NONLINEAR ANALYSIS OF REINFORCED CONCRETE BEAMS WITH SECANT METHOD

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Abstract. This work aims to develop and implement a model to perform structural analysis of reinforced concrete beams using the Finite Element Method, considering the physical nonlinear behavior of materials using moment-curvature diagrams calculated before structural analysis. A finite element model for nonlinear analysis of reinforced concrete beams based on moment-curvature relationships was developed and implemented, adopting the Euler-Bernoulli beam element and using the iterative secant method for solving the resulting system of nonlinear equations. This model was implemented through computational routines and validated by comparisons with results from numerical and experimental examples in the literature. With the results obtained in the examples, it was concluded that the computational routines for nonlinear analysis of reinforced concrete beams proved to be accurate and efficient, and can be used to verify the deformations in service and analysis in ultimate limit state of reinforced concrete beams under bending, provided a finite element mesh sufficiently refined at the points of maximum effort is employed.

Keywords: Reinforced concrete, nonlinear analysis, moment-curvature relation, finite element method, secant method.

1 Introduction

The behavior of steel and concrete materials, which are different, when acting together on reinforced concrete structures, makes the analysis several complex, due to the physical non-linearity of these materials, leading to a non-linear stress-strain relationship and dependent on numerical methods, as the Finite Element Method (FEM), for solution of non-linearity. Although several models using the FEM have been developed to perform nonlinear analysis of structures, it is still a topic that has been widely studied due to the difficulties of modeling nonlinearity in the behavior of materials. Most of these models are complex, as they use incremental iterative methods to solve the nonlinear equilibrium equations, performing stress integrations in the cross sections to evaluate the efforts at each iteration, this requires a large amount of computational resources and the convergence of the calculation numeric is not always guaranteed. Kwak and Kim [1] developed a model for nonlinear analysis of reinforced concrete beams using the moment-curvature relation of reinforced concrete sections previously calculated through the analysis of cross sections, instead of using the sophisticated method of lamellae.

In this context, the present work brought together the questions studied and summarized the methodology and results obtained by de Melo [2], where the objective was to develop and computationally implement a model of nonlinear analysis of reinforced concrete beams using the FEM and considering the physical nonlinear behavior of the materials, through the use of moment-curvature diagrams obtained before structural analysis. The nonlinear equilibrium equations were solved with the iterative secant method. This approach avoids, therefore, the use of iterative-incremental methods that need to integrate the stresses in the cross section at each iteration, greatly reducing computational effort.

That said, the development of models that consider the physical non-linearity of the materials and that combine ease of application, computational efficiency and adequate results, are of great importance for the development of the analysis of reinforced concrete structures in practical design situations, and may be used to check the service limit states and ultimate states of reinforced concrete beams with greater precision and efficiency.

2 Finite element model for analysis of reinforced concrete beams

A model for nonlinear analysis of reinforced concrete beams using the beam finite element and adopting the Euler-Bernoulli theory for mathematical representation of the model was developed. The element with two nodes and four degrees of freedom was adopted, with two displacements per nodal point: transversal displacement and angular displacement or rotation. The previously calculated moment-curvature diagrams approach was used to perform the non-linear physical analysis using the iterative secant method. The method for obtaining the moment-curvature diagram is described in details by de Melo et al. [3].

The stiffness matrix of Euller-Bernoulli finite element beam can be written as [4]

$$\mathbf{k} = \int_0^L \mathbf{B}(z)^{\mathbf{T}} EI(z) \mathbf{B}(z) dz, \tag{1}$$

where **k** is the stiffness matrix of the element, **B** is the vector of the second derived from the shape functions, EI is the flexural stiffness of the cross section, z is the direction of the longitudinal axis and L is the length of the element. To obtain the secant stiffness matrix it isnecessary to employ the secant stiffness

$$EI_{sec} = \frac{M}{\kappa}, \tag{2}$$

where κ is the curvature in the element's nodes and M is the bending moment.

