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An adaptive and selective generalized finite element method for free vibration of trusses

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Abstract. The ability to support large loads over a great span makes trusses structural systems with a variety of applications in engineering. In regards to dynamic analysis, such structures do have few known analytical solutions and, thus, are usually analyzed through approximated methods such as the Generalized Finite Element Method (GFEM). The GFEM is based on the Partition of Unity Method and uses previous knowledge of the problem's solution to expand the traditional Finite Element Method (FEM) approximation space. In previous literature an adaptive GFEM has been proposed for the free vibration problem of bars and trusses and has led to excellent results. This method consists of an iterative process that incorporates in the enrichment an approximated natural frequency result in each step. On the other hand, another technique found in literature is the use of the Friberg error indicator to identify which elements of a mesh have greater impact on the solution in an enrichment process. The indicator has already been shown to be applicable in GFEM analysis. The selective technique aims to reduce the number of degrees of freedom necessary in obtaining good approximations of a determined natural frequency. In this paper a selective adaptive technique is proposed. The Friberg indicator is used to define which bars of the truss will be enriched and the adaptive process is used to improve the accuracy of the solution. Thus, combining the advantages of both methods leads to results with low error and reduced number of degrees of freedom when compared to the traditional GFEM approach. The results obtained by the proposed technique are compared with reference solutions found in literature.

Keywords: Adaptive, Selective Refinement, Generalized Finite Element Method.

1 Introduction

As practical, economical and esthetically pleasing solutions, trusses are commonly used in civil engineering structures. Trusses are especially economically advantageous in comparison to beams when loads must be supported over large spans. Thus, they are regularly used in structures such as bridges, roofs and electricity pylons (Beer and Johnston Jr [1]). Such applications are, in fact, prime examples of structures subject to high dynamic effects.

These analyses are commonly done through approximated methods such as the Finite Element Method (FEM). When working with free vibration problems, FEM approximations tend to present high errors in regards to high frequencies (Arndt et al. [2]). To improve FEM results the approximation space may be enriched with a pre-determined set of functions. These tend to be mainly polynomial (Houmat [3], Ribeiro [4]) or trigonometric (Zeng [5, 6]) functions. The advantage of using the latter is that it mimics the behavior of fundamental vibration modes.

Such technique is the base for the Generalized Finite Element Method (GFEM) (Melenk and Babuska [7], Babuska et al. [8], Duarte et al. [9], Duarte and Oden [10], Oden et al. [11]). The GFEM, which is based on the Partition of Unity Method (Melenk and Babuska [7]), improves local and global results by including knowledge about the problem's solution into the FEM approximation space. The GFEM has already been applied successfully to the dynamic analysis of a variety of structures (Arndt et al. [2], Torii and Machado [12], Arndt et al. [13], Debella et al. [14]).

The first application of GFEM to the free vibration of trusses was presented by Arndt et al. [2], where the authors also presented the adaptive GFEM. This expansion of the GFEM is an iterative process that improves the accuracy of a pre-determined natural frequency. The adaptive GFEM was shown to lead to results with higher precision when compared to traditional h-FEM with a lower number of degrees of freedom. The advantages of the adaptive method for trusses was also shown in transient analysis (Debella et al. [14]).

Up to this point, the adaptive GFEM has always been applied considering the enrichment of a problem's full mesh. However, recent works have tested selectively applying GFEM enrichment functions to a restricted number of elements with the use of error indicators (Malacarne et al. [15, 16]), such as the Friberg indicator (Friberg [17], Friberg et al. [18]). It has been shown that the full mesh does not have to be enriched to obtain low error results.

Therefore, the objective of this paper is to analyze the free vibration of trusses with the adaptive GFEM considering selective enrichments guided by the Friberg error indicator. Thus, improving its accuracy with a reduced number of degrees of freedom.

2 Adaptive GFEM for dynamic analysis of trusses

The Generalized Finite Element Method (GFEM) can be considered an extension of the Finite Element Method (FEM) in which enrichment functions are used to improve local approximations. The enrichments are chosen based on local knowledge of the differential equation solution and are incorporated by modifying basic interpolation functions derived from the Partition of Unity (PU) approach (Melenk and Babuska [7], Babuska et al. [8], Duarte et al. [9], Duarte and Oden [10], Oden et al. [11]). Thus, in GFEM, the displacement approximation u_h^e for a specific finite element is given by:

