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Nonlinear dynamic analysis of a partially fluid-filled tank

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Abstract. The nonlinear dynamic analysis of a partially fluid-filled tank is proposed considering the coupling between the sloshing phenomena and the flexible tank. Donnell non-linear shell theory is considered to describe a simply supported cylindrical tank while the fluid is based on the velocity potential theory that describes an inviscid, irrotational and incompressible fluid. As the fluid is linear, it is possible to consider the velocity potential in two parts: the first one that describes a flexible cylindrical tank without the sloshing and the second one that considers the effect of the sloshing in a rigid tank. These hypotheses are guaranteed with a convenient choice of expressions for the velocity potentials that obey boundary conditions equivalent to these considerations. The equivalent hydrodynamic pressure is obtained by the velocity potential and applied in the equations of the cylindrical tank as an external force while the free surface equilibrium equation must also be considered together with the cylindrical tank's equation. The set of the final discrete equations to be solved is obtained applying the standard Galerkin method in the tank's equations and in the free surface equation, creating a coupled system. First of all, the linear problem is considered to obtain the natural frequencies for the cylindrical shell (bulging modes) and the sloshing. Afterward, the dynamic nonlinear analysis takes place with the special attention to the frequency-amplitude relations, phase-portraits and resonance curves that they evaluate the nonlinear oscillations of the cylindrical tanks.

Keywords: tank, sloshing, nonlinear dynamic.

1 Introduction

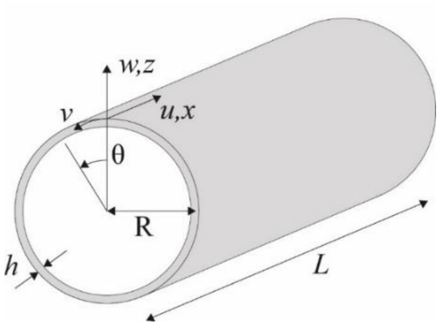
A cylindrical shell is a type of structure that has a complex mechanical behavior when submitted to static or dynamic loads due to the high level of nonlinearity presented in its equilibrium equations, that influences the structure's mechanical response when it develops large displacements. As example of the cylindrical shells' applications, the liquid storage is one of the most important. In this scenario it is highly recommended to take into account the fluid-structure interaction, that will increase the complexity of the problem, changing the structural behavior mostly in the dynamic scenario because of the increase of the structural system's mass. In certain cases, the fluid will have a free surface condition, so it is also possible to study how this surface will oscillate and how it can influence the behavior of the rest of the fluid and the shell as well.

The literature presents several references about the fluid-structure interaction in tanks. A great review of previous studies about this issue was made by Amabili and Païdoussis [1]. Gonçalves and Batista [2] is another work that is also a good example of this subject investigation, where they analyzed the influence of the liquid level in the natural frequencies using the Rayleigh-Ritz technique. In both works mentioned previously, the free surface vibration (sloshing) was not considered. But it can be seen in the study of Kim, Lee and Ko [3], where the velocity potential was used to describe the fluid movement, satisfying the interaction between the fluid and the shell as well as the sloshing condition. Gonçalves, Silva e Del Prado [4] and Pelicano e Amabili [5] are works that performed a nonlinear dynamic analysis of partially fluid-filled tanks but without the considerations of the sloshing phenomena.

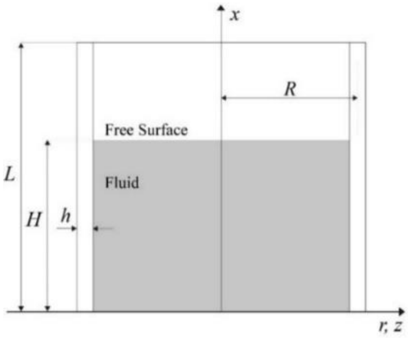
This work presents the dynamic analyses of a partially fluid-filled tank applying the nonlinear Donnell shallow shell theory to model a simply supported cylindrical shell. The fluid is incompressible, non-viscous and linear so a velocity potential is adopted in its discretization, obeying the Laplace equation. The sloshing phenomena is considered by means of a parcel included in the velocity potential. The Galerkin method is applied in the shell equations as wells as in the sloshing equations, obtaining a coupled system. The vibration amplitudes for both shell and sloshing are discretized in time and the system is linearized from where the natural frequencies of bulging and sloshing modes are obtained. The nonlinear dynamic analysis takes place then with the application of a harmonic load from where the frequency-amplitude relations, phase-portraits and resonance curves are obtained to evaluate the nonlinear oscillations of the cylindrical tanks.