This requires evaluation of the curvature κ at the element nodes and the corresponding bending moment M. The curvatures can be obtained from the FEM approximation and result

$$\kappa_1 = \begin{bmatrix} -\frac{6}{L^2} & -\frac{4}{L} & \frac{6}{L^2} & -\frac{2}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = -\frac{6}{L^2}u_1 - \frac{4}{L}u_2 + \frac{6}{L^2}u_3 - \frac{2}{L}u_4, \tag{3}$$

$$\kappa_2 = \begin{bmatrix} \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \frac{6}{L^2}u_1 + \frac{2}{L}u_2 - \frac{6}{L^2}u_3 + \frac{4}{L}u_4, \tag{4}$$

where κ_1 and κ_2 are the curvatures at nodes 1 and 2 of the element, respectively, u_1 is the vertical displacement at node 1, u_2 the rotation at node 1, u_3 the vertical displacement at node 2 and u_4 the rotation at node 2. With the nodal curvvatures it is the possible to obtain the corresponding bendin moments from the moment-curvature diagram. In this work we assume that the moment-curvature diagram is obtained before structural analysis. The routines described by [3] can be employed for this purpose.

Note that Eq. (1) requires integration of the stiffness along the element. In order to work with only node values, here we consider a linear interpolation of the stiffness based on nodal values, given by

$$EI(z) = \left(1 - \tfrac{z}{L}\right) EI_{sec,1} + \left(\tfrac{z}{L}\right) EI_{sec,2}, \tag{5}$$

where $EI_{sec,1}$ is secant stiffness at node 1 of the element,$EI_{sec,2}$ is secant stiffness at node 2. By substituting this interpolation into Equation 1 and integrating along the finite element axis we obtain

$$\mathbf{k} = EI_{sec,1}\mathbf{k}_1 + EI_{sec,2}\mathbf{k}_2, \tag{6}$$

where

$$\mathbf{k}_1 = \begin{bmatrix} \frac{6}{L^3} & \frac{4}{L^2} & -\frac{6}{L^3} & \frac{2}{L^2} \\ \frac{4}{L^2} & \frac{3}{L} & -\frac{4}{L^2} & \frac{1}{L} \\ -\frac{6}{L^3} & -\frac{4}{L^2} & \frac{6}{L^3} & -\frac{2}{L^2} \\ \frac{2}{L^2} & \frac{1}{L} & -\frac{2}{L^2} & \frac{1}{L} \end{bmatrix} \tag{7}$$

$$\mathbf{k}_2 = \begin{bmatrix} \frac{6}{L^3} & \frac{2}{L^2} & -\frac{6}{L^3} & \frac{4}{L^2} \\ \frac{2}{L^2} & \frac{1}{L} & -\frac{2}{L^2} & \frac{1}{L} \\ -\frac{6}{L^3} & -\frac{2}{L^2} & \frac{6}{L^3} & -\frac{4}{L^2} \\ \frac{4}{L^2} & \frac{1}{L} & -\frac{4}{L^2} & \frac{3}{L} \end{bmatrix}. \tag{8}$$

The sum indicated above, used to represent the stiffness variation in the element, shows that the linear stiffness variation along the element is simple to employ in practice. Once the stiffness matrix of each element **k** is obtained, it is possible to form the global stiffness matrix **K** of the structure by superposition of the contribution of all elements [4]. The system of nonlinear equations that represent the structure results [4, 5]

$$\mathbf{K}(\mathbf{U})\mathbf{U} = \mathbf{F}, \tag{9}$$

where **K** is the global stiffness matrix of the system, **U** is the vector of displacements and **F** is the vector of total applied forces. Here we assume that only the stiffness matrix depends on the displacements.

2.1 Solution of nonlinear equations

In order to solve the system of non-linear equation from Equation 9 we employ a Secant iterative method. For this purpose, we start with null displacements

$$\mathbf{U}_0 = \begin{Bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{Bmatrix}. \tag{10}$$

Then, in the first iteration the secatn stiffness matrix is evaluated with null displacements. We then solve the system of linear equations

$$\mathbf{K}(\mathbf{U}_0)\mathbf{U}_1 = \mathbf{F} \tag{11}$$

to obtain an improved approximation $\mathbf{U}_1$ for the displacements. The stiffnes matrix is then evaluated with the updated displacements and the procedure is repeated until convergence. The resulting iterative method can be written as

$$\mathbf{K}(\mathbf{U}_i)\mathbf{U}_{i+1} = \mathbf{F}, \tag{12}$$

where $\mathbf{K}(\mathbf{U}_i)$ is the secant stiffness matrix of the structure obtained with displacements $\mathbf{U}_i$ and $\mathbf{U}_{i+1}$ is the updated displacement vector.

