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The Hierarchical Finite Element Method applied to dynamics analysis of Kirchhoff-Love plates

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Abstract. The theoretical model proposed by Kirchhoff-Love, also referred to as the thin plate model, once applied to the dynamic analysis, is useful on several problems in Engineering, such as seismic effects on slabs, dynamic impacts of aircraft on airport runways and industrial floor structures subject to machinery activities. There is a lot of numerical methods that intend to provide an approximate solution for this problem, for an example the conventional Finite Element Method (FEM), the p-Fourier Method and the Hierarchical Finite Element Method (HFEM). The aim of this paper is to evaluate the efficiency of the HFEM applied to free vibration analysis of thin plates. The HFEM improves the accuracy of the solution by adding hierarchical shape functions of higher order. This does not require a change in the mesh and in the number of element nodes, which would be necessary in the conventional FEM. The results obtained by HFEM are compared to reference analytical solutions found in literature and to other numerical methods, such as the conventional FEM and the p-Fourier Method.

Keywords: HFEM, FEM, Kirchhoff-Love plates, vibration analysis.

1 Introduction

There is a classic literature on the theoretical aspects related to the plate vibration problem such as the process for obtaining the differential equation, different coordinate systems and more. All these points were well compiled and presented by Leissa [1]. The earlier approximated approaches dedicated to solve the plate vibration problem consisted on essentially energy methods [2]. In this context the Finite Element Method (FEM) arose as a particular application of the Rayleigh-Ritz method, in which by chosen a basis of admissible functions related to the node values on convex subdomains, the approximated behavior of the solution is locally described in each subdomain. In order to improve the precision of the numerical solution, some improvements of the conventional FEM have been proposed. One of them is the Hierarchical Finite Element Method (HFEM) that adds high order polynomial functions in the classical FEM shape functions space. Bardell [3] conducted a large study about the HFEM applied to free vibration of plates and cited at least four advantages of this approach: (a) it does not require change of the mesh size; (b) the eigenvalues related to the new stiffness and mass matrices monotonically converge to the analytical ones by upper values; (c) equal order eigenvalue problems leads to results of HFEM better than FEM, also called as the Inclusion Principle [4]; (d) in some cases, it is possible to simulate a simple structure by one hierarchical element mesh, overcoming the continuity problems at the edge.

Some researchers also proposed the inclusion of polynomial and non-polynomial functions in the FEM shape functions basis. The Generalized Finite Element Method (GFEM) [5][6] allows the inclusion of some a priori knowledge about the solution of the problem in the approximate space by the Partition of Unity Method (PUM). Leung and Chan [7] proposed a method, called p-Fourier method, in which trigonometric functions multiplied by polynomial functions are included in the FEM approximate space.

Recently many researches have been focused their efforts to obtain more precise results to the thin plate free vibration problem. Different approaches and methods have been used in order to meet their aim: (a) semi-analytical methods such as the superposition of different in-plane vibration patterns, applied to better describe the in plane propagation of waves on thin plates [8]; (b) the Applied Element Method (AEM) [9]; and (c) the use of a family of two-parameter homogenization functions in a meshless approach [10].

In this paper the free vibration of a square simple-supported plate is analyzed in order to evaluate the HFEM accuracy. The six first plate natural frequencies were calculated analytically, by the h-refinement of FEM, by the HFEM and by the trigonometric enrichment of the p-Fourier method.

2 Free vibration of Kirchhoff-Love plates

First of all, lets deal with the governing mathematical model, i.e. the differential equation of the free vibration of thin plates. A plate is a model of a deformable solid for which one of the normal strains approaching zero, subjected or not to loads essentially in this same direction, what leads to bending moments in both other directions. The governing differential equation known as the differential equation of the classical theory for free vibration of plates is [1]:

$$\left(\boldsymbol{\nabla}^4 - \frac{\rho}{D}\frac{\partial^2}{\partial t^2}\right) w = 0, \tag{1}$$

that is an expression in terms of the vertical displacement w, the biharmonic operator $\boldsymbol{\nabla}^4$, mass density ρ and flexural stiffness $D = Eh^3(12(1-\nu^2))^{-1}$, where E is the Young Modulus, h is the plate thickness and ν is the Poisson ratio. If the plate thickness h can be considered as sufficiently small, the plate corresponds to a Kirchhoff-Love plate. This consideration leads to a formulation in which the rotations and vertical displacements are taken as differentially decoupled what represent an important consideration for when numerical approaches will be used.

