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Richardson Error Estimator and Convergence Error Estimator applied in a buckling analysis by Finite Difference Method (FDM)

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Abstract. In order to reduce the numerical error caused by discretization errors, the Richardson Extrapolation and Convergence Error Estimator were used. The main goal relies on estimating and reducing the numerical error in the analysis of a simply-supported stepped column. The estimate of the discretization error followed the approach proposed by Marchi and Silva [1-2]. The variable of interest was the critical buckling load obtained through the Finite Difference Method (FDM). The main concern regards the verification of the order of convergence for a buckling problem of continuous and stepped columns. The equivalent moment of inertia is determined at the node where the sudden cross-section variation occurs. Different ratios between moments of inertia of the cross-sections were considered. The use of the equivalent moment of inertia in the modeling reached the order of convergence 2 for Richardson Error Estimator and the convergence order 4 using Convergence Error Estimator.

Keywords: Discretization Error, Finite Difference Method, Convergence Order, Buckling, Columns.

1 Introduction

The study of convergence has been increasingly deepened due to its importance in the reliability of the proposed solutions in the most different fields of engineering. Assuming that the numerical model is defined consistently and that there are no modeling or programming errors, discretization errors are the only ones that present concern in relation to increasing precision in numerical solutions. In order to reduce discretization errors, there are some alternatives, such as mesh refinement and increased order in the proposed numerical model. However, these alternatives increase computational cost and complexity in numerical analysis. To overcome these difficulties, extrapolation techniques are useful and applicable, in this sense, Richardson's extrapolation is used as a means of overcoming difficulties in defining reliable and accurate solutions.

The column buckling using the Finite Difference Method (FDM) has been studied by some researchers for this class of problem. Rourke [3] investigated the buckling load for nonuniform columns and presented a method to determine a lower bound buckling load for pinned-pinned and cantilever columns. Iremonger [4]examined buckling loads for tapered and stepped columns that have been determined by FDM and showed that consistent and correct results can be obtained using a small number of subdivisions with extrapolation results. Krishnan *et al.* [5] showed that for the stepped column problem the results are reliable if the FDM is employed for each segment and the continuity conditions are satisfied. Soltani and Sistani [6] analyzed the elastic stability of columns with different flexural rigidities under arbitrary axial load using FDM.

The main objective of this work is to verify the order of convergence for a buckling problem of continuous and stepped columns. FDM is employed to calculate the equivalent moment of inertia at the node where the sudden cross-section variation occurs. This paper is organized as follows: Section 2 presents the numerical formulation (analytical and by FDM) of the buckling problem for continuous and stepped columns; section 3 explains how the convergence analysis is performed and presents the Error Estimators; section 4 presents a brief report on the methodological and numerical procedures; Section 5 outcomes the results and discussions. Finally, section 6 presents the conclusion.

2 The buckling problem formulation

This section presents a brief overview with regards to the formulation of the buckling problem in continuous and staggered columns. According to Timoshenko [7], the continuous columns' solution was solved originally by Leonhard Euler in 1744 and the stepped column' solution used in this work was proposed by Timoshenko and Gere [8] in the book Theory of Elastic of Stability (1963).

2.1 Buckling problem in continuous column

Considering the pinned-pinned column (Fig. 1) with rigidity EI and under axial load P. The maximum axial load that the column can support when it is about to buckle is called critical load (P_{crit}).

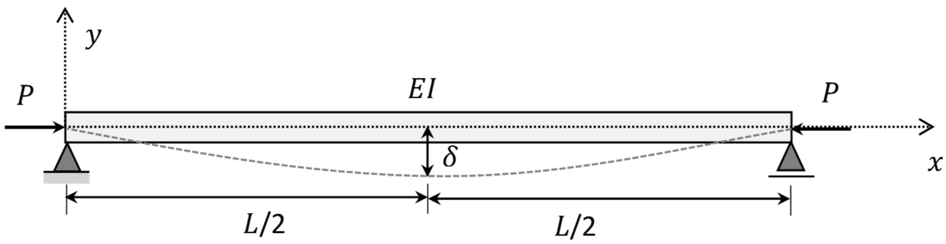

Figure 1. Continuous column buckling under axial load

Assuming the hypothesis that the column is perfectly straight before deflection and that it is made of homogeneous material and that the load is applied to the centroid of the cross-section the governing solution is

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v(x) = 0, \tag{1}$$

being $v(x)$ the lateral displacement and EI is the bending rigidity and P is the axial load. The solution this differential equation is known and defined by

$$v(x) = Asin(\psi x) + Bcos(\psi x), \tag{2}$$

where A and B are constants and

$$\psi^2 = \frac{P}{EI}. \tag{3}$$

Applying the boundary conditions

$$v(0) = 0 \qquad\qquad v(L) = 0 \tag{4}$$

in eq. (2) the critical load is determined by

$$P_{crit} = \frac{\pi^2 EI}{L_b^2}, \tag{5}$$

where L_b depends on support conditions and for this case (pinned-pinned) L_b is equal to the longitudinal length of the column.