The main advantage of the Secant Iterative method, in comparison to Incremental Analysis, is that the resulting problem is better conditioned from the computational point of view. It is known that in the Incremental Analysis the tangent stiffness matrix becomes nearly singular in the verge of collapse [4]. This leads to computational issues that must be addressed with techniques such as arc-length methods. Unfourtunately, these techniques increase the complexity of the computational routines. The secant stiffness matrix, on the other hand, does not become singular in the verge of collapse and no special techniques are necessary to evaluate collapse loads. The resulting computational routines become less complex than those required by Incremental Analysis.

2.2 Secant stiffness method using the moment-curvature curve

In the secant stiffness method, the structure stiffness matrix is updated at each iteration by updating the secant stiffness EI_{sec} that is obtained by the secant line at the point considered in the moment-curvature curve until the convergence of the system. In this method, the bending moment M is a function of the curvature κ. The secant stiffness EI_{sec} was obtained using the following relation

$$EI_{sec} = \frac{M(\kappa)}{\kappa}, \tag{13}$$

where $M(\kappa)$ is a nonlinear function, which can be described by a moment-curvature diagram that relates the curvatures of a cross section to their respective bending moments. EI_{sec} is the secant bending stiffness, as its value is the slope of the secant line in the moment-curvature curve, as shown in Figure 1.

Since the displacements of the nodes are necessary to calculate the curvatures, the first values of M and κ of the moment-curvature diagram, for the first rotation step, were used for the first iteration. After obtaining the displacements in the nodes of the elements, a new value for the curvature was calculated and, with that, updated the value of EI. Then, a new stiffness matrix of the elements was determined to calculate the displacements of a new iteration. The displacements obtained in an iteration are compared with the previous iteration, and if the value was less than that defined as the stop criterion for the iterative method, the result converged, ending the procedure. Thus, the deformed configuration of the beam and its internal forces are obtained. For more details on obtaining the curvature moment diagrams, see de Melo et al. [3].

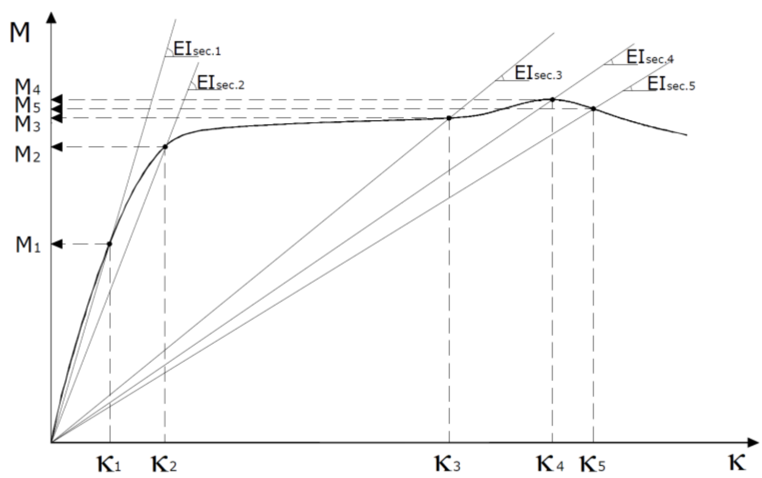

Figure 1. Moment-curvature diagram with updated secant stiffness.
Source de Melo [2].

3 Results

Next, the results of the implementation obtained from the analysis of examples of beams simply supported compared to results available in the literature of theoretical beams and experimentally tested were presented, in order to verify the efficiency and precision of the method of nonlinear analysis of concrete beams armed when compared to others.

3.1 Simply Supported Beam V1

The theoretical beam V1 was studied in Stramondinoli [6] with two different cases of loading on the beam: three point bending and four point bending. The geometric characteristics of the beam and cross section were shown in Figure 2 and the material properties in Table 1.