2 Problem formulation

Consider a thin-walled cylindrical shell of length *L*, thickness *h* and radius *R* as shown in Fig.1a. The shell is made by a linear elastic material with Young's modulus *E*, Poisson's ratio *ν* and density *ρ*. The displacement field is represented by *u*, *v* and *w* in *x*, *θ* and *z* directions, respectively.



(a)



(b)

Figure 1. (a) Cylindrical shell and geometric parameters and (b) partially fluid-filled cylindrical shell

The nonlinear equilibrium equation obtained from Donnell's shallow-shell theory is given by:

$$\rho h\,\ddot{w} + D\nabla^4 w + \beta_1 \dot{w} + \beta_2 \nabla^4 \dot{w} + R\, f_{,xx} - \left(f_{,\theta\theta} w_{,xx} - 2 f_{,x\theta} w_{,x\theta} + f_{,xx} w_{,\theta\theta} \right) - p_H + p_L = 0\,, \quad (1)$$

where β_1 and β_2 are the linear viscous damping and the viscoelastic material damping coefficients, p_H is the hydrodynamic pressure provoked by the internal fluid, *D* (=$Eh^3/[12(1-\nu^2)]$) is the flexural rigidity, p_L is the external time-dependent pressure and the $\nabla^4 w$ operator is given by

$$\nabla^4 w = w_{,xxxx} + \frac{2}{R^2} w_{,xx\theta\theta} + \frac{1}{R^4} w_{,\theta\theta\theta\theta}\,. \quad (2)$$

The stress function *f* is obtained from the equation of compatibility expressed as:

$$\nabla^4 f = \frac{Eh}{R^4}\left(w_{,x\theta}{}^2 - w_{,xx} w_{,\theta\theta} + R\, w_{,xx} \right). \quad (3)$$

The shell is partially filled by fluid, where *H* is the liquid level and the position of the free surface, as can be seen in Fig. 1b. The fluid is irrotational, incompressible and non-viscous, so its motion can be described by a velocity potential $\varphi(x, r, \theta, t)$ that must satisfy the Laplace equation.

It is also considered that the fluid's motion has small amplitude so it can have a linear behavior. Therefore, the fluid acts on the shell by means of a hydrodynamic pressure obtained from the Bernoulli's equation that can be linearized [7]. Then, the hydrodynamic pressure can be written as eq. (4), where ρ_f is the density of the fluid.

$$p_H = -\rho_f \left[\dot{\varphi}\right]\Big|_{r=R}\,. \quad (4)$$

In this linear approach, the velocity potential can be calculated into two parts in such way as $\varphi=\varphi 1+\varphi 2$, where φ_1 is associated to a flexible tank with a rigid bottom while φ_2 concerns the velocity potential of the sloshing in a rigid tank. Each part of the velocity potential must satisfy the proper boundary conditions, so φ_1 is zero at the free surface ($x = H$) and the fluid velocity is zero at the bottom ($x = 0$). The velocity of the shell and the fluid must be the same at the interface ($r = R$). In other words, the displacement is coupled with respect to the radial velocities. These boundary conditions can be written as:

$$\varphi_1\big|_{x=H} = 0 \ , \ \left.\frac{\partial \varphi_1}{\partial x}\right|_{x=0} = 0 \ , \ \left[\frac{\partial \varphi_1}{\partial r} = \dot{w}\right]_{r=R} . \tag{5}$$

In other hand, the bottom and the tank walls must be completely rigid for φ_2, as:

$$\left.\frac{\partial \varphi_2}{\partial x}\right|_{x=0} = 0 \ , \ \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=R} = 0 \ . \tag{6}$$

The Euler equation must be satisfied at the free surface so that the equilibrium can be guaranteed. Therefore, the free surface equilibrium equation, or sloshing equation, can be written by means of the velocity potential as:

$$\frac{\partial^2 \varphi_2}{\partial t^2} + g\frac{\partial}{\partial x}\left(\varphi_1 + \varphi_2\right) = 0 . \tag{7}$$

