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Cohesive crack propagation simulated by different SGFEM strategies

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Abstract. The present work aims to evaluate the performance of the Stable Generalized Finite Element Method (SGFEM), a relatively new approach that derives from a simple modification of enrichment functions used in Generalized/eXtended Finite Element Method (G/XFEM), in the analysis of a three-point bending test. For this, different crack propagation simulations are performed using the standard Heaviside Function, its linear modification as proposed by Gupta et al. [1] and a version that employs a stabilization parameter, presented in Wu and Li [2]. A cohesive crack model is considered and linear elastic material is assumed for the numerical experiments. Equilibrium paths, as well as the scaled condition numbers (SCNs), calculated at each step, are evaluated by SGFEM and compared with the results obtained by G/XFEM. This work is related to a proposal of expansion of the INSANE (INteractive Structural ANalysis Environment) system, an open source project developed at the Structural Engineering Department of the Federal University of Minas Gerais. This platform has enabled the resources that allowed the analysis and discussions carried out in this work.

Keywords: Stable Generalized Finite Element Method, Generalized Finite Element Method, Computational Mechanics, Object Oriented Programming, JAVA.

1 Introduction

Structural collapse is often the consequence of localized failure in solids, i.e., a manifestation of concentration of defects, such as cracks in concrete [2]. It is important, therefore, to properly evaluate residual structural safety once those defects appear to prevent potential catastrophic collapse. Modeling crack propagation can be, however, burdensome when using Finite Element Method (FEM). In order to circumvent this limitation and aiming to combine some of the advantages of Meshless Methods (MM) with the use of a finite element mesh the Generalized Finite Element Method (GFEM) [3–5] was proposed. Since this formulation, as stated by Belytschko et al. [6], can be considered equivalent to the one developed by Northwestern school with the name of eXtended Finite Element Method (XFEM) [7, 8], such approach will be called in this paper Generalized/eXtended Finite Element Method (G/XFEM).

According to Sanchez-Rivadeneira and Duarte [9], G/XFEM can be understood as a FEM with a enriched test/trial space. This space is constructed by augmenting the standard finite element approximation spaces with the enrichment functions, which usually contain a-priori knowledge about the solution of the problem. G/XFEM has been, in the past few decades, widely accepted and applied with success to problems involving cracks. In fact, this approach is available nowadays in commercial software like ANSYS and Abaqus. However, some G/XFEM shortcomings are noteworthy. An important concern is the ill-conditioning of system matrix. This issue may cause reduction of the convergence rates of an iterative solution scheme or severe loss of digits in a direct method [2]. The other drawback here considered is G/XFEM performance in the existence of the so-called blending elements [10], which appear when enrichment functions are only applied locally in the domain. Those elements can not reproduce exactly the enrichment functions. Therefore, in general, the discretization error in the blending elements might be higher than the one verified in the other elements.

Among all the techniques that have been developed to deal with those issues is the modification of enrichment functions. Particularly, the simple enrichment modification proposed by Babuška and Banerjee [11] - which consists in subtracting from a enrichment function its FE interpolant - with the name of Stable Generalized Finite Element Method (SGFEM) has caught attention because of its potentialities. Babuška and Banerjee [11] showed mathematically for 1-D problems that SGFEM yields matrices with a condition number with the same order of FEM, and orders of magnitude smaller than in the G/XFEM case. Gupta et al. [1] and Gupta et al. [12] have obtained numerically similar results for 2-D and 3-D fracture mechanics problems, respectively. Furthermore, SGFEM is able to address with the error due to blending elements [1, 13]. The extension of the ideas presented in [11] for SGFEM to other dimensions was not, however, always straightforward. As stated by Oliveira et al. [13], the search for a stable version of G/XFEM as defined by Zhang et al. [14] - namely, a method that yields the optimal order of convergence with a conditioning that is not worse than that of FEM *independently* of the mesh - has still been pursued. Among the works that dealt with this search, it is noteworthy to mention the one of Wu and Li [2], that handles cohesive crack propagation problems with SGFEM using a modified version of Heaviside function; the one of Zhang et al. [15], in which the SGFEM proposed in [1] shows lack of robustness, considering the relative position between the discontinuity and the mesh, in a Poisson problem; and the ones of Zhang et al. [16] and Zhang and Babuška [17], that deal with the use of different partitions of unity (PoUs) combined to the stable strategy proposed by Babuška and Banerjee [11].