$$u_h^e(\xi) = \sum_{i=1}^{2} \eta_i(\xi)\, u_i + \sum_{i=1}^{2} \eta_i(\xi) \left\{ \sum_{j=1}^{n_l} \left[\gamma_{ij}(\xi)\, a_{ij} + \varphi_{ij}(\xi)\, b_{ij} \right] \right\}, \tag{1}$$

where η_i are PU functions, ξ is the local coordinate system, u_i are nodal degrees of freedom, n_l is the number of enrichment levels, γ_{ij} and φ_{ij} are the enrichment functions, and a_{ij} and b_{ij} are the field degrees of freedom. In this work, as in Arndt et al. [2], Torii and Machado [12], Debella et al. [14] and Malacarne et al. [15, 16], the PU is taken as the linear Lagrangian functions:

$$\eta_1 = 1 - \xi, \tag{2}$$

$$\eta_2 = \xi. \tag{3}$$

As for the enrichment functions, this work utilizes trigonometric functions proposed by Arndt et al. [2] for bars. These functions consist in a pair of sinusoidal and cosine clouds that in the element domain are written as the following two pairs of sine and cosine functions:

$$\gamma_{1j} = \sin(\beta_j\, L_e\, \xi), \tag{4}$$

$$\gamma_{2j} = \sin(\beta_j\, L_e\, (\xi - 1)), \tag{5}$$

$$\varphi_{1j} = \cos(\beta_j\, L_e\, \xi) - 1, \tag{6}$$

$$\varphi_{2j} = \cos(\beta_j\, L_e\, (\xi - 1)) - 1, \tag{7}$$

where L_e is the element length and the element domain is $\xi\ [0, 1]$. Moreover, β_j is a hierarchical enrichment parameter.

The adaptive GFEM (Arndt et al. [2]) is an iterative procedure that aims to increase the accuracy of a determined natural frequency without modifying the chosen mesh or increasing the levels of enrichment. In the adaptive GFEM β_j is taken as:

$$\beta_j = \sqrt{\frac{\rho}{E}}\mu_j, \tag{8}$$

where ρ is the material specific mass, E is the Young modulus and μ_j is a frequency related to the j^{th} enrichment level.

The adaptive GFEM starts by a first approximation for a target frequency (μ_{TARGET}) through the FEM with a coarse mesh. The resulting frequency value is then used as μ_j in eq. (8) and GFEM with $n_l = 1$ is applied to the same mesh previously used in the FEM approximation. This process leads to a new frequency value μ_j, which is iteratively updated until a pre-defined error limit is achieved. More details on the adaptive GFEM formulation and procedure are found in Arndt et al. [2] and Debella et al. [14].

Both in standard and adaptive GFEM the mass and stiffness matrices are obtained by standard FEM procedure (Bathe [19], Hughes [20]). However, when working with trusses, bars may be oriented in any direction in space. Therefore, a coordinate transformation rule must be applied to obtain the system of final governing equations. Once the enrichment functions are null at the nodes, this transformation is given by:

$$\begin{bmatrix} u'_1 \\ u'_2 \\ c'_1 \\ \vdots \\ c'_n \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}, \tag{9}$$

where u'_i are the nodal displacements in local coordinates, c'_i are the enriched (field) degrees of freedom in local coordinates, θ is the bar inclination, u_i and v_i are respectively the horizontal and vertical displacements in global coordinates and c_i are the enriched (field) degrees of freedom in global coordinates. Further details on the coordinate transformation for truss problems are presented in Rao [21].

3 Friberg Error Indicator

With the intent of achieving optimal convergence rates enrichment procedures may be applied together with local error indicators. Such combination defines which enrichment based degrees-of-freedom lead to the best solution improvements (Kelly et al. [22], Gago et al. [23]). In this context, Friberg [17] proposed an error indicator based on approximating the variation in a specific eigenvalue in a hierarchical enrichment process. Given a problem of n and $n+m$ degrees of freedom before and after an enrichment procedure, the Friberg indicator for the i^{th} eigenvalue is given by:

$$\eta_i = \frac{1}{k_i^{(n)}} \frac{\left(\left[\mathbf{K}_{(nm)} - \lambda_i^{(n)}\mathbf{M}_{(nm)}\right]\phi_i^{(n)}\right)^2}{\mathbf{K}_{(mm)} - \lambda_i^{(n)}\mathbf{M}_{(mm)}}, \tag{10}$$

where k_i^n is the modal stiffness, λ_i^n and ϕ_i^n are the i^{th} eigenvalue and eigenvector of an n order approximation and the different $\mathbf{K}$ and $\mathbf{M}$ matrices are submatrices of the hierarchically formed stiffness and mass matrices as in:

$$\mathbf{K}^{(\mathbf{n+m})} = \begin{bmatrix} \mathbf{K}_{(nn)} & \mathbf{K}_{(nm)} \\ \mathbf{K}_{(mn)} & \mathbf{K}_{(mm)} \end{bmatrix}. \tag{11}$$

The Friberg indicator does not depend on the solution of the $n+m$ eigenvalue problem and leads to a dimensionless number that identifies which elements in a mesh most influence the final solution when a determined enrichment is applied. Further details on the indicator formulation may be found in Friberg [17] and Friberg et al. [18].