From the boundary conditions previously defined, the velocity potential φ_1 is assumed as eq. (8) [3], where I_n is the modified Bessel function of first class and order n and $f_m(\theta)$ is the same harmonic function in θ used in the mode m from the transversal displacement field, w.

$$\varphi_1 = \sum_{m=1}^{M}\sum_{\bar{m}=1}^{\bar{M}} A_{m\bar{m}}(\mathrm{t})\cos\left(\frac{(2\bar{m}-1)\pi x}{2H}\right) I_n\left(\frac{(2\bar{m}-1)\pi r}{2H}\right) f_m(\theta) . \tag{8}$$

The velocity potential φ_2, in its asymmetric ($n > 0$) and axisymmetric ($n = 0$) modes can be expressed by eq. (9), where J_n is the Bessel Function of the first kind of order n while J_0 has order zero. $M\theta$ is the number of functions $f_i(\theta)$ that appears in transversal displacement field, w. The terms ε_{ik} and ε_{0k} are obtained from one of the boundary conditions for φ_2, where the velocity at the bottom must be zero, as given in the second equation of eq. (6). Applying this expression in eq. (9), ε_{ik} and ε_{0k} are the respective roots. After defining the expressions for φ_1 and φ_2, the Galerkin method is applied at the sloshing equation, eq. (7), with $J_n(\varepsilon_{kn}\ r/R)r$, or $J_0(\varepsilon_{0k}\ r/R)r$, as weighting function, chosen properly according with the φ_2 definition. These equations must be solved in a coupled system with the shell equilibrium equations.

$$\phi_{2i} = \sum_{i=1}^{M\theta}\sum_{k=1}^{K} B_{ik}(\mathrm{t})\,\omega h \cosh\left(\frac{\varepsilon_{ik} x}{R}\right) J_n\left(\frac{\varepsilon_{ik} r}{R}\right) f_i(\theta) , \ \phi_{20} = \sum_{k=1}^{K} B_{0k}(\mathrm{t})\,\omega h \cosh\left(\frac{\varepsilon_{0k} x}{R}\right) J_0\left(\frac{\varepsilon_{0k} r}{R}\right) . \tag{9}$$

3 Numeric results

The cylindrical shell has simply supported ends and the transversal displacement field, w, for the linear analysis is presented also with the time discretization as:

$$w = c_1 \sin\left(\frac{m\pi x}{L}\right)\cos\left(n\theta\right)\cos\left(\omega t\right) . \tag{10}$$

where m is the longitudinal half-wave number, n is the circumferential wave number and ω is the modal frequency.

Table 1. Comparison of natural frequencies for bulging and sloshing modes (rad/s) ($n = 4$)

Mode	ω_1			ω_2		
m	Ref. [3]	Present	Dif. (%)	Ref. [3]	Present	Dif. (%)
1	14.054	14.396	2.43	1.4427	1.4398	0.20
2	34.672	36.620	5.62	1.9085	1.9081	0.02
3	49.629	48.906	1.46	2.2308	2.2306	0.01
4	61.556	59.224	3.79	2.5029	2.5002	0.11
5	71.476	69.360	2.96	2.7445	2.7613	0.61

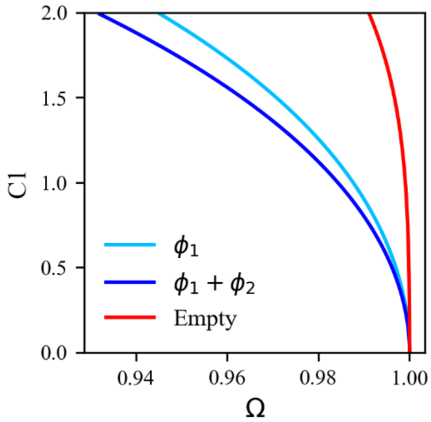
To verify the validity of this formulation, the natural frequencies were obtained and compared with the results presented by Kim, Lee and Koo [3], as shown in Tab.1 where ω_1 is the frequency of the bulging modes and ω_2 is the frequency of the sloshing modes. The cylindrical shell has the dimensions $L = 30$ m, $R = 25$ m and $h = 0.03$ m and it is made of steel with $E = 206$ GPa, $\nu = 0.3$ and $\rho = 7850$ kg/m^3. The internal fluid is water with $H = 21.6$ m and $\rho_f = 1000$ kg/m^3. The agreement between the results is good with the largest difference being 5.62% for the bulging modes and 0.61% for the sloshing modes.