Considering those interesting features concerning SGFEM and its versions, this paper aims to study the performance of different SGFEM strategies to simulate cohesive crack propagation in a three-point bending test. To the best knowledge of the authors, SGFEM has only been applied to cohesive crack propagation in the work of Wu and Li [2]. Those authors employed a modified version of Heaviside function, with a stabilization parameter. The effects of using the linear Heaviside functions, as proposed by Gupta et al. [1] for maintaining the optimal rates of convergence in 2-D fracture mechanics problems, have not been investigated yet. Those results will be compared, in terms of equilibrium paths and condition numbers, to the ones obtained using standard Heaviside function under G/XFEM approach.

2 Model problem

2.1 Governing equations

For conciseness, the strong and the weak forms of the governing equations will not be shown in this paper. The authors used the formulations presented in the work of Wang and Waisman [18] as a reference for the problem here studied.

2.2 Cohesive law

The cohesive law adopted in this work is based on a simplification of the formulation presented by Wells and Sluys [19], where the normal traction force t_n transmitted across the discontinuity is defined as

$$t_n = f_t \exp\left(-\frac{f_t}{G_f}\kappa\right) \tag{1}$$

where f_t is the tensile strength of the material, G_f is the fracture energy and κ is a history parameter, equal to the largest value of normal crack opening $[\![u]\!]_n$ reached. The shear traction t_s acting on the discontinuity surface is computed from

$$t_s = d_{\text{init}}[\![u]\!]_s \tag{2}$$

where d_{init} is the initial crack shear stiffness (when $\kappa = 0$) and $[\![u]\!]_s$ is the crack sliding displacement.

2.3 Enrichment functions

The enrichment function selected to simulate cohesive crack propagation with G/XFEM is the standard Heaviside function, defined as

$$\mathcal{H}(x,y) = \begin{cases} 1, & \text{if } \bar{y} \le 0 \\ 0, & \text{if } \bar{y} > 0 \end{cases} \quad \text{for a local coordinate } \bar{y}\text{, normal to the crack segment.} \tag{3}$$

In the case of SGFEM, two strategies are considered. The first one was proposed by Gupta et al. [1] as linear Heaviside enrichment functions:

$$\mathcal{H}_L^j(x,y) - I_{\omega_j}(\mathcal{H}_L^j(x,y))(\mathbf{x}) = \left\{\mathcal{H}, \mathcal{H}\frac{(x-x_j)}{h_j}, \mathcal{H}\frac{(y-y_j)}{h_j}\right\} - I_{\omega_j}(\mathcal{H}_L^j(x,y))(\mathbf{x}) \tag{4}$$

where $\mathcal{H}$ is expressed by Eq. (3) and h_j is a scaling factor given by the largest distance of node $\mathbf{x}_j$ to the other nodes of cloud ω_j. $I_{\omega_j}(\mathcal{H}_L^j(x,y))(\mathbf{x})$ is the finite element interpolant, defined as [1]:

$$I_{\omega_j}(L_{ji})(\mathbf{x}) = \sum_{k=1}^{n_e} N_k(\mathbf{x}) L_{ji}(\mathbf{x}_k) \tag{5}$$

where L_{ji} is an enrichment function, vector $\mathbf{x}_k$ has the coordinates of node k of element e, N_k is the piecewise linear FE shape function for node k, and n_e is the number of element nodes. The second strategy was proposed by Wu and Li [2] and is computed from:

$$\mathcal{H}_{mod}^j(x,y) = \mathcal{H}(x,y) - [\alpha I_{\omega_j}(\mathcal{H}(x,y))(\mathbf{x}) + (1-\alpha)\mathcal{H}(x_j,y_j)] \tag{6}$$

where $\mathcal{H}(x_j,y_j)$ is the Heaviside function from Eq. (3) computed at the node $\mathbf{x_j}$ and $0 \leq \alpha \leq 1$ is a stabilization parameter.

