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Comparative analysis of the performance of a enriched mixed finite element method with static condensation for Poisson problems

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Abstract. The enriched mixed method is a variant of the mixed finite element method, obtained through a selection and appropriate configuration of the shape functions in the space of flux approximation, increasing the order of approximation just inside the elements, taking care of the balance with the space of approximation of the potential. The purpose of this paper is to analyze this method in the context of the Poisson equation. For spaces of approximation of various orders, we carry out numerical simulations considering two model problems: smooth (low oscillation and low gradient) and strongly oscillatory, in quadrilateral meshes. We conclude that the enriched mixed method of order p achieves a precision practically equivalent, with lower computational cost, to the mixed method of order $p + 1$.

Keywords: Mixed finite elements, Poisson's equation, $Hdiv$ spaces, Balanced spaces, Static condensation.

1 Introduction

Various engineering problems are modeled by second-order elliptic equations with boundary conditions. The Poisson equation is a prototype that preserves essential characteristics of this class, which is why it is often used to validate new numerical methods [1]. The classical formulation according to is the most used to solve problem 1, however, in many engineering problems the flux is of greater interest than the primal variable (potential), so processes are usually applied indirect to find the flux by calculating the gradient of the number solution of the potential; with the consequence of obtaining an approximation of the flux with lower quality than the approximation of the potential, see for example the analysis carried out in [2]. The classical mixed finite element method reformulates equation 1 incorporating the flux as an additional variable, which allows to obtain simultaneously the numerical solution of the primal (potential) and dual (flux) variables. In this paper, in the field of the finite element method (FEM) with meshes of quadrilateral elements, we study the enriched mixed method, proposed by Devloo [3], comparing the errors generated with its use, with the errors obtained when applying the mixed method [4, 5] and the classical method of finite elements. In order to reduce the computational cost, the experiments are carried out applying adequate static condensation to each method one of the three methods. For the experimentation we use the NeoPZ computational environment [6, 7], which allows the implementation of algorithms in finite elements. In [1, 3, 8] characteristics of the order of approximation are presented, among others, we also validate some of their conclusions and increase an analysis from the point of view of the order p. For the experiments we considered two model problems with known exact solutions and representative behaviors: smooth (low oscillation and low gradient) and strongly oscillatory. The paper is organized as follows. The description of computational tools and static condensation are set in section 2. The three finite elements methods used are described in section 3 and the two model problems in section 4. The seccion 5 contains the results of the numerical experiments. Seccion 6 concludes this paper.

2 Computational aspects

2.1 NeoPZ environment

NeoPZ (originally PZ) (`https://github.com/labmec/neopz`) is a general-purpose finite element library organized in modules, open source. It uses advanced object-orientation techniques to implement a wide family of FEM technologies, with the purpose of carrying out numerical simulations of processes originating from various fields of engineering, based on in mathematical models represented by differential or integro-differential equations. During its continuous development, for approximately 30 years, more FEM technologies were incorporated: approximation spaces, hp-adaptivity tools, new types of geometric elements, new variational formulations, among others, which allowed the incorporation of features such as multiscale and multiphysic [6, 7, 9, 10] which has increased their ability to manipulate increasingly complex mathematical models, to obtain more complex simulations close to reality. Some of the simulations carried out in NeoPZ are: flow in porous media, hydraulic fractures, oil reservoirs, dynamics of the grounding line in sea ice layers [11–14]. The bases of the higher order approximation spaces for the flow and potential are implemented in a hierarchical way and designed for the management of conforming or non-conforming hp meshes in dimensions 1, 2 and 3 [5, 15]. In dimension 3, it has implemented elements hexahedrons, tetrahedrons, prisms and even pyramids [16].

2.2 Static condensation

The need to carry out numerical simulations of increasingly complex phenomena has an effect on the increase in the complexity of the mathematical models used in engineering, causing a high computational cost, which persists in leaving lagging behind the continuous and rapid technological increase in processing speed and storage capacity of hardware. One way to reduce this effect is to apply clever degrees of freedom reduction maneuvers in the system of equations. Static condensation is a technique used, appropriately for the method, within the scope of the FEM. In this paper, for each method, we respectively use the static condensation techniques described in [1, 3, 17].

3 Methods

The Poisson equation to be studied is given by:

$$-\Delta u = f \text{ en } \Omega \text{ y } u = g \text{ en } \Gamma = \partial\Omega \tag{1}$$

The approximation spaces that we consider are piecewise polynomial functions, more specifically, given a partition with quadrilateral elements $\tau_h = \{K\}$ de Ω, the approximation space for u (potential) they are subspaces of $U_h = \left\{p \in L^2(\Omega, \mathbb{R}) : \ p|_K \in P_k(K, \mathbb{R}),\ K \in \tau_h\right\}$, where $P_k(K, \mathbb{R})$ is a space of polynomials of maximum degree k in each coordinate.

