XLI CILAMCE
XLI Ibero-Latin American Congress on Computational Methods in Engineering
16-19 November, 2020 FOZ DO IGUAÇU PARANÁ - BRAZIL

Computation of Axial Forces-Bending Moments Interaction Diagrams for Reinforced Concrete Polygonal Cross Sections

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Abstract. This study aimed to calculate the generalized stress (axial forces and bending moments) and their respective interaction diagrams for reinforced concrete cross sections. Most of the approaches to compute the generalize stress are restricted to particular cases, such as rectangular cross sections and simplified stress-strain diagrams. The adopted methodology allows to calculate the generalized stress for any polygonal cross section, and it is also flexible with respect to the choice of the stress-strain diagram of the materials. The main contribution of the present study is to extend such methodology to calculate not only the ultimate axial forces-bending moments interaction diagram, but also the interaction diagrams defined by the yielding of the reinforcements and cracking of the concrete. Such diagrams are important in applications of non-linear analysis of reinforced concrete elements by the lumped damage mechanics for the definition of yielding functions and evolution laws.

Keywords: Reinforced concrete, Ultimate, yielding and cracking interaction diagrams, Polygonal cross sections.

1 Introduction

The reinforced concrete is one of the most adopted structural materials. It allows the conception of structures with a variety of geometrical forms. It also presents a good performance for generating axial forces and bending moments, which balance the external forces that may appear. Generalized stress (axial forces and bending moments) may occur in the element due to the external forces such as the body forces and surface forces. The generalized stresses are calculated as the integrals of the stresses over the cross-section area of the elements.

Interaction diagrams of axial forces and bending moments may be constructed from strength criteria described in terms of generalized strains. They are used in order to verify the structural safety of the element cross sections. There are several ways to compute the interaction diagrams: Rodriguez & Aristizabal-Ochoa [1]; Sfakianakis [2]; Bonet et al. [3]; Sousa Jr. & Muniz [4]. Many of those works are dedicated only to the study of the reinforced concrete cross section, as was the case in Giuseppe et al. [5], which present the study of hollow circular cross sections. Such cross sections may be applied in posts, tunnels, pipes and hollow columns. Other works, as Rodriguez [6], deals with the computation of the interaction diagrams for cross sections composed of small quadrilateral sub-elements. Mendes Neto [7] and Papanikolau [8] presented their studies for some specific cross section's geometries.

The interaction diagram of ultimate generalized stresses (N_u, M_u) is built from the strength criterion of ultimate strain in reinforced concrete cross section, considering the hypothesis of perfect bond between the concrete and the reinforcement. On the other hand, the interaction diagram of reinforcement yielding generalized stresses (N_y, M_y) is built from one criterion defined in terms of the yielding of any reinforcement bar. This criterion is characterized by the yielding stress and represent a boundary between yielding and non-yielding of the reinforcements. Finally, the interaction diagram of concrete cracking generalized stresses (N_c, M_c) is built from one criterion defined by cracking of any concrete fiber in the cross section. This criterion is characterized by the concrete tensile strength, which dictates a value of normal stresses from which cracks are observed in the concrete.

The ultimate interaction diagram allows the objective verification of the cross section structural safety of linear reinforced concrete elements. On the other hand, the yielding and cracking interaction diagrams allows to obtain yielding functions and evolution damage laws required for the constitutive modelling for nonlinear analysis of reinforced concrete elements by the lumped damage mechanics.

2 Objectives

The main goal of the present study is to extend the methodology for the calculation of axial forces and bending moments developed by Mendes Neto [9] to evaluate the interaction diagrams of ultimate, yielding and cracking generalized stresses for reinforced concrete cross sections under axial forces and bending moments. To achieve this goal, the following steps were performed:

-To develop a computational tool in Fortran 90 to compute the ultimate axial forces and bending moments interaction diagrams for reinforced concrete polygonal cross sections. The program is based on the methodology presented by Mendes Neto [9], in which the integrals over the cross-section's areas are transformed into boundary integrals by the Green´s theorem;

-To define a region in the generalized strain space which characterizes the limit between yielding and non-yielding of the reinforcements. The computation of the axial forces and bending moments for points at the boundary of such a region give rises to the yielding interaction diagram;

-To define a region in the generalized strain space which characterizes the limit between cracking and non-cracking of any concrete fiber. The computation of the axial forces and bending moments for points at the boundary of such a region give rises to the cracking interaction diagram;

-To extend the methodology of computation of axial forces and bending moments for a stress-strain curve that takes into consideration the concrete tensile strength. The computation of axial forces and bending moments with such a stress-strain curve is adopted for the evaluation of the cracking interaction diagram.

