A new strategy to evaluate the collapse probability and robustness index using Monte Carlo simulation

Aristides P. Orlandi[1], Flávio de S. Barbosa[1], Alexandre A. Cury[1]

[1] *Graduate Program in Civil Engineering at the Federal University of Juiz de Fora*
José Lourenço Kelmer, w/n, São Pedro District, 36036-900, Juiz de Fora, Minas Gerais, Brazil
aristides.orlandi@engenharia.ufjf.br, flavio.barbosa@engenharia.ufjf.br, alexandre.cury@engenharia.ufjf.br

Abstract. This paper proposes a new strategy to evaluate the collapse probability and its respective robustness idex through the construction of an event tree using Monte Carlo simulations. This event tree expresses - in chronological order - the failure that must occur in different structural elements so that the entire building collapses, characterizing a progressive collapse. Moreover, the possibility of elements failing simultaneously is also considered. The results obtained so far are promising since they have achieved all the proposed objectives and provided a method for assessing progressive collapse. The collapse probability of the structure can be obtained directly in the model or through the application of the theoretical relationship between elements (series or parallel), reaching the same results for both cases.

Keywords: Structural reliability, structural robustness, progressive collapse, event tree, Monte Carlo simulation.

1 Introduction

The design of structures with the adoption of normative safety coefficients is not always adequate to guarantee the stability of constructions, especially in cases where unforeseen actions lead to localized failures in supporting elements. As examples, Seck [1] mention the cases of the World Trade Center, in the United States, where aircraft collisions caused the collapse of both towers, and the Ronan Point Building, in England, where an explosion of a gas cylinder caused the failure of an entire section of the building. Thus, the consideration of the random nature of loads is important to design a safe structure.

There are several methods of structural design that consider uncertainties in the construction's calculation and execution processes. The present paper specifically focuses on the use of probabilistic techniques. Those techniques always admit a small probability of failure and often demand high computational efforts. In this sense, the current challenge is to develop a purely probabilistic structural analysis method with acceptable computational processing costs.

2 Structural reliability

Several reliability studies have been developed in recent decades to assess structural safety. Authors such as Land and Doig [2], Hasofer and Lind [3], Thoft-Christensen and Sorensen [4], Thoft-Christensen and Murotsu [5], Nowak and Collins [6], Montgomery and Runger [7] and others contributed with tools or detailed reviews of structural reliability methods.

According to ISO 2394 [8], structural reliability represents the ability of a structure or structural element to fulfill specified requirements during its designed lifespan. It is usually expressed in terms of probability and encompasses the concepts of safety in-service operation and durability.

As Seck [1] pointed out, a reliability study allows calculating the probabilities of a failure mode in a collapse mechanism after the occurrence of an unpredictable event. This calculation process goes through the following steps:

1) select the random variables that integrate the project's uncertainties;
2) define a global dysfunction or failure mode;

3) calculate the reliability index;
4) calculate the probabilities of failure.

Accordingly to Hasofer and Lind [3], within the structural reliability approach, many methods have been used since the early 1970s. Seck [1] proposed that these methods could be categorized according to three levels:

- Level I: based on semi-probabilistic methods, such as the partial safety coefficients method adopted by current standards;
- Level II: represented by numerical methods, such as FORM (first-order reliability method) and SORM (second-order reliability method). Both methods are based on obtaining a reliability index and then calculating the probability of failure of an element using first and second-order derivatives, respectively;
- Level III: represented by observational methods, such as Monte Carlo simulations. These are based on the generation of pseudo-random scenarios and by the random generation of variables, testing each scenario using a limit-state equation. The probability of failure is then obtained by dividing the number of observations that do not achieve equilibrium over the total number of observations. Lemaire [9] pointed out that the results of these simulations are generally considered as a reference for the other methods, being more accurate if one has enough observations.

Simulation methods evolved along with the improvement of computer's processing power and even today the Monte Carlo simulation is considered the most accurate. Its principle is based on making pseudo-random draws of the design variables and testing them in the limit-state equation. From a simple count, the probability of failure is obtained without needing specific mathematical requirements. However, it is computationally expensive, as it requires a large number of observations to achieve convergence. Particular attention must be paid to the possibility of repetitions in the drawing of variables, as this may influence the calculated probabilities. In practice, the Monte Carlo method can use any set of values as inputs for possible scenarios, including data from real tests or mathematical models that represent known distributions for the variables under study.