3.1 Classical finite element method

Given an approximation space $U_h \subset H^1(\Omega)$, the classical variational formulation $H^1(\Omega)$-as discretized is given by: find $u_h \in U_h \cap H^1(\Omega)$ such that $u_h|_\Gamma = g$ and

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega = \int_\Omega f v_h \, d\Omega, \quad \forall v_h \in U_h \cap H_0^1(\Omega)$$

The approximation spaces $U_h \subset H^1(\Omega)$ that we use are hierarchical, constructed in [15].

3.2 Mixed finite element method

Introducing the additional unknown $\sigma = -\nabla u$ in Poisson's equation 1, we obtain the system

$$\sigma = -\nabla u \quad \text{en } \Omega$$
$$\nabla \cdot \sigma = f \quad \text{en } \Omega \text{ y } u = g \quad \text{en } \Gamma = \partial\Omega$$

known as a mixed formulation of equation 1, which allows to obtain simultaneously the numerical solution of the primal (u) and dual (σ) variables. For a decomposition $\tau_h = \{K\}$ of Ω, consider the approximation subspaces of finite dimension $V_h \subset H\,(\text{div}, \Omega)$ and $U_h \subset L^2(\Omega)$, the corresponding discrete mixed variational formulation is given by: find $(\sigma_h, u_h) \in V_h \times U_h$ such that

$$\int_\Omega \sigma_h \cdot v_h \, dx + \int_\Omega u_h \nabla \cdot v_h \, dx = \int_\Gamma g v_h \cdot \eta \, ds, \quad \forall v_h \in V_h$$
$$\int_\Omega w_h \nabla \cdot \sigma_h \, dx = \int_\Omega f w_h \, dx, \quad \forall w_h \in U_h$$

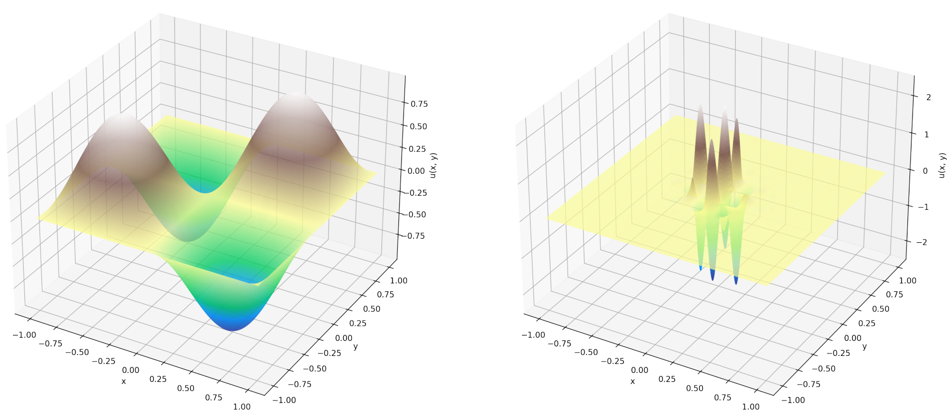
where η is the normal field to Γ, unitary and pointing outward.
It is known that the selection of pairs of approximation spaces (V_h, U_h) must be carried out in a balanced way, to avoid instability or blocking phenomena [5, 18]. In this paper we use the construction of balanced approximation spaces of higher order and hierarchical proposed in [5], in it the functions of form vector $\sigma_h \in V_h$ and the scalar form functions $u_h \in U_h$ are constructed on each element K, from the corresponding polynomial spaces $\hat{V}$ y $\hat{U}$ defined on a master element $\hat{K}$. Furthermore, the flux approximation space has the structure $\hat{V} = \hat{V}^\partial \oplus \mathring{\hat{V}}$, where $\mathring{\hat{V}}$, is generated by functions of form polynomial vectors whose normal component vanishes at the sides of the element (vector functions of type interior) and $\hat{V}^\partial$ is generated by polynomial vector shape functions associated to the sides of the element, whose normal components do not vanish.