-To build the ultimate, yielding and cracking interaction diagrams for reinforced concrete cross sections with available reference solutions in the literature in order to validate the obtained diagrams with comparison with the reference solutions.

3 Axial strains

From the hypothesis of plane cross section remains plane and inextensible in its plane, it is possible to obtain the axial strains $\varepsilon\left(x,y\right)$ for any point in the cross section. In the case of in-plane bending:

$$\varepsilon\left(x,y\right)=\varepsilon\left(y\right)=\varepsilon_0-\kappa_x y\,. \tag{1}$$

where ε_0 is the axial strain at the origin of the xy plane (cross section geometrical center) and κ_x is the cross section curvature. ε_0 and κ_x are also called generalized strains. In the present work the axial strain is given in (‰). Therefore, Eq. (1) must be adjusted (multiplied by a thousand).

4 Stress-strain curves

4.1 Concrete stress-strain curve

In the following, it is presented the stress-strain curves adopted for the concrete. The first one is the curve for the concrete at the ultimate strength, where tensile stresses are disregarded. Such curve is adopted in the computation of the ultimate and yielding interaction diagram. The second curve concerns the non-cracked concrete stress-strain curve. This curve is adopted to compute the cracking interaction diagram. Figure 1 illustrates both curves:

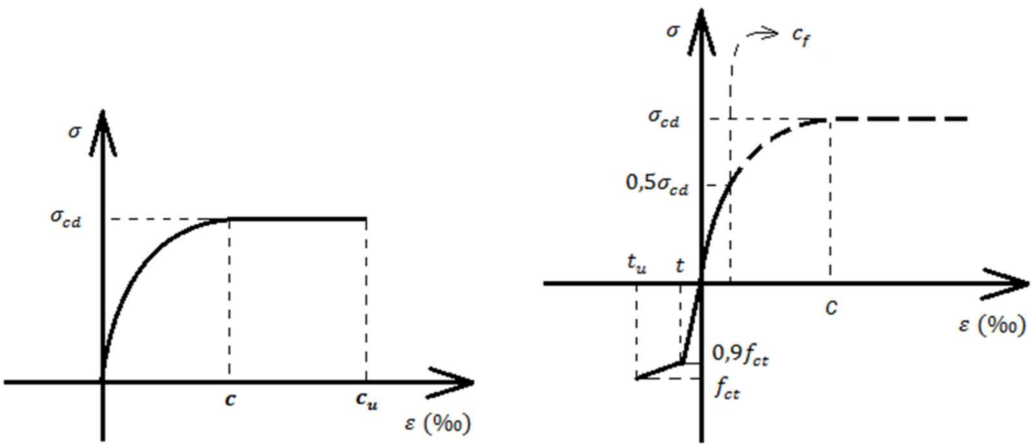


Figure 1. Stress-strain curves for ultimate strength and non-cracked concrete

σ_{cd} is the maximum compression stress admitted for the concrete and f_{ct} is the tensile strength. c and c_u are functions of the concrete characteristic strength f_{ck} and t and t_u are functions of the initial modulus of the non-cracked concrete E_{ci} as described in the Brazilian code NBR-6118/2014.

4.2 Steel stress-strain curve

In the following, it is presented the stress-strain curve adopted for the reinforcement steel. It is a trilinear curve with symmetrically defined yielding levels in both tensile and compression. Such curve was adopted to compute the generalized stress (axial forces and bending moments) in all three interaction diagrams, i.e., ultimate, yielding and cracking interaction diagrams. The steel stress-strain curve is illustrated in Figure 2:

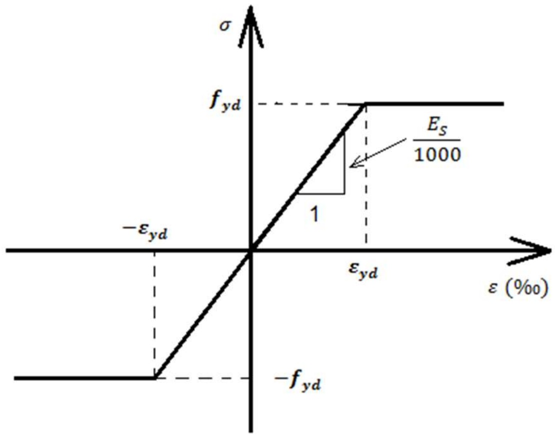


Figure 2. Stress-strain curves for reinforcement steel

E_s is the steel modulus, ε_{yd} is the reinforcement yielding strain and f_{yd} the reinforcement yielding stress.