2.2 Buckling problem in stepped column

Supposing for instance a stepped column as depicted in Fig. 2. The flexural rigidity (EI) changes suddenly and each segment must be calculated separately and then the boundary and continuity conditions must be applied. The governing equation this problem can be written as

$$\frac{d^2 v_i}{dx^2} = \frac{P}{EI_i}(\delta - v_i), \tag{6}$$

being v_i the lateral displacement of the segment i and EI_i is the bending rigidity of the segment i.

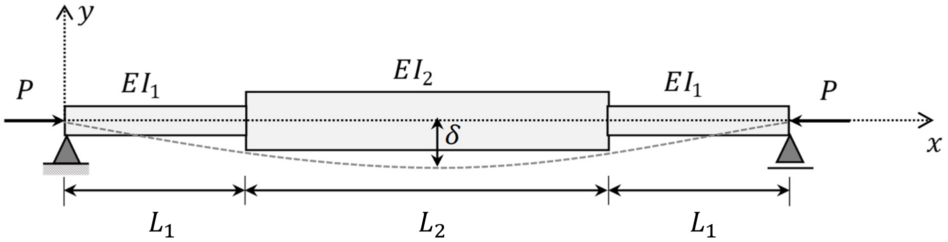

Figure 2. Stepped column buckling under axial load

The solution of the eq. (6) is

$$v_i = C_i \cos(\psi_i x) + D_i \sin(\psi_i x) + \delta, \tag{7}$$

being C_i and D_i are constants the segment i and δ is the maximum deflection. Applying the boundary conditions showed in eq. (4) and the continuity conditions, the solution of the eq. (7) is

$$\tan\left(\sqrt{\frac{P}{EI_1}}L_1\right)\tan\left(\sqrt{\frac{P}{EI_2}}L_2\right) = \sqrt{\frac{I_2}{I_1}}. \tag{8}$$

The transcendental equation (8) can be solved by iterative numerical method and the load P that satisfies the equality is the critical buckling load.

2.3 Buckling problem by Finite Difference Method (FDM)

When applying the FDM it is necessary to discretize the model into elements. The discretization is for continuous and stepped columns are depicted in Fig. 3 and Fig. 4, respectively.

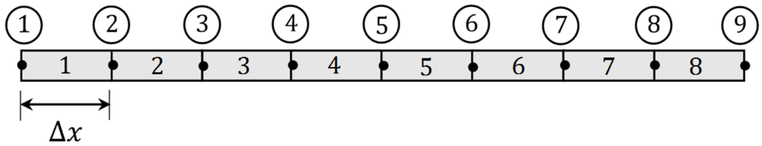

Figure 3. Continuous column discretized with 8 elements

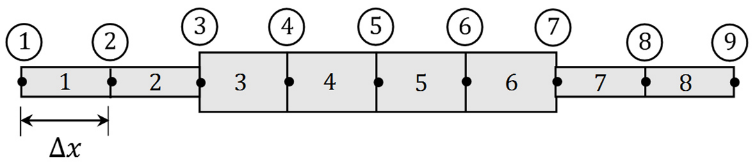

Figure 4. Stepped column discretized with 8 elements

Using the central finite difference operator the problem can be described by

$$-v_{j-1} + 2v_j - v_{j+1} = \lambda v_j, \tag{9}$$

being j the position the node considered in each approximation of the second derivative and

$$\lambda = \frac{P\Delta x^2}{EI_j}. \tag{10}$$

where Δx is the length of the element and EI_j is the flexural rigidity in node j.

