Physical and geometric parameters uncertainty effect on the nonlinear dynamics of cylindrical panels

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Abstract. Some physical and geometric characteristics of a structure such as radius, thickness and Young's might present, within a small margin of error, a value different from the nominal value considered in the project. Therefore, the objective of this work is to investigate the influence of the randomness of these characteristics on the nonlinear dynamic behavior of a simply supported cylindrical panel subjected to a time-dependent loading. The nonlinear equations of motion of the panel are deduced from its total potential energy and the strain-displacements relationships proposed by Donnell's nonlinear shallow shell theory. The physical and geometric parameters mentioned above are inserted individually as random variables in the partial differential equation of motion and the problem becomes stochastic due to the presence of randomness. Thus, the partial stochastic equations of motion of the cylindrical panel are discretized by the Galerkin Stochastic method combined with the Legendre-Chaos Polynomial and integrated over time by the fourth order Runge Kutta method to obtain several results to forced nonlinear vibrations of the panel for a given load. These results are compared with those obtained by the Monte Carlo simulation performed with the deterministic equations, showing the Legendre-Chaos Polynomial as a good tool for obtaining the statistical responses of the present stochastic system without the need for a sampling process.

Keywords: Random parameters, Stochastic Galerkin Method, Nonlinear dynamic analysis, Cylindrical panel.

1 Introduction

Slender structures have a light and resistant format that enable innovative designs in the fields of civil, mechanical, aerospace, naval engineering. The cylindrical panels coming from the circular sector of the cylindrical shell and it is a slender structure sensitive to change from the stable to the unstable regime, due to external excitations or excessive vibrations. The geometric non-linearity described in the non-linear relations of the deformations as a function of the displacement fields is the focus of several works of slender structures such as the cylindrical panel that present large displacements in the order of their thickness under the influence of several factors such as: initial geometric imperfections, types of loading, different geometries and materials [1-4].

Additionally, the system can also be influenced by uncertainties in its physical and geometric parameters resulting from a manufacturing error, inaccuracy or incompleteness of data, for example, and it becomes essential to investigate the impact of these random variables on nonlinear dynamic behavior of the structure. In recent years, the quantification of these uncertainties has been the subject of several studies as there is still a vast unknown field about how random parameters can influence the behavior of the system in question.

Stochastic systems resolutions are basically divided into simulation based-methods such as Monte Carlo Simulation (MCS) for example and expansion-based methods such as perturbation methods, spectral approach and stochastic reduced based methods [5]. One of the most widely used spectral approach methods is the polynomial chaos (PC) using a Galerkin scheme [6] which consists of a generalization of the expansion of Wiener's classic polynomial chaos. The generalization of the polynomial chaos (gPC) developed in [7] demonstrates that Askey's polynomials can be used to represent stochastic processes with a good convergence rate, since they have a weight function identical to weight functions of certain probability distributions [8]. The great advantage of the polynomial chaos expansion (PCE) of the system random variable is obtaining statistical information of the whole group of samples without the need for a sampling process optimizing time and computational effort.

The incorporation of randomness into systems through the polynomial chaos has already been the subject of research in some studies [5,9], but there is a lack in the research field of structural dynamic systems involving this stochastic method. The aim of this work is to contribute to this field of studies demonstrating that the PCE can be used to obtain stochastic results from uncertain systems in an efficient and accurate manner. First the stochastic equation of motion of the cylindrical panel is derived. Second, it is discussed how the stochastic and deterministic results are obtained. Third, the presentation of the results obtained by PCE and MCS are compared to each other for the specific dynamic system of this work. Finally, the influence of different uncertain parameters on the dynamic behavior of the structure is shown.

2 Mathematical formulation

Consider a thin cylindrical panel with thickness *h*, radius *R*, length *L*, opening angle θ, simply supported, composed by a linear, homogeneous and isotropic elastic material with an Young's modulus, *E*, Poisson's coefficient ν and material density ρ. Figure 1 illustrates the presented panel geometry as well as the cylindrical coordinates x, θ, z, and their respective displacement fields: u, v and w.