3.3 Enriched mixed finite element method

This method was proposed in [3, 19] and consolidated in [8], in which two new balanced pairs of approximation spaces are proposed for the potential and the flux, one for triangular meshes and the other for quadrilateral meshes. In the case of quadrilateral meshes, these spaces can be interpreted as enriched versions of Raviart-Thomas RT_k spaces[4]. Enrichment procedures are applied by space increments using additional bubble terms. These bubble terms are scalar functions supported by a single element (in the case of H^1-conformal approximations) or vector functions whose normal components vanish at the edges of the elements (in the case of $H(div)$-compliant spaces). The advantage of using bubbles as stabilization correctors is based on the fact that all the corresponding degrees of freedom can be condensed, so that the number of equations to be solved and the structure of the matrix are not affected. by the enrichment process [8]. In this paper, we consider enriched approximation spaces for flux and potential structured respectively as follows: $\hat{V}_k^{1+} = \hat{V}_k^\partial \oplus \mathring{\hat{V}}_{k+1}$ y $\hat{U}_k^{1+} = \nabla \cdot \hat{V}_k^{1+}$

4 Model problems

For numerical experiments, we consider two model problems of the Poisson equation 1, with exact solution, each one with a homogeneous Dirichlet boundary condition on the domain $\Omega = [-1, 1] \times [-1, 1]$. These problems respectively have the following representative characteristics: smooth (low oscillation and low gradient) and strongly oscillatory.



(a) Smooth solution (b) Strongly oscillating solution

Figure 1. Exact solutions to model problems

For the Poisson equation 1, the problem with smooth solution that we consider has an exact solution: $u(x, y) = \sin(\pi x)\sin(\pi y)$. Figure 1a. The problem with strongly oscillatory solution that we take has the exact solution: $u(x, y) = 0.4\sin(9\pi x)(1 + \cos(9\pi y))\Big(\frac{\pi}{2} + \arctan(10 - 200(x^2 + y^2))\Big)$. Figure 1b.

5 Numerical results

To validate the ideas discussed in this paper the numerical experiments for the three methods were performed in a Macbookpro with a six-core Intel Core i7 processor (2.2GHz), 16 GB of DDR4 2400 MHz ram memory. We have used the NeoPZ library to implement a finite element numerical simulation program, In this section, we show results for three experiments.

5.1 The smooth model problem: h-refinement

The soft model problem was solved by applying the three methods considered in section 3; for orders $p = 2$; 3 and 4. The convergence rates for the variable u in L^2 are shown in tables 1, 2 and 3. It is observed that the theoretical convergence rates are achieved.

Table 1. Errors and convergence rates for u in the smooth model problem with h-refinement and p = 2

	Classic FEM		Mixed		Enriched Mixed	
h	Error in L^2	Convergence rate	Error in L^2	Convergence rate	Error in L^2	Convergence rate
0.25000	3.86E-03	-	2.14E-03	-	1.25E-04	-
0.12500	4.90E-04	2.9787	2.69E-04	2.9921	7.90E-06	3.9813
0.06250	6.15E-05	2.9950	3.37E-05	2,9980	4.95E-07	3.9951
0.03125	7.69E-06	2.9988	4.21E-06	2.9995	3.10E-08	3.9988

Table 2. Errors and convergence rates for u in the smooth model problem with h-refinement y p = 3

	Classic FEM		Mixed		Enriched Mixted	
h	Error in L^2	Convergence rate	Error in L^2	Convergence rate	Error in L^2	Convergence rate
0.25000	1.76E-04	-	1.06E-04	-	4.52E-06	-
0.12500	1.11E-05	3.9854	6.66E-06	3.9933	1.41E-07	4.9994
0.06250	6.97E-07	3.9963	4,17E-07	3.9983	4.42E-09	4.9999
0.03125	4.36E-08	3.9991	2.61E-08	3,9996	1.38E-10	5.0000

Table 3. Errors and convergence rates for u in the smooth model problem with h-refinement and p = 4

	Classic FEM		Mixed		Enriched Mixed	
h	Error in L^2	Convergence rate	Error in L^2	Convergence rate	Error in L^2	Convergence rate
0.25000	6.70E-06	-	4.19E-06	-	1.48E-07	-
0.12500	2.11E-07	4.9906	1.32E-07	4.9942	2.33E-09	5.9911
0,06250	6,60E-09	4.9976	4.11E-09	4.9985	3.64E-11	5.9977
0,03125	2,06E-10	4.9993	1.29E-10	4.9996	5.70E-13	5.9994

5.2 The strongly oscillatory model problem: p-refinement

Considering a refined mesh with $h = 0.0625$, the strongly oscillatory model problem was solved by applying the three methods described in section 3. For p-refinement, in table 4 the numerical results are shown, respectively, of the errors in L^2 for u and H^1. In addition, for comparison purposes, we select the respective p orders in each method; as highlighted in dark gray. Table 5 shows the respective condensed degrees of freedom. In Figures 2 and 3 we show a comparison of the convergence curves as a function of p-refinement.