In nodes 3 and 7 at the intersection of the sections (see Fig. 3) the equivalent moment of inertia was used as shown in Iremonger [4] and Krishnan *et al*. [5] and defined by

$$I_{eqv} = \frac{2I_1 I_2}{I_2 + I_2}. \tag{11}$$

Its well known that

$$(\boldsymbol{K} - \lambda \boldsymbol{M})\{\boldsymbol{v}\} = 0, \tag{12}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{v}$ is the nodal displacement vector. The eq. (12) is eingevalue problem and the critical load can be obtained by

$$P_{crit} = \frac{\lambda_{min} EI}{\Delta x^2} \tag{13}$$

being λ_{min} is the smallest eigenvalue.

3 Convergence Analysis

This paper used the approach proposed by Marchi and Silva [1-2] using the Richardson Error Estimator and Convergent Error Estimator to determine the reliability of the results obtained. The true error is determined by

$$E = \Phi - \phi, \tag{14}$$

where Φ is the analytical solution and ϕ is the numerical solution.

Using initially three mesh: fine (ϕ_1), medium (ϕ_2) and coarse (ϕ_3), the estimate of the numerical solution of the asymptotic order of the error is

$$\phi_\infty(PL) = \phi_1 + \left(\frac{\phi_1 - \phi_2}{r^{PL} - 1}\right), \tag{15}$$

being r is the grid refinement mesh ratio and PL is the asymptotic order of the error.

And the estimate of the numerical solution of the apparent order of the error is

$$\phi_\infty(PL) = \phi_1 + \left(\frac{\phi_1 - \phi_2}{r^{PU} - 1}\right), \tag{16}$$

where PU is

$$PU = \frac{\log\left(\frac{\phi_2 - \phi_3}{\phi_1 - \phi_2}\right)}{\log(r)}. \tag{17}$$

The error estimate of the numerical solution ϕ_1 must then be calculated for asymptotic (PL):

$$U_{Ri}(\phi_1, PL) = \left|\frac{\phi_1 - \phi_2}{r^{PL} - 1}\right| \tag{18}$$

and apparent order (PU):

$$U_{Ri}(\phi_1, PU) = \left|\frac{\phi_1 - \phi_2}{r^{PU} - 1}\right|. \tag{19}$$

Then, the Richardson Error Estimator (U_{Ri}) can be calculated by

$$U_{Ri}(\phi_1) = \ sg(\phi_1 - \phi_2) \max\{|U_{Ri}(\phi_1, PL)|; |U_{Ri}(\phi_1, PU)|\}. \tag{20}$$

The estimated error is considered reliable if the condition

$$\left|\frac{U_{Ri}(\phi_1)}{E\ (\phi_1)}\right| \geq 1 \tag{21}$$

was satisfied.

Using the same procedures, Marchi and Silva [1-2] proposed a convergent numerical solution given by

$$\phi_c = \frac{\phi_\infty(PL) + \phi_\infty(PU)}{2}. \tag{22}$$

The convergent error estimative of the convergent numerical solution (ϕ_c) can be defined by

$$U_c(\phi_c) = \left|\frac{\phi_\infty(PL) - \phi_\infty(PU)}{2}\right| \tag{23}$$

and the true discretization error of the convergent solution is

$$E(\phi_c) = \Phi - \phi_c. \tag{24}$$

Then, the condition showed in eq. (21) must also be checked.

4 Numerical Procedures

Initially, the exact solutions to the buckling problem were defined by eq. (5) and (8). The simulations were performed for continuous column and for a stepped column with inertia ratio $I_2/I_1 = 1.5$ and 2.0, and the meshes were obtained through a refinement ratio (r) equal to 2.

The column considered in simulations has a square section with a moment of inertia: $I_1 = 0.5$, in all cases, and $I_2 = 0.75$ and 1, respectively. The material properties used in the modeling were: Young's modulus equals 200 MPa and Poisson's ratio $\nu = 0.3$.

The simulations were performed by MATLAB using FDM with modeling starting with 8 elements up to a limit of 16384 elements. Firstly, the algorithm discretized the structure and assembled the mass and stiffness matrices as shown in the eq. (12). Then, the eigenvalue problem was defined and solved using MATLAB's predefined *'eig'* function. Finally, the critical buckling load was defined for each structure as explained in the eq. (13). The results obtained for each mesh during simulations were used in post-processing for the analysis of convergence through the error estimators described in section 3 of this paper. Finally, the results obtained are shown in section 5.

5 Numerical results

This section presents the results obtained according to the procedures shown previously and makes some important and relevant considerations for the analyzed problem. Figure 5 shows the results for continuous columns. Figures 6 and 7 present the convergence analysis for a stepped column with inertia ratio equal to 1.5 and 2, respectively. The results using the equivalent moment of inertia showed consistent and convergent as explain in Iremonger [4] and Krishnan *et al*. [5].