3 Numerical Examples

In this section, the performances of the SGFEMs (namely, strategies that employ the FE interpolant subtraction from enrichment functions, as in [11]) proposed by Gupta et al. [1] and Wu and Li [2], with different Heaviside functions, are compared to the one of G/XFEM with standard Heaviside function (Eq. (3)) through two numerical experiments performed with the beam of Fig. 1.

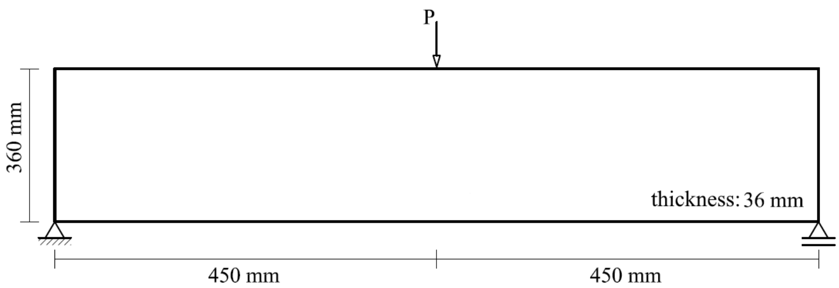

Figure 1. Three-point bending test used in the experiments carried out in this section [20].

As in Wu and Li [2], linear elastic material is considered. The following parameters were used: Young's modulus $E = 44000.0\ N/mm^2$, Poisson's ratio $\nu = 0.2$, tensile strength $f_t = 3.8\ N/mm^2$, fracture energy $G_f = 0.164\ N/mm$ and initial crack shear stiffness $d_{\text{init}} = 61.111\ N/mm^3$. This last parameter was computed considering that $d_{\text{init}} = \beta_r G/l_c$, with $\beta_r = 0.05$, $G = E/2(1+\nu)$ and where l_c is the crack increment size. For the simulations performed in this paper, the crack increment is equal to finite elements height, $15\ mm$. Plane stress condition is assumed. Both experiments in sections 3.1 and 3.2 were made using Q4 elements. Numerical integration in the elements crossed by the discontinuity was performed using triangular subdivision implemented in [21], with three-point and six-point Gaussian quadrature in each triangular sub-domain for G/XFEM and SGFEM strategies, respectively. The remaining elements were integrated with 4×4 points. Finally, integration on the crack boundary was performed with two and three points for G/XFEM and SGFEM strategies, respectively.

For experiments presented in sections 3.1 and 3.2, three values of the stabilization parameter α, proposed in [2], were studied: 0.1, 0.5 and 1.0 (named SGFEM $\alpha = 0.1$, SGFEM $\alpha = 0.5$ and SGFEM $\alpha = 1.0$ at the graphics). The G/XFEM and the SGFEM as formulated by Gupta et al. [1] are identified with the labels G/XFEM and

SGFEM Lin. The non-linear analysis is performed with displacement control, with an increment of $0.0015\ mm$ in the horizontal displacement of the right support (Fig. 1), a tangent constitutive tensor approximation, a convergence absolute tolerance of 1×10^{-4} in terms of force, and a reference load $P = 1.0\ N$. Conditioning is evaluated, at each step of the analyses, using the scaled condition number, as defined by Gupta et al. [1]. The crack propagates from a initial notch, which is inserted using Heaviside enrichment functions (Figs. 2 and 5).

3.1 Experiment 1

The major goal of this section is to analyze the performances of the SGFEMs that employ the stabilization parameter (determined in a heuristic way), since in the work of Wu and Li [2], for large values of α, they do not conduct to good results. The mesh used in the experiments here presented is depicted in Fig. 2 (with 15 x 24 divisions). Fig. 3 shows the equilibrium paths obtained for each of the methods here studied, while Fig. 4 depicts the scaled condition number at each step.