5 Methodology for the computation of axial forces and bending moments

In this section it is presented the adopted methodology for the computation of the generalized stress (axial forces and bending moments) in reinforced concrete polygonal cross sections. The generalized stress may be decomposed in contributions from the concrete and from the reinforcement:

$$N = \iint_A \sigma(\varepsilon)\, dA = \iint_{A_c} \sigma_c(\varepsilon)\, dA_c + \iint_{A_s} \sigma_s(\varepsilon)\, dA_s = N_c + N_s\,. \tag{2}$$

$$M = -\iint_A \sigma(\varepsilon)\, \mathrm{y} dA = -\iint_{A_c} \sigma_c(\varepsilon)\, \mathrm{y} dA_c - \iint_{A_s} \sigma_s(\varepsilon)\, \mathrm{y} dA_s = M_c + M_s . \tag{3}$$

where N_c, M_c are contributions from the concrete and N_s, M_s contributions from the reinforcement. Here the indexes c and s represents concrete and steel, respectively. Due to the small area occupied by the reinforcement, A_c may be considered as the full area of the cross section and A_s may be considered as pointwise contributions. The origin of the coordinate system is adopted at the geometrical center of the cross section.

5.1 Concrete contributions

The goal of the present topic is to present the concrete contributions N_c and M_c as a function of the generalized strains ε_0, κ_x. In order to do so, consider the infinitesimals $dN_c = \sigma(\varepsilon)dxdy$ and $dM_c = \sigma(\varepsilon)\, ydxdy$ defined for a point in the cross section. Thus, N_c and M_c can be presented as:

$$N_c = \iint_A \sigma_c(\varepsilon)\, dxdy . \tag{4}$$

$$M_c = -\iint_A \sigma_c(\varepsilon)\, \mathrm{ydxdy} . \tag{5}$$

In order to obtain N_c and M_c, the Green´s Theorem is adopted. By following the developments presented in Mendes Neto [9], for non-null cross section curvatures $\kappa_x \neq 0$, the concrete contribution for the axial force results:

$$\begin{aligned} N_c &= \frac{1}{\kappa_x}\sum_{i=1}^{N} \Delta x_i f_{1i} \\ f_{1i} &= \begin{cases} \Delta I_{1i}, & \Delta\varepsilon_i = 0 \\ \dfrac{\Delta I_{2i}}{\Delta\varepsilon_i}, & \Delta\varepsilon_i \neq 0 \end{cases} \end{aligned} \cdot \tag{6}$$

in which:

$$\begin{aligned} \Delta x_i &= x_{i+1} - x_i \\ \Delta\varepsilon_i &= \varepsilon_{i+1} - \varepsilon_i \\ \Delta I_{1i} &= I_1(\varepsilon_{i+1}) - I_1(\varepsilon_i) \\ \Delta I_{2i} &= I_2(\varepsilon_{i+1}) - I_2(\varepsilon_i) \\ I_1(\varepsilon) &= \int_0^{\varepsilon} \sigma(\xi) d\xi \\ I_2(\varepsilon) &= \int_0^{\varepsilon} I_1(\xi) d\xi \end{aligned} . \tag{7}$$

Where x_i and x_{i+1} are the x coordinates at consecutive vertex of the polygonal cross section and ε_i and ε_{i+1} are the axial strains at those vertex. For $\kappa_x = 0$, the problem of integration the stresses over the cross section becomes more simple as the stresses are constant and, thus, can get out of the integrand.