2.1 Structural systems

Nowak and Collins [6] observed that in structural system analysis, the different elements can be considered in parallel or in series, depending on the presence of structural redundancy or not, respectively. Their functioning depends on the interaction between them. However, the temporal order in which the failure of a system element occurs is also significant. Thus, it is essential to study the concept of progressive collapse where different elements of a system may fail in chronological order or simultaneously. In the latter case, it is called intersection events, as pointed out by Seck [1].

A typical example of a serial system are the links of a chain. The chain is as strong as its weakest link. The probability of failure of this type of system can be calculated using the following equations:

$$P_f = 1 - [1 - P\,(R_1 \le F_1)][1 - P\,(R_2 \le F_2)] \ldots [1 - P\,(R_n \le F_n)] \tag{1}$$

$$P_f = 1 - \prod_{i=1}^{n}[1 - P_{fi}] \tag{2}$$

In Equations 1 and 2, P_f stands for the probability of failure of the system and P_{fi} refers to the probability of failure of the i^{th} element. R_i and F_i are, respectively, the resistance and the action applied to the i^{th} element.

A typical example of parallel systems is a cable composed of several strands of wire. Thus, a system failure occurs only if all elements fail (redundancy). The probability of failure is obtained by directly multiplying the probability of failure of all elements in parallel, as shown in Equations 3 and 4.

$$P_f = P[\,(R_1 < F_1) \cap (R_2 < F_2) \cap \ldots \cap (R_n < F_n)] \tag{3}$$

$$P_f = \prod_{i=1}^{n}[\,P_{fi}] \tag{4}$$

2.2 Quantification of robustness

The concept of robustness comes from the characteristic of "robust". Several authors such as Haberland

[10], Starossek [11], Kagho-Gouadjio [12], Seck [1] and others have similar definitions about the concept of structural robustness. It can be said that robustness is defined by the proportionality of the consequences of structural damage to its cause. Also, according to ISO 2394 [8], structural robustness can be defined as:

"The ability of a structure to withstand adverse and unforeseen events (like fire, explosion, impact) or consequences of human errors without being damaged to an extent disproportionate to the original cause."

The present work uses the formulation presented by Kagho-Gouadjio [12] that allows comparing the local risk to the global risk, expressed in Equation 5:

$$IR = 1 - \frac{P_{global}}{P_{local}} \tag{5}$$

where, IR represents the robustness index, P_{global} represents the collapse probability of a path and P_{local} is the probability of failure of the first element of that path. This approach has the advantage of being easily applied and its result can be simply compared between different models. The index IR compares the probability of local failure with the collapse probability, ranging from 0 to 1, where a result equal to 0 represents a non-robust structure, and a result equal to 1 represents an infinitely robust structure.

2.3 Normative principles

Currently, Brazilian and international standards design structures using deterministic or semi-probabilistic methods. However, concepts of structural reliability are being gradually incorporated into their texts, albeit only qualitatively in some cases. Eurocode [13] is the standard that shows the highest evolution in terms of structural reliability. It differentiates structures into three categories, dividing them according to the consequences in an event of failure. These outcomes are translated in terms of loss of human life, material, and environmental damage. Minimum values are defined considering the lifespan of each structure. For example, it is defined a reliability index $\beta = 3{,}8$ that represents a probability of failure of $7{,}2 \times 10^{-5}$ for an average structure considering a 50 years life span.

3 Case study: fixed-fixed beam

The present work can be considered as a continuation of Seck's [1] work. Thus, a one-dimensional case study developed by that author was initially considered to present the central aspect of the proposed methodology. The objective of this study is to create an event tree to obtain the collapse probabilities and the robustness indexes. The model is a 3 meter-long fixed-fixed beam subjected to a concentrated force F applied at 1/3 of the span. The considered failure mode is the formation of a plastic hinge due to the bending moments in sections A, B, and C. The structural scheme is depicted in Figure 1.

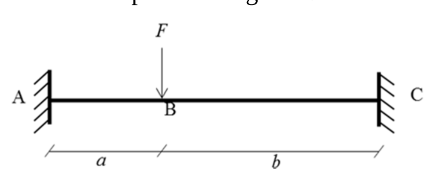

Figure 1 - Fixed-fixed beam, considering a = 1.0 m and b = 2.0 m.

The applied force F and the resistant bending moment of each section are random variables with normal (Gaussian) distribution, defined by their means and standard deviations (see Table 1).