The analysis of the nonlinear free vibration takes place then, where the frequency-amplitude relations are obtained by applying the Galerkin-Urabe Method [10] on the undamped equations of the system. As a nonlinear analysis, a more accurate transversal displacement field, w, that it can consider the modal coupling, and the time discretization for each mode employed are required. From the perturbation method [11], w is defined in the eq. (11), and the nonlinear discretization in time for each one of the amplitudes is presented in eq. (12).

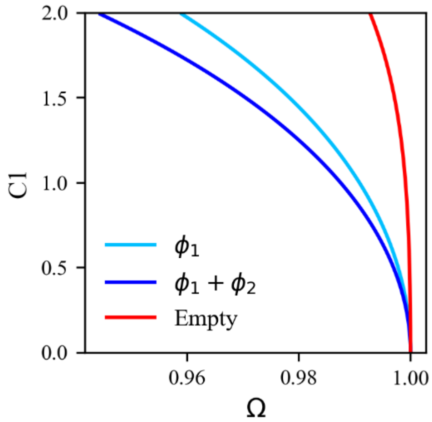
$$
\begin{aligned}
w = {} & C_1(t)h\,\mathrm{sen}\left(\frac{m\pi x}{L}\right)\cos(n\theta)+C_2(t)h\,\mathrm{sen}\left(\frac{m\pi x}{L}\right)\cos(3n\theta)+C_3(t)h\,\mathrm{sen}\left(\frac{3m\pi x}{L}\right)\cos(n\theta) \\
& +C_4(t)h\,\mathrm{sen}\left(\frac{3m\pi x}{L}\right)\cos(3n\theta)+C_5(t)h\left[\frac{3}{4}-\cos\left(\frac{2m\pi x}{L}\right)+\frac{1}{4}\cos\left(\frac{4m\pi x}{L}\right)\right] \\
& +C_6(t)h\left[\frac{3}{4}-\cos\left(\frac{2m\pi x}{L}\right)+\frac{1}{4}\cos\left(\frac{4m\pi x}{L}\right)\right]\cos(2n\theta)
\end{aligned}
\quad . \qquad (11)
$$

$$
C_1(t)=C_2(t)=C_i\cos(\omega t),\; C_3(t)=C_4(t)=C_i\cos^3(\omega t),\; C_5(t)=C_6(t)=C_i\cos^2(\omega t). \qquad (12)
$$

The problem is studied, from now, based in two different geometries for the cylindrical shell. The first one, called G1, has $L = 30$ m, $R = 25$ m and $h = 0.03$ m and the second one, called G2, has $L = 30$ m, $R = 25$ m and $h = 0.025$ m. The internal fluid level is $H = 21.6$ m for both geometries. The vibration mode of the lowest natural frequency is (m, n) = (1, 11) for G1 and G2 with the natural frequencies $\omega_1=4.923$ rad/s (bulging mode) and $\omega_2=2.229$ rad/s (sloshing mode) for G1 and $\omega_1=4.231$ rad/s (bulging mode) and $\omega 2 = 2.221$ rad/s (sloshing mode) for G2. The results are obtained for an empty tank and a partially fluid-filled tank with, or without, the sloshing consideration. Then, the frequency-amplitude relations are presented in Fig. 2 with a parameter of frequency Ω ($=\omega/\omega_0$), where it can be seen a similar behavior for both cases with a softening behavior when the fluid is considered, which is even sharper when the sloshing phenomena is take into account.