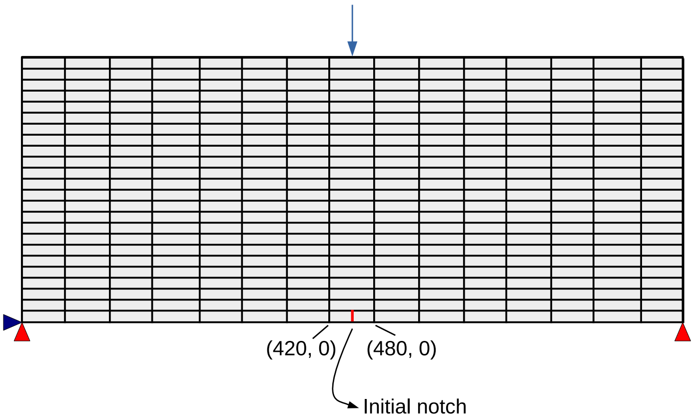

Figure 2. Mesh used in the experiments carried out in this section.

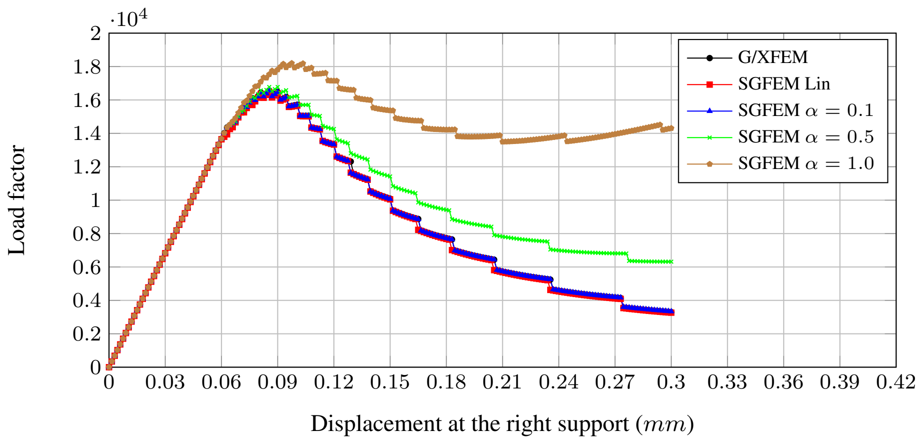

Figure 3. Equilibrium paths obtained with the mesh of Fig. 2, using different Heaviside enrichment functions.

It is possible to see in Fig. 3 that the equilibrium paths obtained with G/XFEM, SGFEM Lin and SGFEM $\alpha = 0.1$ are almost concordant. Moreover, these responses can be considered accurate in the sense of what Wu and Li [2] call stress locking-free response: the vanish load capacity is well represented during the entire path. This is not always true for larger α values as we can see, for example, with SGFEM $\alpha = 1.0$, where the response shows eventually increasing load capacity, even though the crack is propagating and opening widely. Wu and Li [2] ascribe such behavior to the incapability of the SGFEM $\alpha = 1.0$ for reproducing the relative rigid body rotations.

Fig. 4 reveals that, despite some oscillations, the best results for conditioning are obtained by SGFEM $\alpha = 0.1$. Although SGFEM $\alpha = 0.5$ and SGFEM $\alpha = 1.0$ start the simulation with the lowest values for scaled condition numbers, they suffer some instabilities during the analyze and reach comparatively large values