The use of the Friberg indicator in GFEM analysis was first evaluated by Malacarne et al. [15, 16]. The authors demonstrated, in the free vibration analysis of bars and trusses, that the use of the indicator allows GFEM enrichments to be applied selectively. The proposed adaptive and selective process is described in Fig. 1. Such procedure leads to approximations with high converge rates and a smaller global system sizes when compared to standard GFEM.

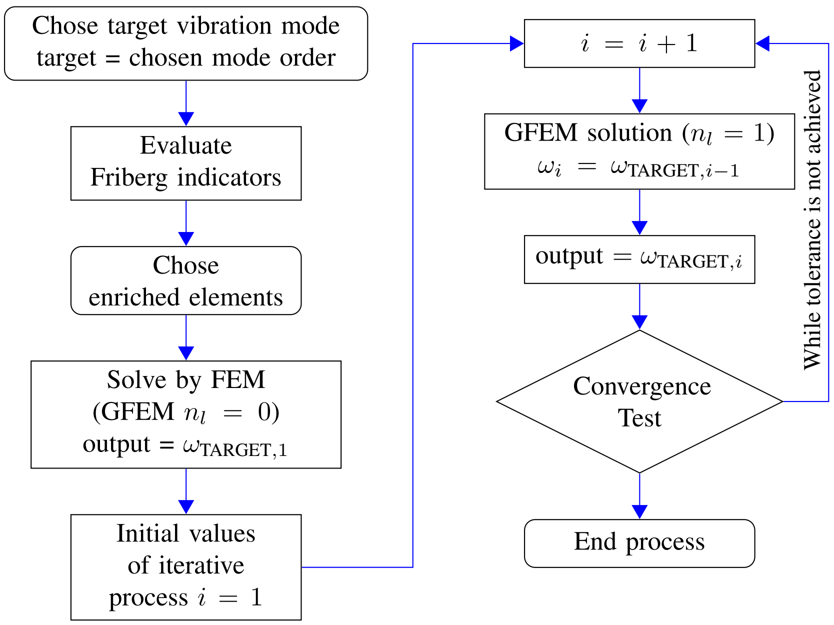

Figure 1. Adaptive and Selective GFEM flowchart.

4 Numerical Results

In order to evaluate the proposed technique, the adaptive and selective GFEM was applied in the seven bar truss presented in Fig. 2. All bars have Young's Modulus of 2.1×10^{11} N/m^2, cross section area of 0.001 m^2 and material density of 8000 kg/m^3. The same truss was evaluated in Zeng [5] and Malacarne et al. [16].

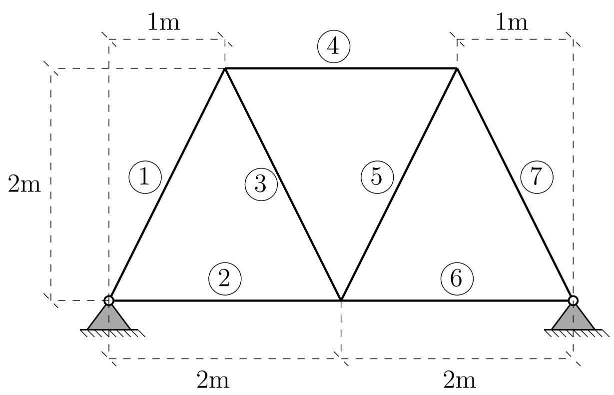

Figure 2. Seven bar truss and bar numbering.

In order to compare the proposed technique results, the first five frequencies, obtained by GFEM with all bars with one enrichment level and $\beta_1 = \pi$, the Composite Element Method (Zeng [5]), the *h*-FEM (according with the technique described in Debella et al. [14]) and an exact solution (Lahe et al. [24]), are presented in Table 1.

To evaluate the technique efficiency, two frequencies are chose as target (1st and 5th frequencies). The results and are presented in next subsections.

Table 1. Frequencies reference results (rad/s).