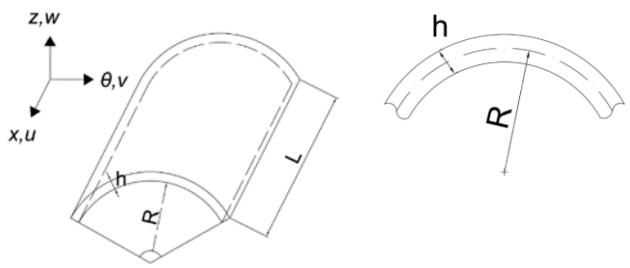


Figure 1. (a) Geometry and displacement field of the cylindrical panel and (b) detailing of the cross section

The mathematical formulation is based on Donnell's nonlinear shallow shell theory [10] which is widely used in cylindrical shell problems because it is relatively simple and precise, considering that the thickness *h* is relatively smaller than the radius *R* and length *L*, obeying the ratio $h / R \leq 20$ in an homogeneous, isotropic material with linear elastic behavior. The nonlinear equations of motion of the cylindrical panel are reduced to a set of two equations as a function of Airy stress function, $f(x,\theta)$ and the transversal displacement field:

$$\begin{gathered}\rho h\ddot{w}+\beta_1\dot{w}+D\nabla^4 w-\frac{1}{R}f_{,xx}-\frac{1}{R^2}\left[f_{,\theta\theta}\left(w_{,xx}\right)-2f_{,x\theta}\left(w_{,x\theta}\right)+f_{,xx}\left(w_{,\theta\theta}\right)\right]-p(t)=0\\ \frac{1}{Eh}\nabla f^4\left(x,\theta\right)=\frac{1}{R^2}\left(2w_{,x\theta}-Rw_{,xx}+w_{,x\theta}^2-w_{,xx}w_{,\theta\theta}-w_{,\theta\theta}-w_{,xx}\right)\end{gathered} \tag{1}$$

where *w* is the transversal displacement, β_1 is the linear viscous damping and $p(t)$ is the external lateral pressure that the cylindrical panels were subjected, as following:

$$p(t)=P_L\,\mathrm{sen}\left(\frac{m\pi x}{L}\right)\mathrm{sen}\left(\frac{n\pi\theta}{\Theta}\right)\cos\left(\omega_f t\right) \tag{2}$$

where P_L is the magnitude of the lateral pressure applied; *m* and *n* are the half-wave in the axial and circumferential directions, respectively; ω_f is the excitation frequency; and *t* is the time.

The modal expansion used to describe the transverse displacement field of the simply supported cylindrical panel is composed by two degrees of freedom, eq. (3), that represents the nonlinear behavior of the cylindrical panel for the studied geometries satisfactorily [4], minimizing both deterministic and stochastic analysis in relation to computational effort and analysis time.

$$w(x,\theta,t)=A_1(t)sen\left(\frac{m\pi x}{L}\right)sen\left(\frac{n\pi\theta}{\Theta}\right)+A_2(t)\left[\frac{3}{4}-\cos\left(\frac{2m\pi x}{L}\right)+\frac{1}{4}\cos\left(\frac{4m\pi x}{L}\right)\right]$$
$$\left[\frac{3}{4}-\cos\left(\frac{2n\pi\theta}{\Theta}\right)+\frac{1}{4}\cos\left(\frac{4n\pi\theta}{\Theta}\right)\right] \tag{3}$$

The transversal displacement field eq. (3) is replaced in the compatibility equation and the Airy stress function, *f*, is found, analytically, and replaced in the motion eq. (1). The classic Galerkin method for the discretization of the equilibrium equation is applied, giving equations, EQ_i (i = 1,2), such as:

$$EQ_i\left(A_1(t),A_2(t),p(t),\zeta_k\right)=0 \qquad (i=1,2) \tag{4}$$

dependent on the modal amplitudes of the transversal displacement field, $A_1(t)$ and $A_2(t)$, the load, *p(t)*, and the random variable, ζ_k, that will be considered in the problem.