Table 4. p-refinement for the strongly oscillatory model with $h = 0.0625$

	Error for u in L^2			Error in H^1		
p	Enriched Mixed	Mixed	Classic FEM	Enriched Mixed	Mixed	FEM
1	**4.25E-02**	2.26E-01	2.44E-01	**2.196**	3.234	7.663
2	3.65E-02	**4.15E-02**	2.17E-01	1.455	**1.965**	3.981
3	**2.86E-02**	3.65E-02	**3.68E-02**	**0.942**	1.435	**2.026**
4	6.73E-03	**2.86E-02**	3.65E-02	0.609	**0.939**	1.687
5	**5.32E-03**	6.73E-03	**2.84E-02**	**0.410**	0.608	**1.044**
6	4.99E-03	**5.32E-03**	6.65E-03	0.271	**0.409**	0.717
7	**3.43E-03**	4.99E-03	**5.30E-03**	**0.185**	0.271	**0.461**
8	-	**3.43E-03**	4.87E-03	-	**0.185**	0.322
9	-	-	**3.41E-03**	-	-	**0.210**

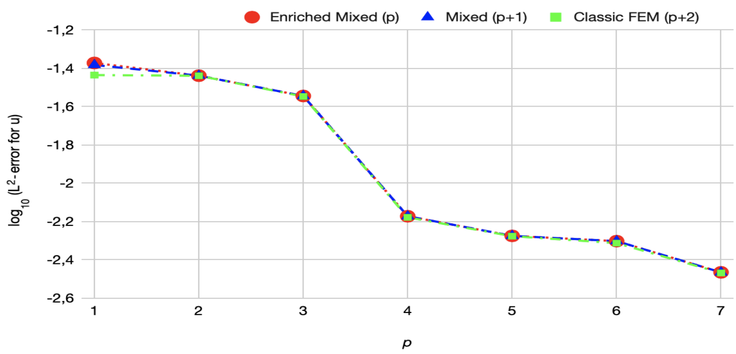


Figure 2. Error of u in L^2 norm for the strongly oscillatory model with $h = 0.0625$

Table 5. Degrees of freedom through p-refinement for the strongly oscillatory model with $h = 0.0625$

	Degrees of Freedom Condensed		
p	Enriched Mixed	Mixed	Classic FEM
1	**5248**	5248	1089
2	7360	**7360**	3201
3	**9472**	9472	**5313**
4	11584	**11584**	7425
5	**13696**	13696	**9537**
6	15808	**15808**	11649
7	**17920**	17920	**13761**
8	-	**20032**	15873
9	-	-	**17985**

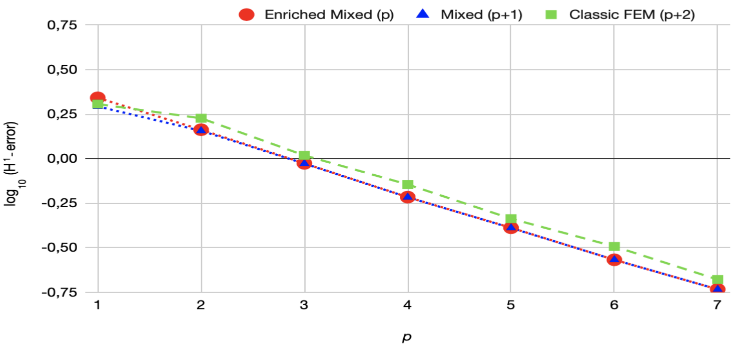


Figure 3. Error in norm H^1 for the strongly oscillatory model with $h = 0.0625$

6 Conclusions

This paper has used an empirical approach to compare the performance of three methods in finite element methods. This contribution is important to establish a benchmark when using a method and obtain low computational costs without affecting the precession in a high way.

For the problem with a strongly oscillatory solution approached, the enriched mixed method of order p has a lower computational cost than the mixed method of order $p + 1$, since, at the master element level, the former has fewer polynomial vectors associated to the sides of the element. However, in the potential variables, their errors in the L^2 norms are practically the same. On the other hand, for the H^1 error, the p order rich mixed method is slightly less accurate, in thousandths, than the $p + 1$ order mixed method; which is reasonable, because, on the master element, the first one has fewer vector polynomials in the flow.

Likewise, in the case of the classical finite element method od order $p + 2$, for the highly oscillatory model problem, we observe that the approximation error for the variable u in the norm L^2 is practically the same as that the enriched mixed method of order p, being relegated in the case of the error in norm H^1 because, in this case, the flux calculation is carried out indirectly.

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