3 Weak form, the eigenvalue problem and numerical approaches

Before one starts the numerical approaches to deal with eq. (1), such as the conventional FEM, HFEM and p-Fourier method, it is necessary to obtain the integral equation on its weak form. Let us consider the energy functional of a bending thin plate, that represents the weak form of the integral equation of the problem, as [11]:

$$\Pi = \frac{1}{2}\int \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, dA + \frac{\rho h}{2}\int (\dot{w})^2\, dA, \tag{2}$$

where h is the plate thickness, $\dot{w}$ is the first temporal derivative of the displacement field w, i.e. the velocity field. The constitutive relation of the homogenous and isotropic material is given by:

$$\boldsymbol{D} = \frac{h^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{3}$$

where ν is the Poisson ratio. And **B** contains the linear operators that represent the differential relations between the displacement field and the strains:

$$\boldsymbol{B} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -\frac{2\partial^2 w}{\partial x \partial y} \end{bmatrix}. \tag{4}$$

Now vanishing the energy functional first variation and considering harmonic movement (free vibration), the problem becomes a generalized eigenvalue problem in the form:

$$([\boldsymbol{K}] - \lambda^2[\boldsymbol{M}])\emptyset = \boldsymbol{0}, \tag{5}$$

where [**K**] is the stiffness matrix, [**M**] is the mass matrix, λ are the eigenvalues related to the natural frequencies and $\emptyset$ are the eigenvectors (natural mode shapes).

3.1 Finite Element Method

Petyt [11] describes the construction of the CR element proposed by Bogner, Fox and Schmit [12]. The CR element is a four-node element with sixteen degrees of freedom. It is formulated by the product between the Hermite's polynomial shape functions of the Euler-Bernoulli beam element in two in-plane directions. This formulation leads to an element with one vertical displacement and two rotations per node. However, in order to meet the conforming condition, an additional degree of freedom has to be included at each node. This degree of freedom is called "twist" and represents the cross derivative of the displacement [11]. The displacement field in dimensionless coordinates is given by:

$$w(\xi,\eta) = \sum_{i=1}^{4}\sum_{j=1}^{4} f_i(\xi) f_j(\eta) w_{ij}, \tag{6}$$

where f_1, f_2, f_3 and f_4 are the cubic Hermite polynomials and w_{ij} are the degrees of freedom.

3.2 Hierarchical Finite Element Method (HFEM)

To achieve the FEM hierarchical refinement, the Legendre polynomials in Rodriguez form given by [3]:

$$f_r(\alpha) = \sum_{n=0}^{\frac{r}{2}} \frac{(-1)^n(2r-2n-7)!!}{2^n n!(r-2n-1)!}(\alpha)^{r-2n-1}; r > 4, \tag{7}$$

are used. In eq. (7), $r/2$ denotes the integer part of r and !! the double factorial operator. The hierarchical polynomials in each direction are obtained replacing α for each dimensionless coordinate.

3.3 p-Fourier Method

The p-Fourier approach for plate vibration analysis was presented by Leung and Chan [7]. In the p-Fourier Method the non-polynomial enrichment functions used are the trigonometric functions given by:

$$f_r(\alpha) = \frac{1}{2}(1-\alpha^2)sin\left(\frac{j\pi}{2}(1+\alpha)\right), \tag{8}$$

where the j index represents the number of semi-waves on each direction and increases by j=1, 2, and so on.