Thickness, Young's modulus and radius are considered as a random variable independently, with uniform probability distribution. Then, the results are obtained from the use of the Legendre-Chaos Polynomial in an intrusive approach and compared with the results generated from Monte Carlo Simulation (MCS) method.

Equation (4) will be solved by the numerical method chosen to generate the deterministic samples of random variable in MCS. For the stochastic results, one more step is increased in the process. The response of the stochastic system is obtained by solving the eq. (4) expanding the modal amplitudes of the transversal displacement in terms of Legendre-Chaos Polynomial. An approximation is given by truncating the sum in a finite number of terms, *NT*:

$$A_j\left(t\right)=\sum_{i=0}^{NT}a_i\left(t\right)\Phi_i\left(\zeta_k\right) \tag{5}$$

where A_j(t) (j = 1,2) are the modal amplitude of the degree of freedom, a_i(t) (i = 0, …, *NT*) are the term of the expansion of the polynomial chaos that will be determined, and Φ_i are the term of the Legendre polynomial chaos that will be expanded as a function of the random variable ζ_k.

After incorporating randomness into the problem to rewrite the modal amplitudes as a function of the Legendre-Chaos Polynomial terms, the Galerkin Stochastic method is applied to ensure that the approximation error is orthogonal to the functional space measured by the finite polynomial base $\{\Phi_i\}$. Deterministic equations (*NT* + 1) are generated that consider the randomness of the problem, but that no longer depend on the random variable (ζ_k), only on the PCE coefficients $a_i(t)$. Then, an equivalent system of deterministic equations is set up that can be discretized in space and time by traditional numerical techniques such as the fourth order Runge-Kutta method.

According to [9], the first and second order statistical moments are obtained directly from the first order polynomial and the sum of the other polynomials corresponding to the mean and variance, respectively, according to the following relationships:

$$E\left[w(x,\theta,t)\right]=a_0(t) \qquad Var\left[w(x,\theta,t)\right]=\sum_{i=1}^{NT}a_i^2\left\langle\Phi_i^2\right\rangle \tag{6}$$

In this way, unlike the deterministic problems that illustrate the variation over time of a given modal amplitude, PCE obtains the variation of the mean and the variance over time of the whole set of samples without the need to perform a space sampling as in the MCS.

The dimensionless parameters used in the equations of this work are defined as $\tau = \omega_0 t$ and $\Omega_f = \omega_f / \omega_0$, where ω_0 is the natural frequency of the cylindrical panel, Ω_f is the excitation frequency parameter of the applied lateral load.

3 Numerical Results

In this section, for the nonlinear analysis, consider a cylindrical panel of nominal geometry with circumferential length, a_θ = 1.00 m, axial length, L = 1.00 m, thickness h = 0.01 m and radius R = 4.1665 m. The material has the following properties: $E = 2.1 \times 10^{11}$ N m^{-2}, $\nu = 0.3$, $\rho = 7860$ kg m^{-3}. The linear viscous damping is given by $\beta_1 = 2\eta_1\rho h\omega_0$, with $\eta_1 = 0.01$ and the magnitude of the lateral pressure applied eq. (2), is P_L = 9000

N m^{-2}.

Bifurcation diagrams are presented as results of nonlinear forced vibrations of cylindrical panels and they were obtained by the brute force method, which is one of the simplest methods for obtaining the bifurcation diagrams of stable permanent solutions of dynamic systems, which allows to find different types of solutions as points fixed, periodic or quasi-periodic solutions and chaotic displacements. For each loading configuration, a set of initial conditions is chosen, and the equilibrium equations of the discretized cylindrical panel are integrated over a sufficiently long time by the fourth-order Runge-Kutta method to arrive at a permanent response. After establishing the control parameter interval, at the end of each integration over the time for each control parameter, the coordinates of the Poincaré sections are stored and the bifurcation diagram is plotted [3].