Seck [1] traced all possible paths of structural failure by adding plastic hinges on nodes A, B, and C to create an event tree. Seck's results were determined by using FORM method to obtain probabilities of failure and validating only the initial local failures using a simplified Monte Carlo model.

The event tree was divided into three levels. At the first level, the initial local probabilities of failure were calculated. In the second level, the local probabilities of failure considering one failed section were evaluated. Finally, in the third level, local probabilities of failure were calculated with two failed sections. The failure in the three nodes leads to an unstable configuration (collapse).

Table 1 - Random variables considered.

Random variables			
Variables	Distribution	Mean (μ)	Standard deviation (σ)
MrA	gaussian	1.600 Nm	500 Nm
MrB	gaussian	1.600 Nm	500 Nm
MrC	gaussian	1.600 Nm	500 Nm
F	gaussian	2.800 N	1.000 N

When analyzing Seck's [1] model, for each possible path a collapse probability and a robustness index are calculated. These values are obtained considering several local failures aligned in parallel, demanding that all local failures in a path occur to achieve collapse. The different paths are organized in series, in which any failure path leads to the system's collapse. It is noteworthy that Seck [1] has chosen the random variables respecting the typical coefficient of variation (CoV) of 30% (standard deviation equal to 30% of the mean value). Thus, for the simulation of 10^8 scenarios with Monte Carlo, the resistance variables were found with negative values for 68.610 cases. From an engineering's perspective, these results are equivalent to 0,07% of the entire sampling space, becoming significant when compared to the collapse probabilities of some paths. If a CoV of 20% had been adopted, the number of negative resistance scenarios would drop to 28 in 10^8 scenarios, becoming negligible in the analysis. Therefore, although the present example is didactic, the importance of knowing the random variables and adopting calibrated parameters is crucial.

In order to reproduce Seck's results, Monte Carlo simulations were applied as follows:

1) A set of random data are generated for the first level of the tree;

2) For the second (until the last level), the same logic is applied: new values are generated for the random variables, and the ones from the previous levels are discarded. Then, the collapse probabilities and robustness index are calculated for each path.

Practically identical results were achieved using the FORM method, which approximates the probabilities of failure with less computational cost than Monte Carlo. Both strategies, as applied by Seck's [1], lead to structural analysis imprecisions that may be detected by using a slightly different Monte Carlo simulation as will be explained in the next section.

4 Proposed approach

In order to reproduce Seck's [1] results, one must initially calculate the bending moments due to a unitary force and considering the event tree without any plastic hinge. The results obtained are considered as structural coefficients presented in Figure 2 that will be later multiplied by the random variables.

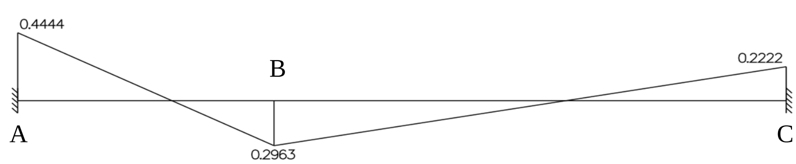

Figure 2 - Diagram of bending moments for unit load without hinges in the structure.

Then, let us suppose that the Monte Carlo simulations drew 3.000 N for the applied force and 1.600 kN.m, 1.500 kN.m and 1.820 kN.m for the resistant moments in each section respectively. Thus, the bending moments acting in sections A, B, and C due to the force F are, respectively:

F_a = 3.000 × 0,4444 = 1333,2 $kN.m$ < MR_a (1.600 $kN.m$)

F_b = 3.000 × 0,2963 = 888,9 $kN.m$ < MR_b (1.500 $kN.m$)

F_c = 3.000 × 0,2222 = 666,6 $kN.m$ < MR_c (1.820 $kN.m$)

It is noted that the proposed scenario does not cause failure in any of the sections.

At this point, let us consider that level 2 of the event tree is calculated with a plastic hinge inserted in section A. Assuming that the algorithm randomly selected the same values proposed for the first hypothetical level, the new structural coefficients for initial failure in A would be those presented in Figure 3.