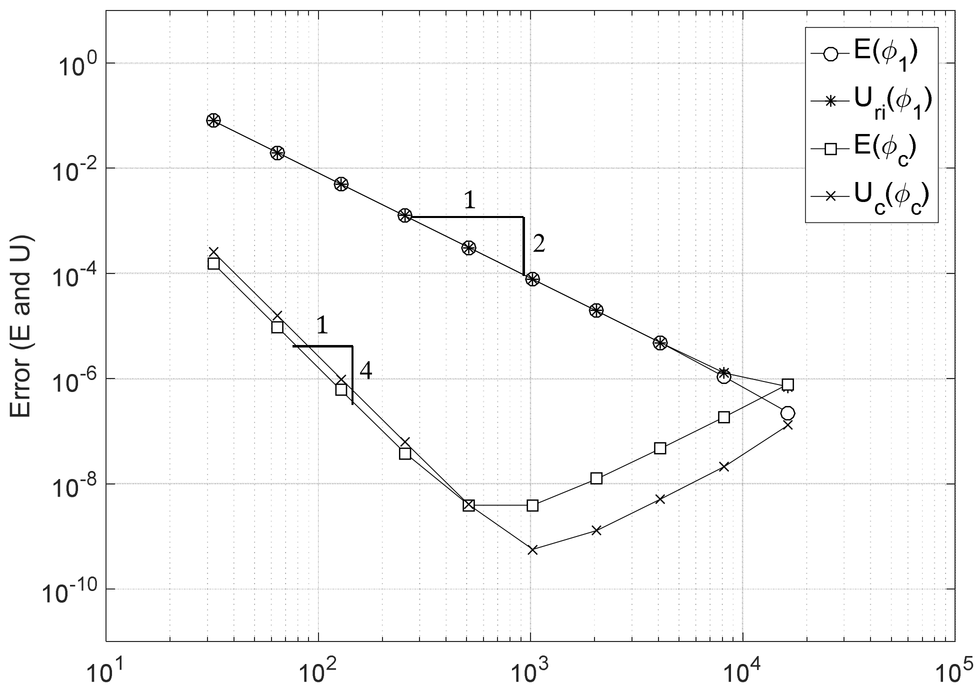

Figure 5. Convergence analysis for continuous column by FDM

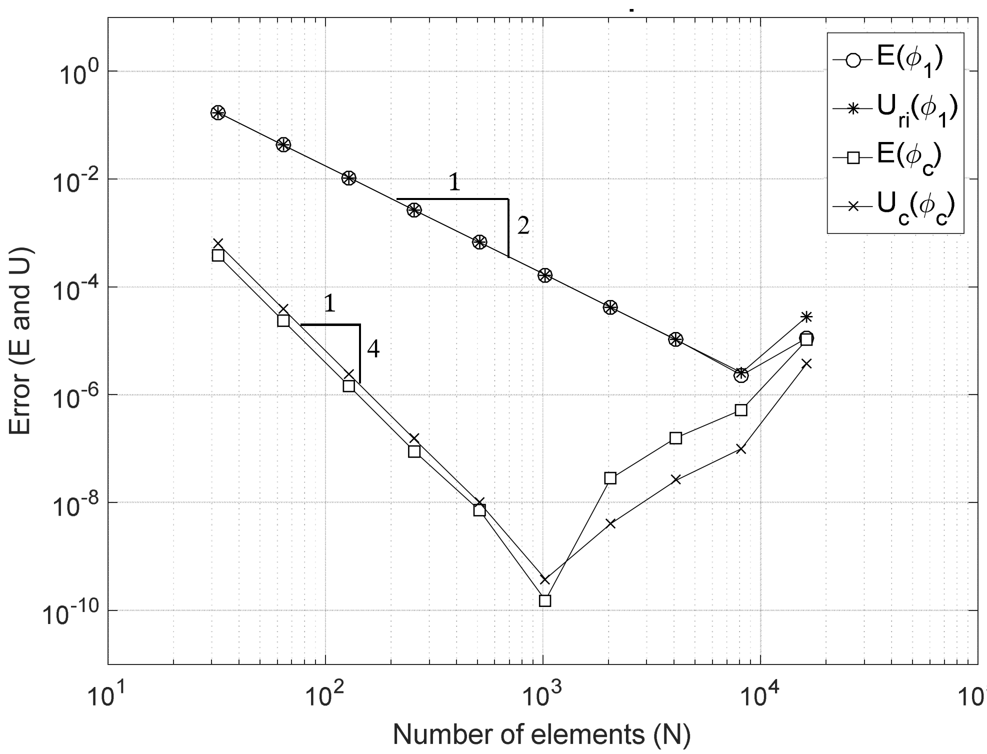

Figure 6. Convergence analysis for stepped column with inertia ratio equal 1.5 by FDM