(a) G1



(b) G2

Figure 2. Frequency-Amplitude Relations for the geometries G1 and G2

When an external time-dependent pressure is applied to the tank with the shape of the first mode, as shown in eq. (13), the resonance curves are obtained for geometries G1 and G2 with respect to the variation of the load frequency ω_f, applying the fourth order Runge-Kutta method. The results are shown in Fig. 3 for $P_L = 10$ N/m^2, $\beta_1 = 2\eta_1\rho h\omega_0$ with $\eta_1 = 0.01$, $\beta_2 = 0$ and ω_0 is the lowest natural frequency. The higher values for $C1_{max}$ appear next to the frequency of resonance, with a similar behavior for both geometries. It can also be noted that, for this amplitude $C1$, there is no change in the behavior of the cylindrical tank in the sloshing resonance region. Near to

$\Omega = \omega_f / \omega_0 = 0.45$ for G1 and near to $\Omega = \omega_f / \omega_0 = 0.52$ for G2, where the resonance of the sloshing frequency would occur, the amplitude for the fundamental vibration mode is not excited. It is important to notice that the consideration of the free surface in the problem, as already observed in Fig. 2, change the resonance curves (blue curves), leaving them with more loss of stiffness when compared to the same problem that does not consider the free surface of fluid (cyan curves).

$$p_L(t) = P_L \operatorname{sen}\left(\frac{m\pi x}{L}\right)\cos(n\theta)\cos(\omega_f t). \tag{13}$$

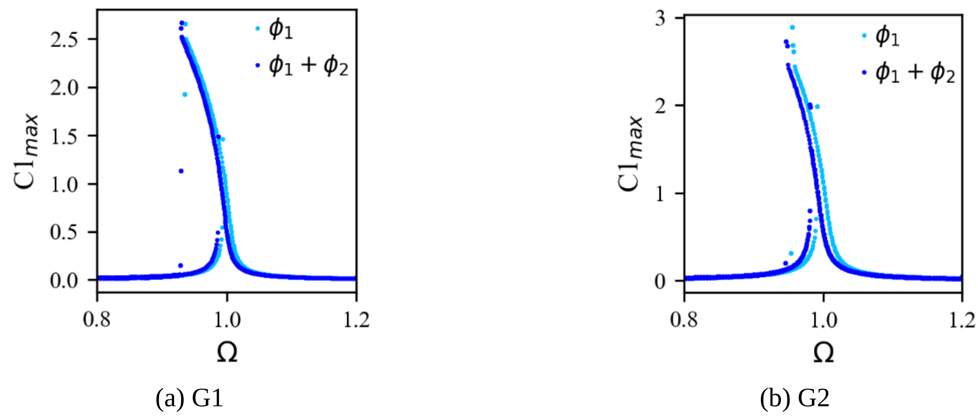


Figure 3. Resonance curves to fundamental vibration mode for the geometries G1 and G2