4 Free vibration of a simple-supported square thin plate

The simple supported condition means null displacements at all the boundary. These boundary conditions allow us to take the separable-variable method assumption as solution hypothesis [1], leading to an eigenfunction solution ϕ for the differential equation (eq. (1)). Considering $m, n \in \mathbb{N}$ and L_x and L_y as the length and the height of the plate respectively, then the eigenvalues λ related to the eigenfunction ϕ of eq. (1) are [11]:

$$\lambda = \left(\frac{L_y{}^2 m^2 + L_x{}^2 n^2}{L_x{}^2 L_y{}^2}\right)\sqrt{\frac{\pi^4 \rho}{D}}. \tag{9}$$

This is a classical analytical solution and shows that the whole natural frequencies spectrum of a simple-supported plate can easily be determined only by knowing the constitutive and geometrical properties of the plate. For the present study, both L_x and L_y are 0,.3048 m, the density ρ is 2821 kg/m³ and the flexural stiffness D is 235.43 N.m.

4.1 Analytical frequencies

The analytical dimensionless frequencies λ in function of parameters m and n are presented in Tab. 1 and will be used as benchmark results. The same analytical results are presented by Leissa [1], Bardell [3] and Petyt [11].

Table 1. Analytical natural frequencies of simple-supported square plate

Mode	m	n	λ
1	1	1	19.7392088
2	1	2	49.34802201
3	2	1	49.34802201
4	2	2	78.95683521
5	3	1	98.69604401
6	1	3	98.69604401

4.2 *h*-refinement of FEM error analysis

Following the formulation proposed by Bogner, Fox and Schmit [12], one proposes to perform an h-refinement of the plate problem with different configurations of mesh, starting with a four-element mesh and stopping at a hundred element mesh. Figure 1 presents in a log-log graph the relative error related to the 1st, 3rd, 4th and 6th frequencies obtained by the h-refinement of FEM and the number of degrees of freedom in each mesh.

It can be seen that h-refinement presents a monotonic upper convergence and that as the number of degrees of freedom increases the numerical value approaches the analytical value for all the frequencies. Once 2nd and 3rd, and 5th and 6th modes present equal frequencies, they do not appear on the graphical representation since the FEM provides an equal result to each mode.

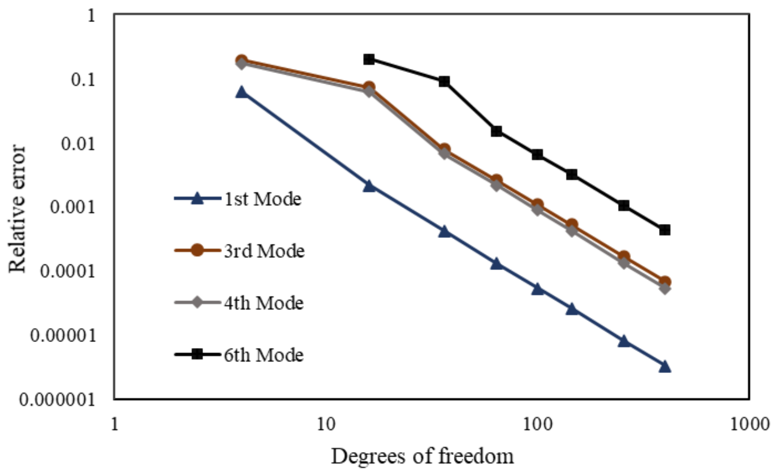

Figure 1. Relative error obtained by the h-refinement of FEM

4.3 HFEM error analysis

The HFEM analysis was developed as follows: as each new hierarchical polynomial is added in the formulation of the shape functions in each direction, new edge and internal degrees of freedom consequently arise increasing the order of the matrix eigenvalue problem without modify the mesh size. The relative errors of the results for the nine first degrees of hierarchical polynomials and just one finite element are presented in Tab.2.

Table 2. HFEM error analysis

Polynomial Order	4th	5th	6th	7th	8th	9th
Error λ_1	0.000164	0.000164	7.18E-08	7.18E-08	1.1E-10	1.1E-10
Error λ_2	0.189657	0.00291	0.002894	1.22E-05	1.21E-05	1.61E-08
Error λ_3	0.189657	0.00291	0.002894	1.22E-05	1.21E-05	1.61E-08
Error λ_4	0.17233	0.002659	0.002659	1.05E-05	1.05E-05	1.34E-08
Error λ_5	0.414438	0.414438	0.014394	0.014394	0.000204	0.000204
Error λ_6	0.414439	0.414439	0.014394	0.014394	0.000204	0.000204

It is observed that when the 8^{th} degree hierarchical polynomial is included the relative errors of the first six frequencies are smaller than 0.025%. As described by Bardell [3], the HFEM provides accurate results only if the hierarchical polynomial used has a number of semi-waves equal to or greater than the largest mode to be approximated.