$$N_c = \sigma(\varepsilon_0)\iint_A dxdy = \sigma(\varepsilon_0) A . \tag{8}$$

On the hand, the concrete contribution for the bending moment, for $\kappa_x \neq 0$, results:

$$M_{cx} = -\frac{1}{\kappa_x}\sum_{i=1}^{N}\Delta x_i f_{3i}$$
$$f_{3i} = \begin{cases} I_{1i}\dfrac{y_i + y_{i+1}}{2} + \dfrac{I_{2i}}{\kappa_x}, & \Delta\varepsilon_i = 0 \\ \dfrac{g_i\ \Delta I_{2i} + \Delta y_i\ \Delta K_{1i}}{\Delta\varepsilon_i^{\ 2}} + \dfrac{1}{\kappa_x}\dfrac{\Delta I_{3i}}{\Delta\varepsilon_i}, & \Delta\varepsilon_i \neq 0 \end{cases} . \tag{9}$$

in which

$$\begin{aligned} &\Delta y_i = y_{i+1} - y_i \\ &g_i = y_i\ \varepsilon_{i+1} - y_{i+1}\varepsilon_i \\ &\Delta I_{3i} = I_3\left(\varepsilon_{i+1}\right) - I_3\left(\varepsilon_i\right) \\ &\Delta K_{1i} = K_1\left(\varepsilon_{i+1}\right) - K_1\left(\varepsilon_i\right) \\ &I_3\left(\varepsilon\right) = \int_0^{\varepsilon} I_2\left(\xi\right) d\xi \\ &K_1\left(\varepsilon\right) = \int_0^{\varepsilon} \xi I_1\left(\xi\right) d\xi \end{aligned} . \tag{10}$$

For $\kappa_x = 0$ the stress distribution $\sigma\left(\varepsilon_0\right)$ is constant and can get out of the integrand:

$$M_c = -\sigma\left(\varepsilon_0\right)\iint_A ydxdy = \sigma\left(\varepsilon_0\right) I . \tag{11}$$

The values of the integrals $I_1\left(\varepsilon\right)$, $I_2\left(\varepsilon\right)$, $I_3\left(\varepsilon\right)$ and $K_1\left(\varepsilon\right)$ depends on the adopted stress-strain curve adopted for the concrete (Figure 1).

5.2 Reinforcements contributions

The goal of the this topic is to present the reinforcement contributions N_s and M_s as a function of the generalized strains ε_0, κ_x. In order to do so, consider the infinitesimals $dN_s = \sigma\left(\varepsilon\right)dxdy$ and $dM_s = \sigma\left(\varepsilon\right) ydxdy$ defined for a point in the reinforcement area A_s. Thus, N_s and M_s can be presented as:

$$N_s = \iint_{A_s} \sigma_s\left(\varepsilon\right) dxdy . \tag{12}$$

$$M_s = -\iint_{A_s} \sigma_s\left(\varepsilon\right) \mathrm{ydxdy} . \tag{13}$$

where A_s means the total aera of the reinforcement, i.e., the summation of the area of all bas. Once the areas of the reinforcement bars are much smaller in comparison with the area of the cross section, it is possible to consider the reinforcement contributions in a discrete manner. Thus, the integrals become summations.

Accounting for the areas A_{si} of the $i = 1 \ldots n$ reinforcement bars it is possible to obtain the reinforcement´s contribution for the axial force:

$$N_s = \sum_{i=1}^{n} \sigma_s\left(\varepsilon_i\right) A_{si} . \tag{14}$$

Considering that each bar A_{si}, $i = 1 \ldots n$, is located at a distance y_{si} from the origin of the coordinate system, it is possible to obtain the reinforcement´s contribution for the bending moment:

$$M_s = -\sum_{i=1}^{n} \sigma_s\left(\varepsilon_i\right) y_{si} A_{si} \;. \tag{15}$$

6 Strain envelopes

The strain envelopes characterize a region in the space of the generalized strains $\kappa_x \times \varepsilon_0$ in which the axial strains respect some criteria. The ultimate strain envelop is defined by the criteria of failure and non-failure of the reinforced concrete cross section. The yielding envelop is defined by the criteria of yielding and non-yielding of the reinforcements. Finally, the cracking envelop is defined by the criteria of cracking and non-cracking of the concrete fibers. The envelopes are constructed by comparing the axial strains at the cross-section fibers, Eq. (1), with the established criteria.