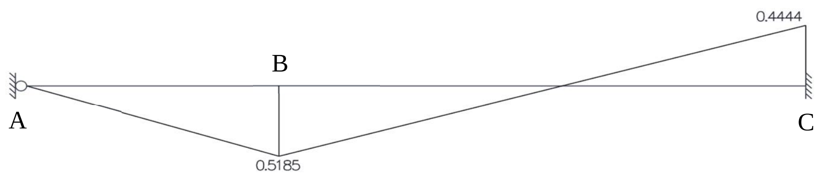

Figure 3 - Diagram of bending moments for unit load with a hinge in section A.

This situation leads to the following bending moments acting in sections B and C due to the force F:

F_b = 3.000 × 0,5185 = 1.555,5 $kN.m$ > MR_b (1,500 $kN.m$)

F_c = 3.000 × 0,4444 = 1.333,2 $kN.m$ < MR_c (1.820 $kN.m$)

In this case, section B fails, but section C does not. The problem with Seck's [1] approach is that for a real structure, the resistant bending moments and acting forces will not change from the first to the second level, as simulated here (at least in a linear case), so it considers an impossible scenario where section A would not have failed at level 1. Consequently, level 2 could not have been reached. In other words, it is not an achievable collapse scenario. Similar scenarios, even if they do not present failure in later stages, would be equally unfavorable to the analysis since they would count in the total number of analyzed scenarios, changing the probability of local failure of each section.

The approach proposed in this paper assembles the event tree by calculating local probabilities of failure at a given level, but only considering the scenarios that caused ruin in the previous steps. This strategy has the advantage of running the Monte Carlo algorithm only once, instead of once for each level, which significantly reduces the processing time. Likewise, the robustness index is now calculated only once for each initial local failure, instead of once for each path, seeking to agree with the robustness definitions in the literature. The results achieved for the complete tree with the proposed approach are presented in Figure 4.

It is observed that the probabilities of local failure at level 1 are equal to those obtained by Seck [1]. These results are expected since the FORM method accurately approximates the probability of local failure. However, there is a significant difference in the results for the rest of the tree, as shown in Table 3. Such divergences arise from the exclusion of the unfeasible scenarios by applying the proposed Monte Carlo approach.

Table 3 - Comparison of results only on the path started by failure in A.

Event	**Probability**	
	Seck (2018)	**Present work**
Failure in A	0,233	0,234
Failure in A > B	0,224	0,273
Falha in A > B > C	0,973	0,998
Global failure	0,051	0,064

For the remaining of the tree, the achieved collapse probabilities are higher than those obtained by Seck [1]. These results are justified and based on the following statements adopted by Seck [1]:

1) The probability of failure of each element after the first level of the event tree was calculated considering all scenarios as possible scenarios, including unachievable ones;

2) The collapse probabilities were calculated by the product of each element's probability of failure in the path, accepting the relationship of parallelism or redundancy between the elements of the same path. In that way, the imprecision described before was propagated to the collapse probability.

The present methodology calculates the probability of failure of each element, considering only the scenarios that cause ruin in the previous steps. Consequently, fewer failure scenarios for each path are achieved, as well as the total of possible scenarios. Besides, the probability of failure is calculated by dividing the number of failure scenarios by the total number of analyzed scenarios. In this case, the number of excluded scenarios is much higher than the amount of eliminated failure scenarios, thus leading to a higher probability of failure.