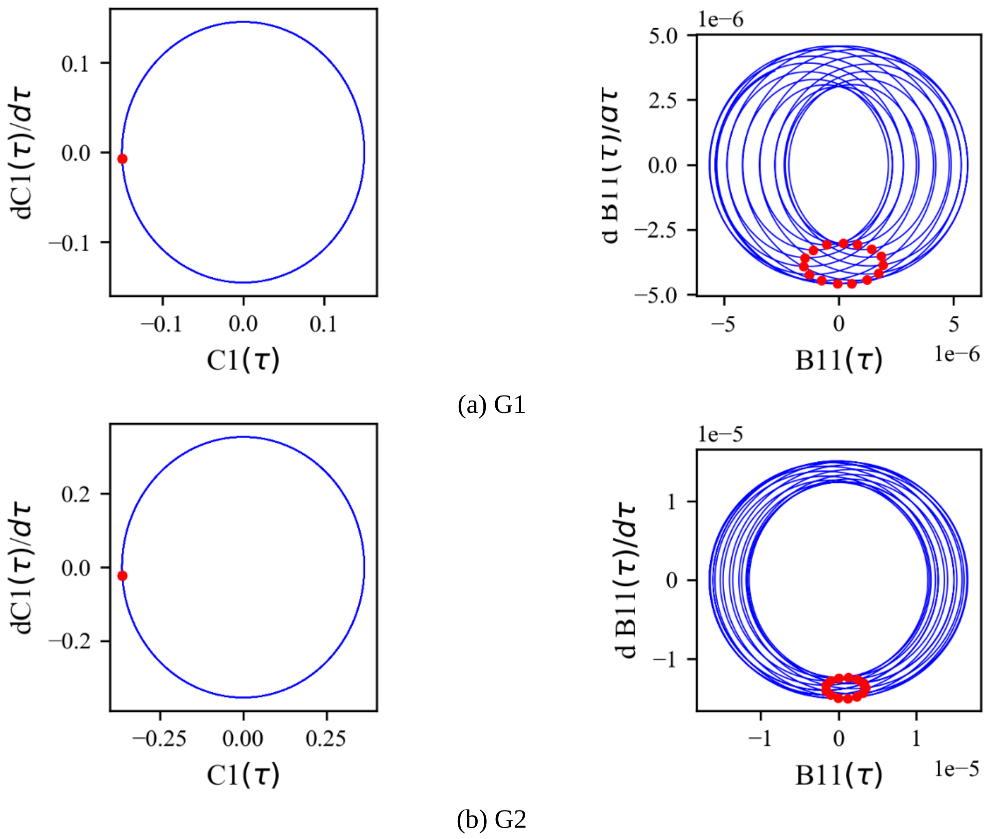
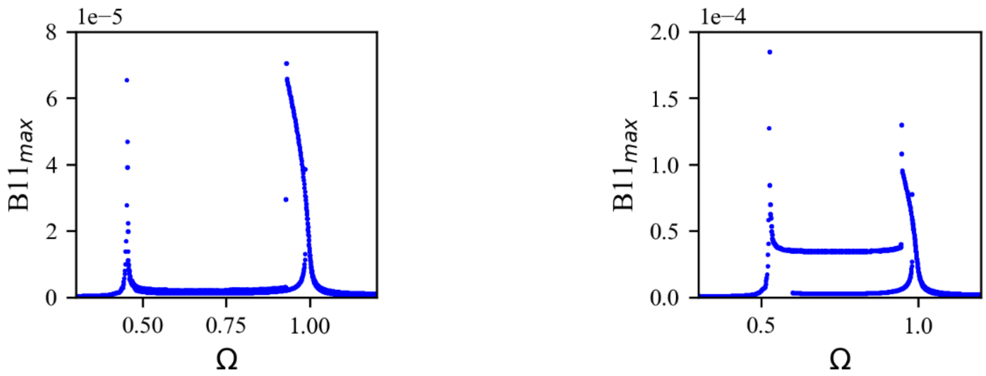


Figure 4. Frequency-Amplitude Relations for the geometries G1 and G2

The phase portraits are shown in Fig 4. for the *C1* (shell amplitude) for *B11* (sloshing amplitude). Both of them are related with the fundamental vibration mode. The initial conditions were the equivalent Poincaré sections from the resonance curves for *Ω = 0.975* in both geometries. In each one of the studied cases, a harmonic variation over time for *C1* is observed while for *B11* it is observed a quasi-periodic oscillation.

Figure 5 presents the resonance curves obtained for the *B11* sloshing amplitude for both geometries. It can be noted that there are two regions where the resonance occurs: the sloshing natural frequency where the *Ω* parameter assumes the value of *0.45* for G1 and 0.52 for G2, and the cylindrical shell natural frequency, with *Ω* near to one. Thus, different from *C1*, the sloshing amplitude can be excited on the two resonance regions of the problem.



(a) G1

(b) G2

Figure 5. Resonance curves to sloshing fundamental vibration mode for the geometries G1 and G2

4 Conclusion

In this work, a partially fluid-filled simple supported cylindrical tank is studied by applying the Donnell's nonlinear theory to describe the shell's displacement field and a velocity potential to treat an inviscid, irrotational and incompressible internal fluid, considering also the sloshing. The fluid inserts a hydrodynamic pressure on the internal surface of the shell and it is also necessary to guarantee the fluid free surface equilibrium. The standard Galerkin Method was applied and a system of equations, coupling the cylindrical shell's amplitudes and the sloshing amplitudes, was solved. The presented formulation obtained results in agreement with other works for the linear free vibration analysis. The importance of the sloshing consideration can be seen because of the changes in the both stiffness and mass of the structural system on the frequency-amplitude relations and resonance curves. The phase-portraits showed a periodic response for the cylindrical tank amplitude and a quasi-periodic response for the sloshing amplitude. On the resonance curves, the cylindrical shell amplitude is excited only on the resonance region of the cylindrical shell while the sloshing amplitude is excited on the resonance region of both sloshing and cylindrical shell.

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