The ultimate envelope is illustrated in Figure 3a. According to the classical reinforced concrete theory, the ultimate envelop is characterized by three poles of failure I, II and III, which results in six straight edges in the space $\kappa_x \times \varepsilon_0$, three for $\kappa_x \geq 0$ and three for $\kappa_x \leq 0$. The yielding envelope is illustrated in Figure 3b. It is characterized by the criteria of yielding and non-yielding of the reinforcement bars. According to the classical theory of plasticity, the yielding occur when $\|\varepsilon\| \geq \varepsilon_{yd}$ .Therefore, this criteria results in four straight edges in the space $\kappa_x \times \varepsilon_0$, two for $\kappa_x \geq 0$ and two for $\kappa_x \leq 0$. The cracking envelope is illustrated in Figure 3c. It is characterized by the criteria of cracking and non-cracking of the concrete fibers. According to the classical theory, brittle materials such as the concrete, cracks when $\sigma = -f_{ct}$. Considering in compression $\varepsilon_{elastic} = \varepsilon\left(0.5\sigma_{cd}\right) = c_f$ and in traction $\varepsilon_{elastic} = \varepsilon\left(-f_{ct}\right) = -t_u$, this criteria results in four straight edges in the space $\kappa_x \times \varepsilon_0$, two for $\kappa_x \geq 0$ and two for $\kappa_x \leq 0$.

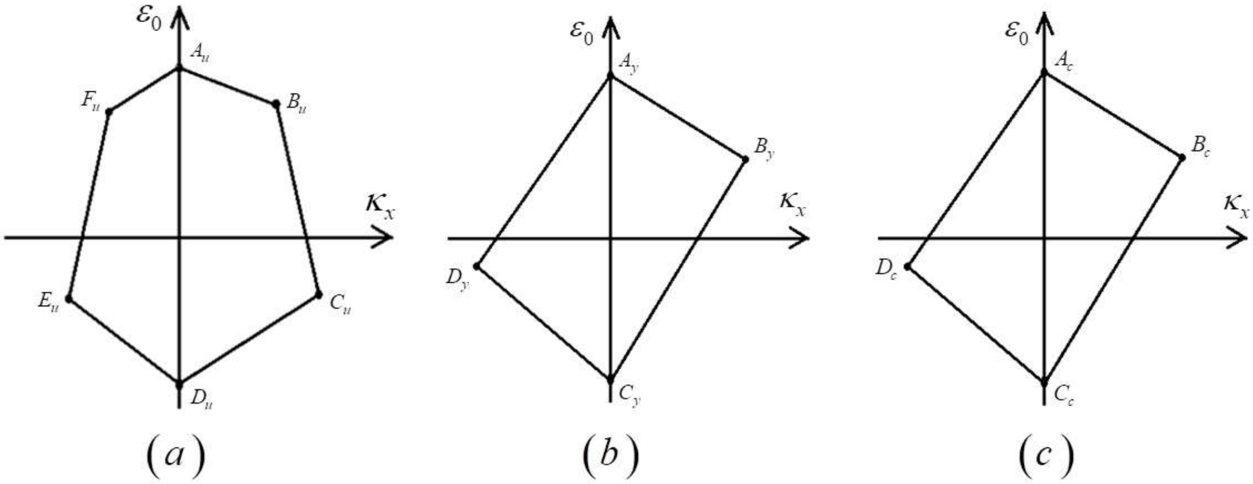


Figure 3. Strain envelopes: (a) Ultimate envelope (b), Yielding envelope, (c) Cracking envelope

7 Results

One example of a cellular reinforced concrete cross-section under bending moment and axial force is analyzed. The results of ultimate, yielding and cracking interaction diagrams were compared with results available in the literature.

7.1 Interactions diagrams for a cellular cross section

Th**is** example deals with a cellular cross-section with dimensions $b = 250$ cm x $\mathrm{h} = 140\mathrm{cm}$ and with square holes of sides 80cm as illustrated in Figure 4.