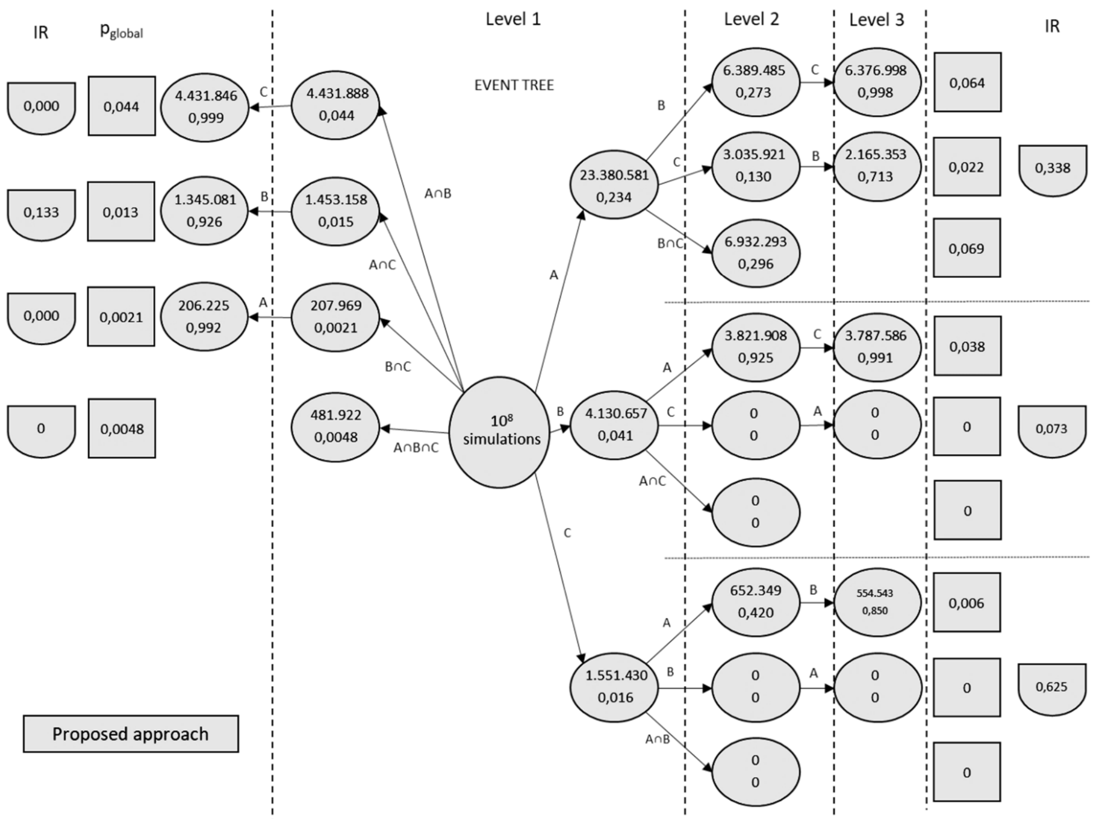

Figure 4 - Event tree resulting from the proposed approach.

Equations 6 show the collapse probability used by the previous authors. In contrast, Equations 7 present the proposed methodology, based on counting, where c_i represents the failure scenarios found, n the total of tested scenarios, c_{i-1} the total failure scenarios from the previous step, and $P_{element}$ the probability of failure of each element.

$$P_{element} = \frac{c_i}{n} \qquad P_{global} = P_1 \times P_2 \times \ldots \times P_i \tag{6}$$

$$P_{element} = \frac{c_i}{c_{i-1}} \qquad P_{global} = \frac{c_3}{n} \tag{7}$$

By analyzing these equations, one observes that even though c_i is higher in Equations 6, n is still much more significant than c_{i-1}, leading to a lower probability of element failure than those achieved in the present paper. As the previous authors calculate the collapse probability by multiplying the probability of each element, the result ends up being smaller than expected, justifying the larger values found in the present work.

As a result of the proposed approach, the collapse probability can be obtained exactly as the previous authors, i.e., multiplying the probabilities of each element from a path. The same collapse probability can also be obtained as taking the number of failure scenarios on the last level of the tree divided by the total number of scenarios from the first level. This result confirms the hypothesis of parallelism between events on the same path and, consequently, the collapse probability obtained in the proposed methodology.

Another important conclusion is that based on the proposed method, it is possible to count all scenarios in all paths that lead to collapse. Thus, the probability that the structure will collapse is obtained regardless of the path that it will occur. In this example, the structure's probability of failure is 26.3% (26.281.847 scenarios in a sampling space of dimension 10^8). The same result can be obtained when considering that the different paths are aligned in series using Equation 2 (thus confirming such theoretical hypothesis).

5 Conclusions

This paper presented a Monte Carlo based strategy to analyze structural reliability and robustness. The proposed strategy proved itself advantageous in the sense that it allowed confirming the hypothesis of parallelism or redundancy between the elements of the same failure path and the serial relationship between different paths. The proposed method also obtained a collapse probability for the entire structure by counting all failure scenarios, regardless of the path taken. The processing time for solving the proposed models was also adequate. Unfortunately, it could not be compared with the bibliography's counterpart, again due to the differences in each approach.

Acknowledgements. The authors would like to gratefully acknowledge UFJF (Universidade Federal de Juiz de Fora), CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, PROCAD 88881.068530/2014-0), CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) – Grants 304329/2019-3, 311576/2018-4-PQ, and FAPEMIG (Fundação de Amparo à Pesquisa do Estado de Minas Gerais) – Grants “PPM-0002-16”, “PPM-00106-17” and “PPM-0001-18” for the financial support.

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