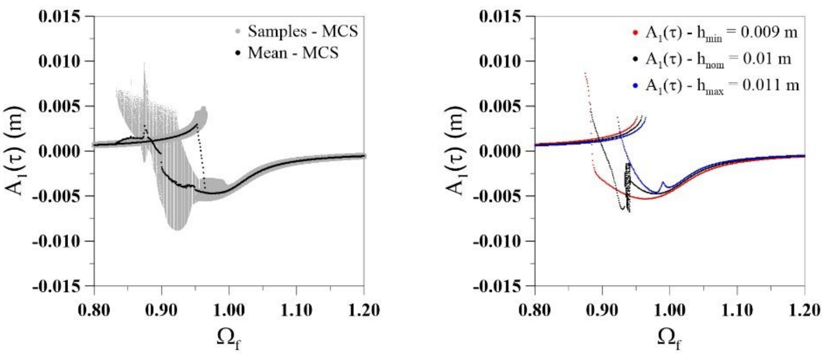
The construction of the bifurcation diagram is made from two paths, one in which the control parameter is increased to the maximum limit of the interval (forward path), and in the other one, the control parameter is decreased to the minimum limit established (backward path). In this work, the control parameter interval in both segments for all diagrams were $0.80 < \Omega_f < 1.20$. It is worth mentioning that the number of samples (n) used in the MCS to compare with PCE results was 1000 samples for each case analyzed in the present work.

For each analyzed random variable, a set of values was generated described by a uniform distribution in the interval [A, B], in which limits A and B correspond to 90 and 110% of the nominal value of the random variable, respectively, as shown in Table 1.

Table 1. Uniform distribution range of the random variables considered

Case	ζ_k	A	Nominal value	B
1	h (m)	0.009	0.01	0.011
2	E (N m^{-2})	1.89 x 10^{11}	2.1 x 10^{11}	2.31 x 10^{11}
3	R (m)	3.74985	4.1665	4.58315

Case 1 considers thickness as the random variable of the problem and Fig. 2 shows the bifurcation diagrams of the n deterministic samples in gray and the mean of all these samples for each value of the control parameter in black for the both paths. Also, the deterministic bifurcation diagram was obtained for the minimum, nominal and maximum thickness values, according to Table 1, in order to get a better visualization to the influence on nonlinear dynamic behavior when the random variable assumes values different from the nominal value.



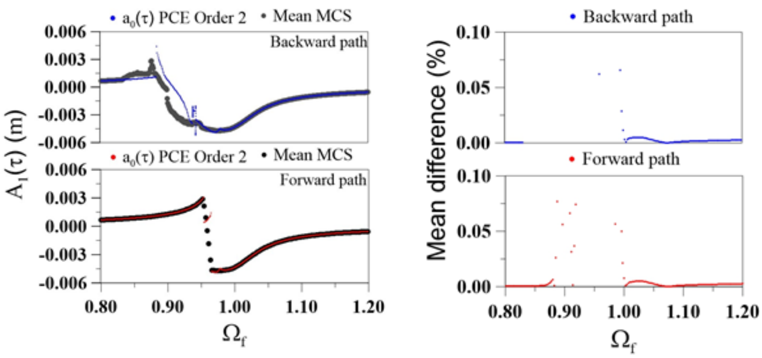
(a) (b)

Figure 2. Deterministic bifurcation diagrams considering (a) randomness in h (MCS) and (b) minimum, nominal and nominal values of thickness

It can be observed from Fig. 2 that within this uncertainty interval adopted for the random variable in panel's thickness, the bifurcation diagram may be different with bifurcation points in different positions, and for some values such as the nominal thickness value, the solutions can be observed as chaotic or quasi-periodic around the control parameter $\Omega_f = 0.93$. Therefore, depending on the value that the random variable assumes, the nonlinear behavior may be different for certain control parameters making it difficult to obtain the deterministic samples mean at those specific points.

In all cases, from the deterministic bifurcation diagrams considering $n = 1000$ samples in MCS, the mean of each segment (forward and backward paths) were obtained and compared with the first term of the PCE that represents the first order statistical moment, as shown in Fig. 3a. The percentage difference between the results obtained by the PCE and MCS methods for each control parameter is shown in Fig. 3b. The stochastic results were obtained by the PCE orders 0, 1, 2, 3 and 4 for the amplitude $A_1(\tau)$ and the results are presented for the PCE order of greatest efficiency in terms of computational effort, time and percentage differences with the MCS method.