Mode	GFEM 36 d.o.f.	Zeng [5] 30 d.o.f.	*h*-FEM 706 d.o.f.	Lahe et al. [24]
1	1647.819119	1648.26	1647.814217	1647.784428
2	1740.882082	1741.32	1740.806703	1740.839797
3	3111.534742	3113.83	3111.32946	3111.322715
4	4562.658836	4567.69	4561.894189	4561.817307
5	4823.794071	4829.70	4823.328158	4823.248678

4.1 First Frequency as Target

In this analysis, the first frequency is chosen as the target. Thus, for this frequency the Friberg indicator is calculated for each element when independently enriched. The results are presented in Table 2 in descending order.

Table 2. Friberg influence factors for the first frequency.

Element	Friberg indicator	Percentage cumulative sum
3	0.0202	44.8%
5	0.0202	89.7%
1	0.0023	94.8%
7	0.0023	99.9%
4	4.69×10^{-05}	$\approx$ 100.0%
2	1.26×10^{-33}	$\approx$ 100.0%
6	1.26×10^{-33}	$\approx$ 100.0%
Total	0.045047	100.0%

It can be observed that elements 2, 4 and 6 have Friberg indicators with a much smaller order of magnitude than the other elements, so only elements 1, 3, 5 and 7 are enriched in the adaptive process. These correspond to almost 100% of Friberg sum. The results of the adaptive and selective procedure are presented in Table 3.

Table 3. Adaptive and selective GFEM for 1^{st} frequency.

Enriched elements	1, 3, 5 and 7		
Iteration	1 (8 d.o.f.)	2 (24 d.o.f.)	3 (24 d.o.f.)
1^{st} frequency (rad/s)	1683.521	1647.818	1647.818

The comparison between results shown in Table 1 and Table 3 demonstrates that the adaptive selective process results in a frequency value lower than that of standard GFEM and the Composite Element Method. Such result is obtained with only 24 degrees of freedom. This represents 66,6% and 80% of the degrees-of-freedom respectively required by such methods. Thus, as elements 2, 4 and 6 presented low indicator values when individually enriched (Table 2), these may be neglected in the enrichment process for adaptive GFEM reducing the total number of degrees of freedom necessary in the analysis.

4.2 Fifth Frequency as Target

Next, the fifth frequency is chosen as the target for the indicator calculation and adaptive process. The Friberg indicator values obtained, for the enrichment of each element, are presented in descending order in Table 4.

Table 4. Friberg influence factors for the fifth frequency.

Element	Friberg indicator	Percentage cumulative sum
1	0.3149	27.3%
5	0.463	67.5%
7	0.3149	94.9%
3	0.0463	98.9%
4	0.013	$\approx$ 100.0%
2	6.70×10^{-32}	$\approx$ 100.0%
6	6.70×10^{-32}	$\approx$ 100.0%
Total	1.1521	100.0%

In this case elements 2 and 6 have Friberg indicators with a much smaller order of magnitude than the other elements, so only elements 1, 3, 4, 5 and 7 are enriched in adaptive process. These correspond to almost 100% of Friberg sum. The results of the adaptive selective process are presented in Table 5.

Table 5. Adaptive and selective GFEM for 5^{th} frequency.

Enriched elements	1, 3, 4, 5 and 7		
Iteration	1 (8 d.o.f.)	2 (28 d.o.f.)	3 (28 d.o.f.)
5^{th} frequency (rad/s)	5678.185	4823.363	4823.270

When Table 5 results are compared with those from Table 1 it is observed that the adaptive selective process leads to lower frequency values than GFEM and the Composite Element Method. This occurs with a reduced number of degrees of freedom of 28. This represents 77% and 93% of the degrees-of-freedom respectively required by the GFEM and Composite Method. Therefore, as elements 2 and 6 present significantly lower influence (Table 4) when compared to other elements, their enrichment is not necessary in the adaptive procedure.

5 Conclusion

In this paper an adaptive and selective GFEM technique was presented. The proposed technique first determines through the Friberg indicator the influence values of each element's enrichment in the final result. Subsequently, the enrichment is not applied to elements with indicator values with order significantly lower than those of most mesh elements. Based on such selectively enriched mesh the adaptive procedure is used to approximate the value of a pre-defined target frequency.

The adaptive and selective GFEM was tested in a seven bars truss. For both target frequencies evaluated the process leads to accurate results with a reduced number of degrees of freedom. Hence, the technique shows potential. However, it still needs more studies both to verify its accuracy in other examples and to determine if the computational cost of the iterative process is compensated by the reduction of the problem size.

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