overline{P}_f}{d\rho} = \int_{-\infty}^{+\infty} \frac{\left[I(g(\rho+h,x)) - I(g(\rho-h,x))\right]}{2h} f_X(x)dx, \tag{20}$$

that is clearly a central difference formula for the derivative of $I(g(\rho,x))$ with respect to ρ. For $h \to 0$ we then get

$$\lim_{h\to 0}\frac{d\overline{P}_f}{d\rho} = -\int_{-\infty}^{+\infty}\frac{dI(g(\rho,x))}{d\rho}f_X(x) = \frac{dP_f}{d\rho}. \tag{21}$$

This results demonstrates that a design parameter that do not affect the random variable can be approximated by a random variable with very small variance for the sake of sensitivity analysis. From the computational point of view, this allow routines developed to evaluate Eq. (4) to be adapted to evaluate Eq. (11) approximately, by including a random variable Y with very small variance to represent the deterministic design parameter ρ.

It also important to emphasize that the results of this work were obtained for Y with uniform distribution, in order to simplify the mathematical treatment of the problem. However, the same results can be obtained for Y with other distributions as long as

$$E[Y] = \rho \tag{22}$$

and

$$Var[Y] \to 0, \tag{23}$$

as can be verified by the reader. In other words, the results above hold provided the expected value of the random variable Y is the value of the design parameter ρ and the variance of Y is reduced. For this reason, here we employ Y with Normal distribution for computational purposes.

4 Computational approach

In the previous section we demonstrated that design parameters that do not affect the distribution f_X can be represented as random variables with small variance for the sake of approximate sensitivity analysis. This allows evaluation of probability of failure sensitivity with the score function method for Case II, thus unifying the framework of local sensitivity analysis. Consider then the approximation to the probability of failure given by

$$\begin{aligned}\overline{P}_f &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(g(y,x))f_X(x)\phi(y,\rho,h)dxdy \\ &= E\left[I(g(Y,X))\right]\end{aligned}, \tag{24}$$

where $\phi(y,\rho,h)$ is the Normal distribution with expected value ρ and variance h^2. Since the design parameter ρ now only affects ϕ, the approximate derivative can be obtained by direct differentiation and results

$$\frac{d\overline{P}_f}{d\rho} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(g(y,x))f_X(x)\frac{d\phi(y,\rho,h)}{d\rho}dxdy. \tag{25}$$

The above expression can evaluated with the score function method by writing

$$\frac{d\overline{P}_f}{d\rho} = E\left[I(g(Y,X))s(Y,\rho,h)\right] \tag{26}$$

where the score function results, from the definition of the Normal distribution [1–3],

$$s(y, \rho, h) = \frac{y - \rho}{h^2}. \tag{27}$$

Thus in order to evaluate sensitivity of Case II it is enough to employ the score function method with very small h, i.e. by approximation of the design parameter ρ by a random variable Y with $E[Y] = \rho$ small variance $Var[Y] = h^2$. This allows employment of the score function method for Case II and avoids weak derivative approaches.

5 Numerical Example

Consider the limit state function

$$g(\rho, X) = 1 - \exp(X - \rho), \tag{28}$$

where X has Normal distribution with parameters $\mu = 0$, $\sigma = 1$ and $\rho \in \mathbb{R}$ is the design parameter. Note that the equation $g(x, \rho) = 0$ has unique solution given by $x^* = \rho$. The probability of failure and its sensitivity then result

$$P_f = \Phi(-\rho), \tag{29}$$

$$\frac{dP_f}{d\rho} = -\phi(\rho), \tag{30}$$

where Φ and ϕ are the standard Normal CDF (Cumulative Distribution Function) and PDF, respectively. In this example we take $\rho = 1/2$ and the exact derivative results $dP_f/d\rho = -0.3521$.

The sample size was taken as $N = 10^8$ and the derivative was evaluated with $h = 2.0000$, 1.0000, 0.5000, 0.2500, 0.1250 and 0.0625. A large sample was employed because in this work we are not interested in studying sampling errors, but only the conceptual accuracy of the results obtained in the previous sections. The relative error of the approximate sensitivities were evaluated as

$$e = \left| \frac{d\overline{P}_f/d\rho - dP_f/d\rho}{dP_f/d\rho} \right|. \tag{31}$$

The sensitivities and the relative errors obtained are presented in Table 1. The relation between the parameter h and the relative errors is also presented in Figure 1 in log-log scale. We observe that the proposed approach indeed converges to the exact solution as the parameter h is reduced, as expected.

Table 1. Sensitivities and relative errors obtained with the proposed approach

	$h = 2.0000$	$h = 1.0000$	$h = 0.5000$	$h = 0.2500$	$h = 0.1250$	$h = 0.0625$	Exact
$d\overline{P}_f/d\rho$	-0.1740	-0.2650	-0.3229	-0.3437	-0.3495	-0.3512	-0.3521
Relative error e	0.5057	0.2472	0.0828	0.0237	0.0074	0.0024	-

6 Conclusions

Local sensitivity analysis can be broadly classified in two cases. In this paper it was demonstrated that sensitivity of Case II can be evaluated with Case I, by approximation of the design parameter with a random variable with small variance. The results are also verified in a numerical example. This allows employment of the score function method for evaluation of sensitivities of Case II. Consequently, sensitivity analysis of probability of failure can be cast in a unified framework based only on the score function method from Eq. (4), without need for implementing the weak derivative approach form Eqs. (11) and (12). This approach simplifies computational routines, because only Case I must be implemented in computational routines. Besides, these results also indicate how existing computational routines based on the score function method can be extended to evaluate sensitivities of Case II. The main disadvantage of this approach, however, is that a new random variable must be included in the problem, in order to represent the design parameter.

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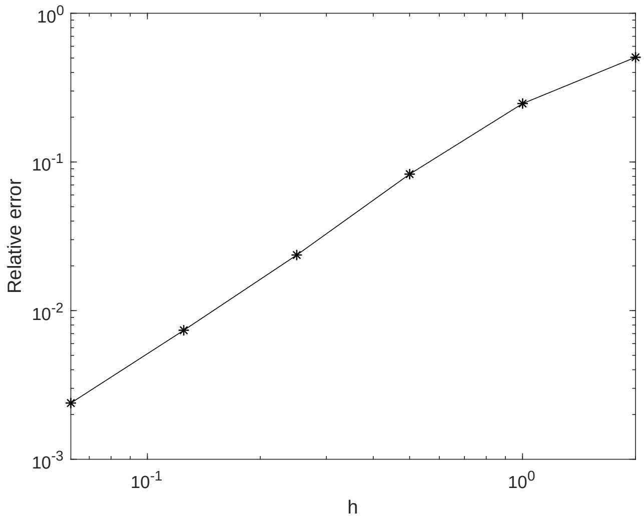

Figure 1. Relation between h and the relative error

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