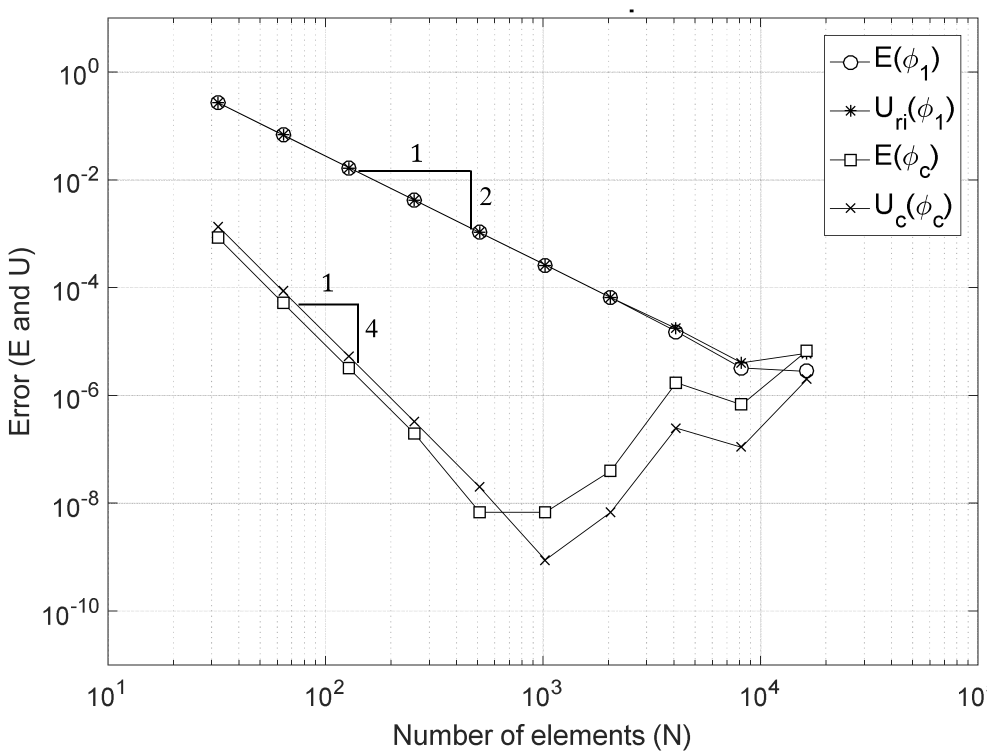

Figure 7. Convergence analysis for stepped column with inertia ratio equal 2.0 by FDM

In all cases, the true error of the numerical solution, $E(\phi_1)$, was about 10^{-6}. Checking the continuous column and the stepped column with an inertia ratio equal to 2, it turns out that the true error of the convergent numerical solution, $E(\phi_c)$, was approximately 10^{-8}. For the stepped column with inertia ratio equal to 1.5, the $E(\phi_c)$ was about 10^{-10}. In all simulations, the Richardson Error Estimator obtained an asymptotic convergence of 2nd order and followed the order of convergence of the true error, which is an important tool in the validation of the problem that also has order 2. The Convergence Error Estimator presented results with asymptotic convergence of 4th order and approached the real error until reaching a computational limit close to 10^{-9}. Lastly, the results prove what was explained by Marchi and Silva [7-8]: in a convergent interval of Pu, is recommended using a numerical convergent solution ϕ_c instead using ϕ_1, because the true error $E(\phi_c)$ is smaller than $E(\phi_1)$.

6 Conclusions

This paper studied the buckling problem in continuous and staggered columns through FDM and used the Richardson error estimator and the convergence error estimator to analyze the problem convergence. The results using the equivalent moment of inertia were convergent and consistent and the error estimators proved to be adequate for the problem analyzed. The Richardson Error Estimator obtained a convergence of 2nd order and the Convergence Error Estimator had a convergence of 4th order. Finally, the study showed the need for the study of convergence and presented some numerical tools capable of reducing computational time and validating the results obtained.

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