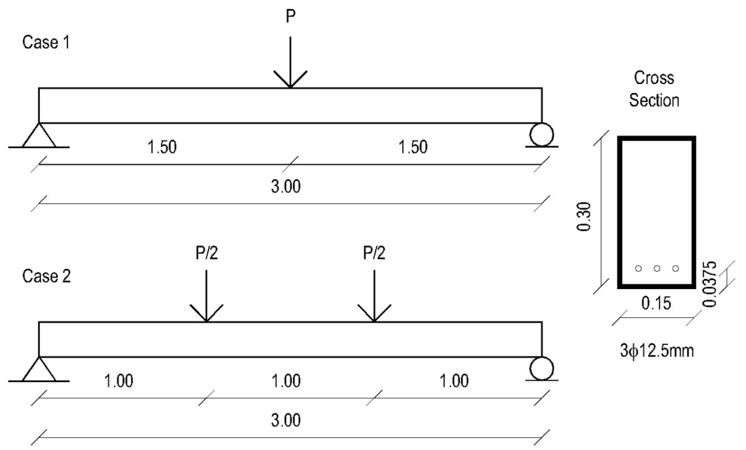

Figure 2. Dimensions (in meters) and cross section of beam V1.
Source: de Melo [2].

Table 1. Material properties of beam V1.

fcm (MPa)	ftm (MPa)	$\varepsilon 0$ (m)	α	fy (MPa)	Es (MPa)	εu (m)	Sh
20	2	0,002	0,088	500	200000	0,03	0,05

The same constitutive laws used in Stramondinoli [6] were adopted, where for concrete in compression the Hognestad [7] model was adopted and in traction the trilinear model considering the *tension-stiffening* effect developed in Stramondinoli and La Rovere [8]. For steel, the bilinear hardening model (*strain hardening*) was adopted, as proposed by La Rovere [9].To evaluate the influence of the convergence of the results, different meshes were adopted for each loading case: case 1 were 4 types of finite element meshes, with 10, 32, 60 and 100 elements and case 2 were 3 types of meshes, with 6, 12 and 36 elements.

The moment-curvature relationship for the cross section is shown in Figure 3. The results obtained for the vertical displacement in the center of the span according to the total load are shown in Figure 4. It is noticed that the results obtained with a mesh composed of 10 finite elements accurately reproduce the results of Stramondinoli [6] only for a load up to 55 kN. After this loading, only meshes with 60 and 100 elements show similar results. This indicates that the present methodology requires refining the mesh, which should be investigated in future works.

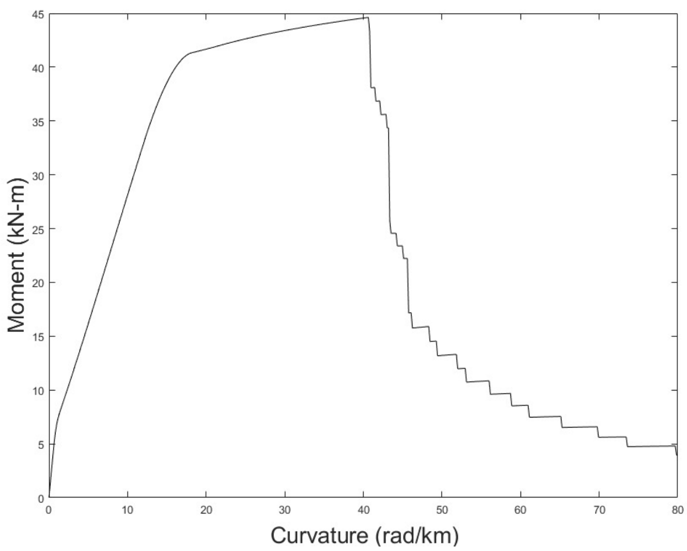

Figure 3. Moment-curvature relationship of beam V1.
Source: de Melo [2].

The displacements in the center of the span for case 2 are shown in Figure 5. It was noticed that the results are similar up to 80kN. After this loading, the beam collapses and the results are slightly different from those obtained by Stramondinoli [6].