4.4 p-Fourier trigonometric enrichment

The six lowest frequencies relative errors obtained by one element and increasing the number of trigonometric enrichment functions in p-Fourier Method are presented in Tab.3.

Table 3. p-Fourier Method error analysis

j	1	2	3	4	5	6
Error λ_1	0.003314	0.000418	7.88E-05	7.88E-05	2.79E-05	2.79E-05
Error λ_2	0.011165	0.001176	0.000688	0.000389	0.00027	0.000194
Error λ_3	0.011165	0.004503	0.004469	0.001151	0.001146	0.000465
Error λ_4	0.008805	0.003939	0.003939	0.000958	0.000958	0.000377
Error λ_5	0.27987	0.268364	0.009017	0.009017	0.002707	0.002707
Error λ_6	0.27987	0.268366	0.009017	0.009017	0.002707	0.002707

5.5 Convergence analysis

In order to evaluate the convergence of the three methods presented in this paper, Fig. 2 presents the relative errors and the respective number of degrees of freedom in each analysis for 1st and 6th modes in log-log graphs.

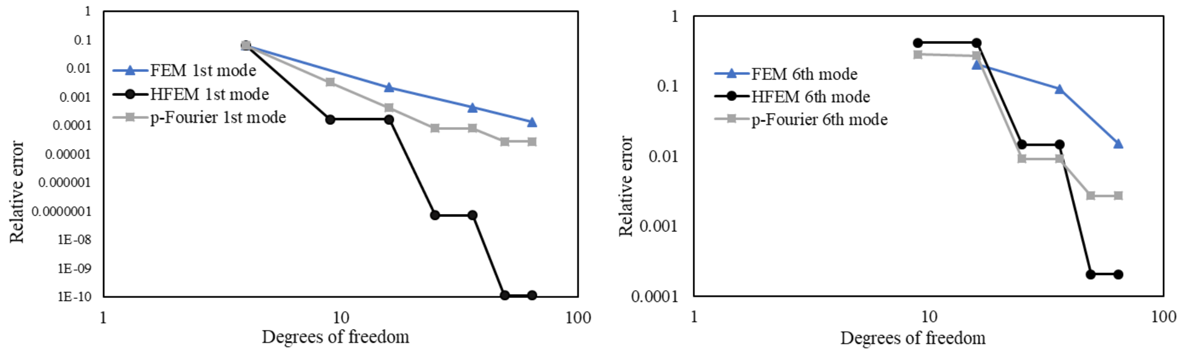

Figure 2. Convergency analysis of 1st and 6th modes

By the graphical analysis of Fig.2 , one observes that either for the 1st and 6th modes the HFEM yields better results as the number of degrees of freedom increases. For the 1st mode such behavior was observed for each and every number of degrees of freedom and, for the 6th mode, even that the p-Fourier Method shows better results comparing to HFEM for lower numbers of degrees of freedom, this behavior does not hold for the highest numbers since the HFEM provides the better results in this case.

5 Conclusions

One can realize that either the p-Fourier Method and the HFEM provided better results than the h-refinement of FEM for the free vibration of a thin plate problem once both presented convergence rates greater than the h refinement of FEM. Comparing to the HFEM, the p-Fourier Method improved its results getting better results on the 6th mode for lower numbers of degrees of freedom. However, as the number of degrees of freedom increased, the accuracy of HFEM overcame the p-Fourier Method.

In other words, one can conclude that the HFEM provided more precise results for an equal order eigenvalue problem than the h refinement of FEM and the p-Fourier Method, what means a very interesting behavior by the computational view.

Finally, it is proposed to extend the analysis for other boundary conditions, such as clamped and pointwise supported plates, in addition to comparing the results with other methods, such as the Generalized Finite Element Method (GFEM) and the Finite Differences Method (FDM), for example.

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