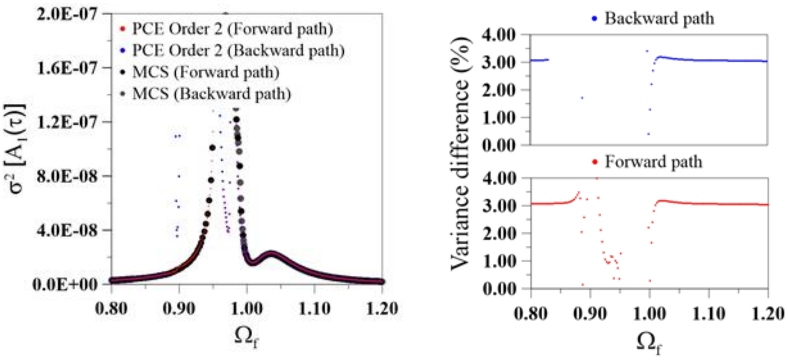
It can be noticed in Fig. 3b the good convergence of the mean obtained by the PCE order 2 with the results obtained by the MCS in the two paths with percentage differences less than 0.10% in 79% and 70% of the analyzed points in the forward path and backward path, respectively, excluding the points within the range of the control parameter $0.85 < \Omega_f < 1.00$ as already observed in the deterministic bifurcation diagrams due to the difficulty to obtain the mean in that region that has quasi-periodic solutions.



(a) (b)

Figure 3. (a) Stochastic bifurcation diagrams of amplitude $A_1(\tau)$ (PCE) and (b) percentage differences between MCS and PCE results considering randomness in h

For the second-order statistical moment Fig. 4a, good results were also observed with percentage differences less than 4.00% to 84% and 71% of the points analyzed in the forward and backward paths, respectively, Fig. 4b. These results are presented in the same control parameter range, observing larger percentage differences between the deterministic and stochastic methods for variance in the same regions of Fig. 3, extrapolating the percentage difference of 4.00% in the Fig. 4b.

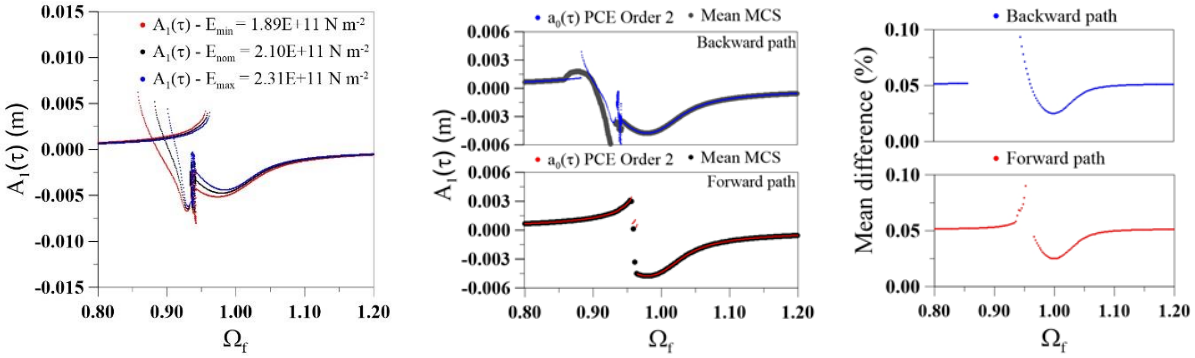


(a) (b)

Figure 4. (a) Variance of amplitude $A_1(\tau)$ considering randomness in h (PCE) and (b) percentage differences between MCS and PCE results

The second and third cases considers the Young's modulus and radius as the random variable of the problem, respectively. The loading of Case 1 remains to facilitate comparison within the same geometry. The bifurcation diagram for the three values of the random variable (minimum, nominal and maximum) obtained by the MCS method is also shown for these cases in Figs. 5a and 6a.