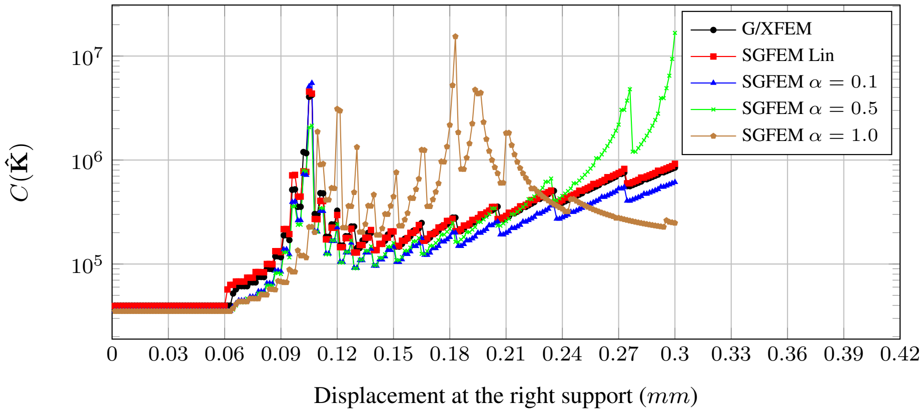

Figure 4. Scaled Condition Numbers $C(\hat{\mathbf{K}})$, at each step, computed through the mesh of Fig. 2. Different Heaviside enrichment functions are employed.

for $C(\hat{\mathbf{K}})$. This result differs from the ones obtained by Wu and Li [2], where those two approaches not only achieved relative low values for condition number but also were stable with respect to this aspect while the crack propagated. Fig. 4 also shows that G/XFEM and SGFEM Lin had similar behaviors.

3.2 Experiment 2

This section aims to study the robustness of the approaches here considered. Since Zhang et al. [15] and Oliveira et al. [13] have observed that SGFEM using linear Heaviside functions may not be well-conditioned for any mesh configuration, the authors tried to created a mesh with an unfavorable design, as shown in Fig. 5(with 14 x 24 divisions). Furthermore, the standard Heaviside enrichment function with G/XFEM can also be sensitive to the location of the discontinuity relative to element nodes [22]. Fig. 6 shows the equilibrium paths obtained for each of the methods here studied, while Fig. 7 depicts the scaled condition number at each step.

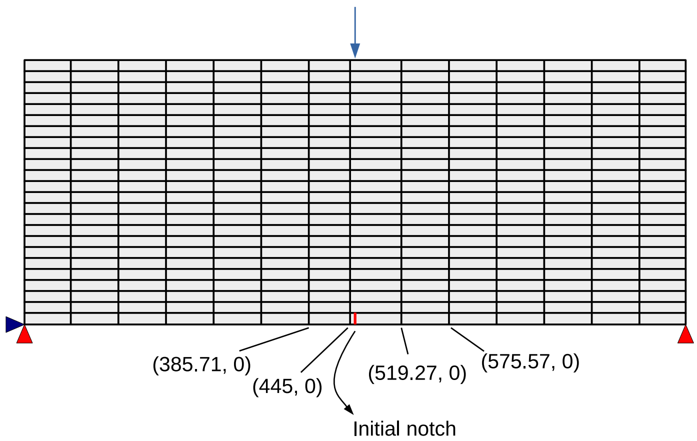

Figure 5. Mesh used in the experiments carried out in this section.

The behaviors shown in Fig. 6 are quite similar to those discussed in section 3.1. The major difference is that, in the new mesh configuration, SGFEM $\alpha = 1.0$ provided a more consistent response, even though a stress locking-free behavior is not completely achieved. The equilibrium paths obtained by SGFEM approaches with larger α are still a little more ductile than the ones provided by the other strategies.

On the other hand, Fig. 7 shows that SGFEM $\alpha = 0.1$ is robust considering the relative position between the discontinuity and the mesh. Once more, this approach reaches the best results for conditioning. It is interesting to note, however, that G/XFEM seems robust as well. In fact, the authors could not see, in the experiments made so far, ill-conditioning/instability related to the use of standard Heaviside functions in G/XFEM regardless the mesh configuration. Nevertheless, the lack of robustness of SGFEM Lin can be easily seen in Fig. 7, specially in the last