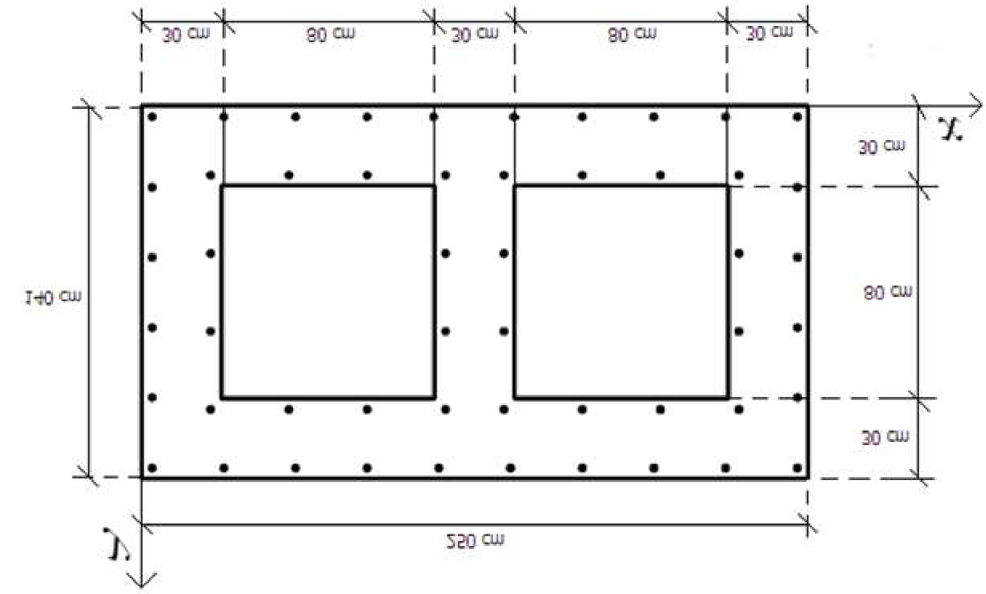
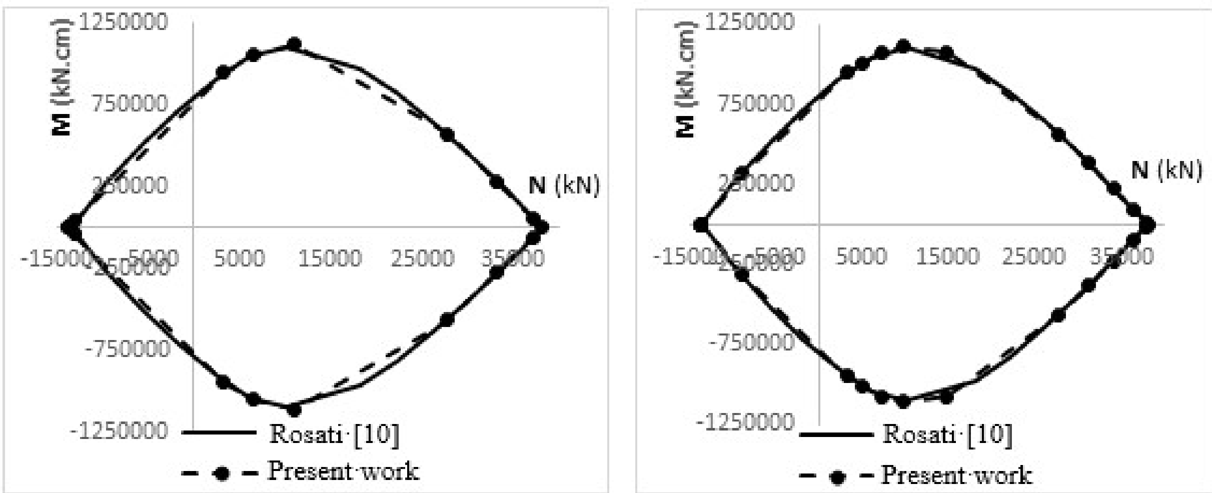


Figure 4. Cellular cross section

The adopted concrete has a characteristic strength of $f_{ck} = 20,75$ MPa . The reinforcement steel is of the type $f_{yd} = 375$ MPa and the bars are $\phi 32mm$. The Rüsch effect was considered by the factor 0.85 . The concrete and the steel strengths were reduced by the factors $\gamma_c = 1,4$ and $\gamma_s = 1,15$, respectively.

This example was analyzed in Rosati et al. [10] and it is the reference solution for the validation of the results. Once again, the ultimate interaction diagrams were computed with 5, 10, 20 and 100 points per straight edge of the ultimate strain envelope. These results are presented in Figure 5.