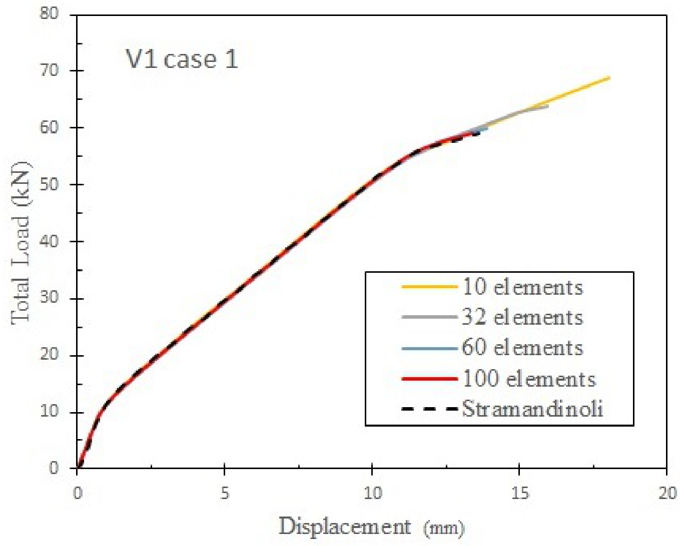

Figure 4. Load-displacement diagram for case 1 of beam V1 loading.
Source: de Melo [2].

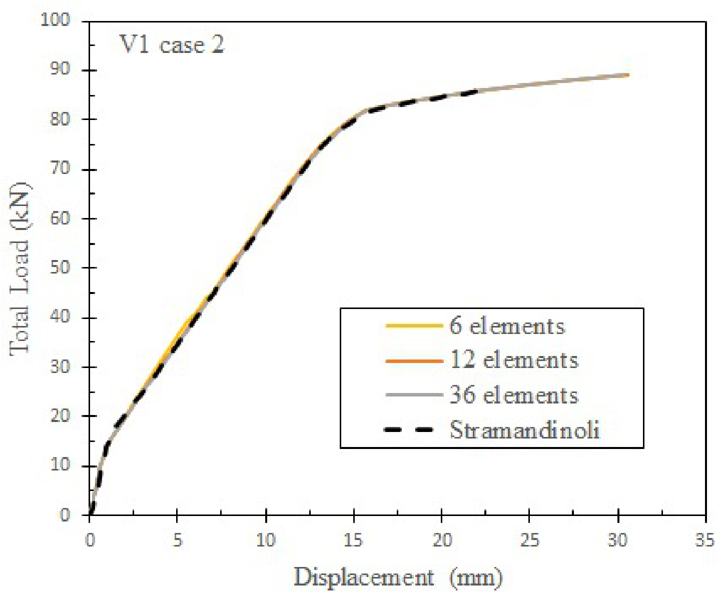

Figure 5. Load-displacement diagram for case 2 of beam V1 loading.
Source: de Melo [2].

In case 1 of loading, it was possible to notice that, in the case of a single concentrated load, a large amount of elements is necessary to capture the final part of the curve until it rupture. This is due to the formation of plastic hinges, which are captured more precisely when there are a lot of elements in the maximum stress zones of the beam. For case 2 of loading, there was a faster convergence of the results, where from 12 elements all curves were very close. The results also coincided with those obtained by Stramondinoli [6], however, the breaking load was greater for the curves obtained with the method developed in this work, presenting a result of 89 kN for all meshes, while Stramondinoli [6] obtained 86 kN.

4 Conclusion

It was possible to conclude that the model developed in this work was able to predict the nonlinear behavior of the studied beams under predominantly bending loading. However, the uniform refining of the mesh is not efficient and should be replaced adaptive refining in regions of great curvature (plastic hinges) in future works.

In comparison with incremental-iterative approaches, the secant method employed in this work proved to be efficient because it avoids incremental analysis. Consequently, several difficulties related to incremental analysis and near singular tanget stiffness in the verge of collapse are avoided. This makes non-linear analysis easier and reduces the complexity of the required computational routines.

It can be concluded that the computational routines for nonlinear analysis developed in this work proved to be efficient and can be used to check the in-service deformations and analysis in the ultimate limit state of reinforced concrete beams predominantly under bending, provided that is used a mesh of finite element sufficiently refined at the points of maximum effort. Thus, it was noticed the need for further investigation through an adaptive refining technique, in search of obtaining accurate results and with less elements in the analysis.

Acknowledgment. The authors would like to thank CNPq-Brazil for financial support of this research.

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