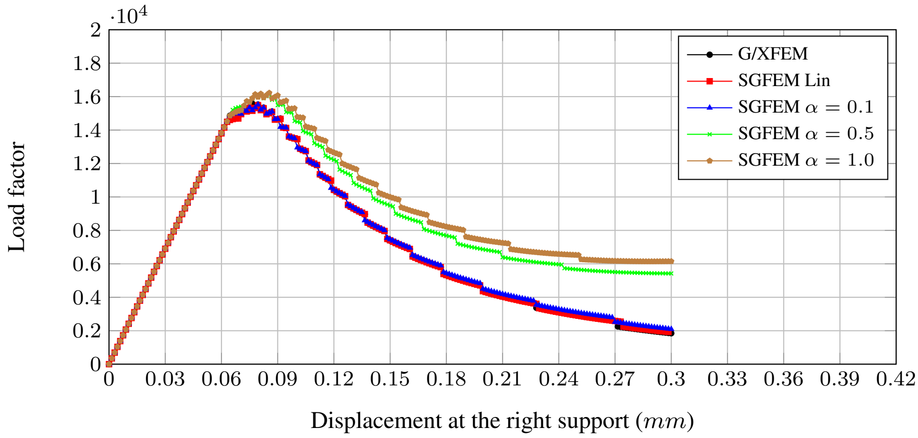

Figure 6. Equilibrium paths obtained with the mesh of Fig. , using different Heaviside enrichment functions.

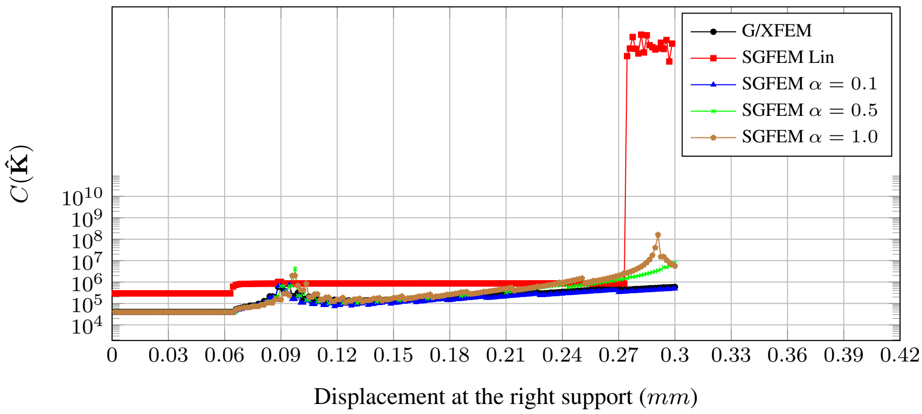

Figure 7. Scaled Condition Numbers $C(\hat{\mathbf{K}})$, at each step, computed through the mesh of Fig. . Different Heaviside enrichment functions are employed.

steps of the analysis. SGFEM $\alpha = 0.5$ and SGFEM $\alpha = 1.0$ had similar results considering the ones discussed in section 3.1.

4 Conclusions

The use of the linear Heaviside function [1] under the SGFEM strategy in a cohesive crack propagation model was investigated. This investigation was motivated by the significant accuracy obtained by this approach in linear fracture mechanics problems, in the works of Gupta et al. [1] and Oliveira et al. [13], for example. However, the lack of robustness of such strategy was once more verified. The approach proposed by Wu and Li [2] demonstrated to be, for a small value of stabilization parameter α, in the experiments carried out in this work - specially chosen for validating computational implementation - a trade-off between accuracy and conditioning. Since this parameter is determined in a heuristic way, however, more experiments are necessary to study its stability. G/XFEM using standard Heaviside function strategy has also had a good and balanced performance considering conditioning and accuracy. Since its lack of robustness [22] could not been seen in the experiments made so far, this approach can also be considered an interesting option for the simulation of cohesive crack propagation problems. Computational efficiency of both strategies may be studied in future works, as well as the influence of mesh refinement on G/XFEM conditioning, or of using polynomial and trigonometric enrichment functions, for different numerical experiments.

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