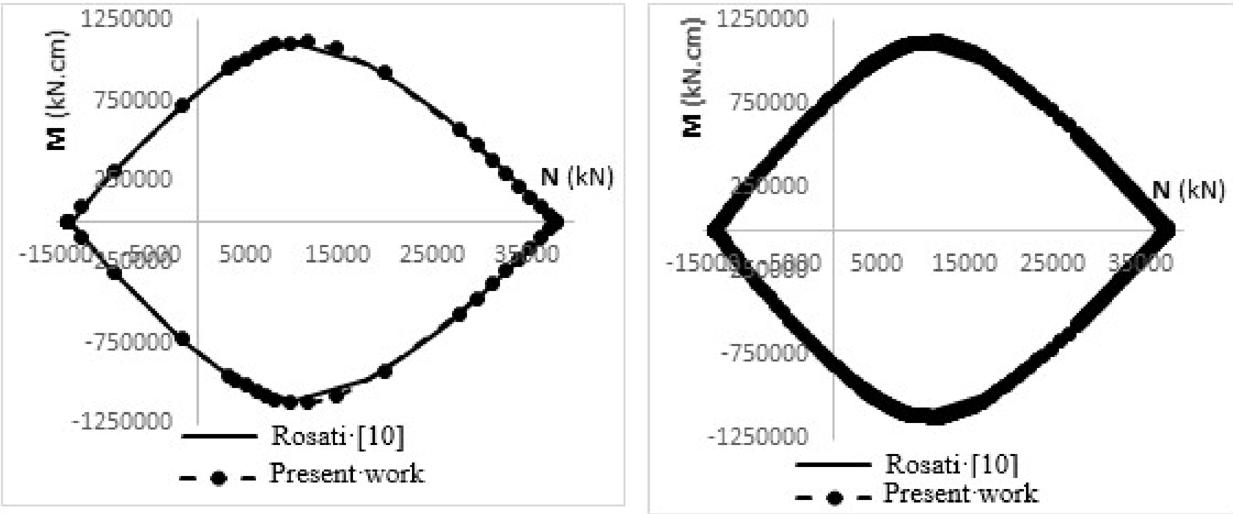


Figure 5. Ultimate interaction diagrams

Notice the great agreement between the responses of ultimate interaction diagrams. On the other hand, Figure 6 illustrates the ultimate, yielding and cracking interaction diagrams obtained with the present methodology for 1000 points per straight edges of the strain envelopes. As expected, the yielding and cracking diagrams relies within the ultimate diagram. Thus, those results are consistent.

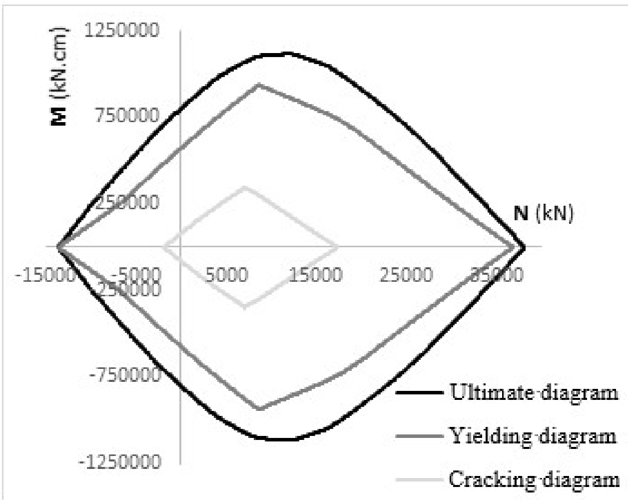


Figure 6. Ultimate, yielding and cracking interaction diagrams

8 Conclusions

The present work was dedicated to the calculation of axial force and bending moment interaction diagrams for reinforced concrete cross sections. The adopted methodology allows the calculations of the axial forces and bending moments for any polygonal cross section and it is flexible with respect to the choice of the stress-strain material curves. The main contribution of the present work was to extend such methodology to calculate not only the ultimate interaction diagram but also the yielding and cracking interaction diagrams.

One example was used to validate the methodology developed by Mendes Neto [9] and implemented as a Fortran code in the present work. The ultimate interaction diagram for the example is available in the literature. Thus, it was possible the validation of the methodology for the calculation of the ultimate axial forces and bending moments. For the yielding and cracking interaction diagrams there were no reference solutions available in the literature. However, the yielding interaction diagram relied close to the ultimate diagram, as expected by the definition of the ultimate and yielding strain envelopes. On the other hand, the cracking interaction diagram relied at the interior of the other diagrams, as expected by the definition of the cracking strain envelop.

In conclusion, the extension of the methodology presented by Mendes Neto [9] can be adopted to compute the ultimate, yielding and cracking interaction diagrams for any polygonal cross section. Such diagrams are important in applications of nonlinear analyses of reinforced concrete frames by the lumped damage mechanics.

Acknowledgements. The authors would like to thank the financial support provided by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) for this research project.

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