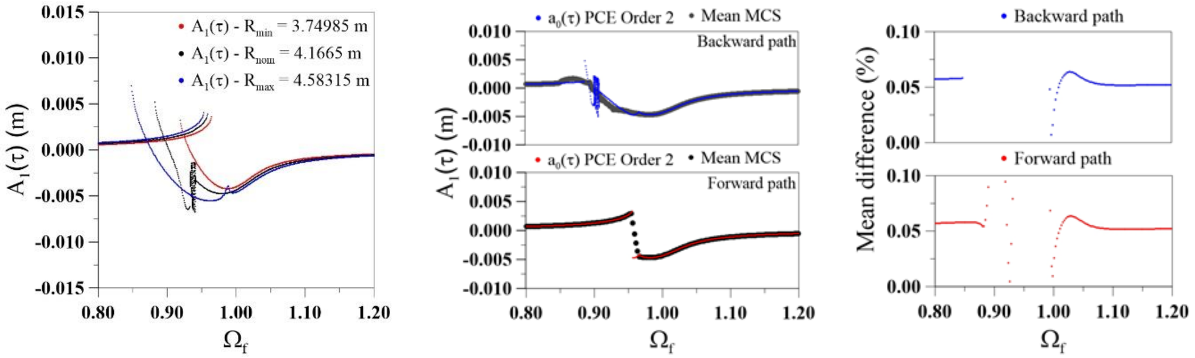
It is observed in Case 2, Fig. 5, differently from the previous case, deterministic samples generate bifurcation diagrams with the same shape and only around the control parameter $\Omega_f = 0.93$ the Poincaré sections might correspond to a quasi-periodic solution in this region, Fig. 5a. However, the randomness in Young's modulus caused the structure to have the same type of solution for the various samples of Young's modulus. This may be associated with the fact that this variable is a linear term in the nonlinear equilibrium equation of the problem, having less influence on the nonlinear dynamic behavior of the cylindrical panel. As in Case 1, the PCE order 2 showed excellent results for the mean, Fig. 5b, with almost 98% and 89% of the analyzed points over the control parameter range with percentage differences between the two methods less than 0.10%, as shown in Fig. 5c.



(a) (b) (c)

Figure 5. (a) Deterministic bifurcation diagrams (MCS), (b) Stochastic bifurcation diagrams of amplitude $A_1(\tau)$ (PCE) and (c) percentage differences between MCS and PCE results considering randomness in E

Case 3 considers the radius as the random variable and as in Case 1, for the nominal value of the random variable, the bifurcation diagram presents next to the control parameter Ω_f = 0.93 a quasi-periodic solution, Fig. 6a. Also, in Case 3 the PCE order 2 was the one that generated the best results in comparison with the other orders. Thus, in the Fig. 6b are presented the mean results with approximately 78% and 71% of the analyzed points with percentage differences less than 0.10% for the one-way and way back paths, respectively, Fig. 6c.



(a) (b) (c)

Figure 6. (a) Deterministic bifurcation diagrams (MCS), (b) Stochastic bifurcation diagrams of amplitude $A_1(\tau)$ (PCE) and (c) percentage differences between MCS and PCE results considering randomness in R

4 Conclusions

In the present work the influence of the randomness of some physical and geometric characteristics such as radius, thickness and Young's modulus on the nonlinear dynamic behavior of a simply supported cylindrical panel under a time-dependent lateral pressure was investigated. The panel's nonlinear equations of motion were deduced by Donnell's non-linear shallow shell theory. The physical and geometric parameters mentioned above were inserted individually as random variables in the partial differential motion equation that were discretized by the Galerkin Stochastic method combined with the Legendre-Chaos polynomial and integrated along time by the fourth order Runge Kutta method to obtain the results of forced nonlinear vibrations of each analyzed case. These results were compared with those obtained by Monte Carlo simulation performed with deterministic equations. It was possible to notice from the analyzed cases the excellent convergence to the first order statistical moment of the stochastic results obtained by the PCE method with the results obtained by the MCS in a large part of the analyzed control parameter range, except for the regions where the quasi-periodic solutions are presented, demonstrating in this way that the intrusive Polynomial Chaos Expansion associated with the Galerkin Stochastic method can be a good tool for obtaining statistical moments results in a dynamic nonlinear analysis without the need for a sampling analysis.

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