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Periodic Signal Generation using an Approximation to the Analytical Wave Equation Solution

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Abstract. A major issue in analysing wave propagation problems is the geometric and time domain discretizations, which if it is not properly performed, may lead to poor results or computational issues such as memory overflow. Ideally, an analytical solution for the Partial Differential Equation is desired since it may provide results for any given pair $(\vec{x}, t)$ independently of mesh size, time step and without any previous iteration. In this paper, an approximate analytical solution of the wave equation is used in order to simulate the behavior of any periodic signal in one dimensional domains. The periodic signal is approximated using Fourier series with the sine and cosine terms evaluated analytically and their respective coefficients evaluated numerically using Gauss-Legendre integration rule. A Python 3.7 Application Programming Interface was developed using an object oriented approach, allowing user defined periodic input function, spatial domain size, time range, number of Fourier terms used and static and dynamic solution plotting. Dirichlet Boundary Conditions defined as time periodic functions are considered and three different functions, rectangular pulse, sawtooth wave and Gaussian pulse, are evaluated and their respective results compared with the corresponding Finite Difference Method solution presenting a mean-squared-errors of order 10^{-3}.

Keywords: Periodic signals, Analytical solution, Application Programming Interface, Dirichlet, Wave Equation

1 Introduction

The Wave Equation is an important second order partial differential equation which describes several physical phenomena such as mechanical, electromagnetic, acoustic and thermal waves and its solution is related to the problem dimension, physical domain, boundary and initial conditions. Some practical applications of the Wave Equation is in modeling the propagation of acoustic waves from ultrasound transducers, for example, in Simões et al. [1], which may lead to new imaging techniques and non-invasive treatments.

Due to the impossibility of solving the Wave Equation analytically for an abritrary domain, several numerical methods have been developed over the years with a vast literature such as the Finite Element Method (FEM) (Atalla and Sgard [2], Benner and Heiland [3]), the Boundary Element Method (BEM) (Gimperlein and Meyer [4]) and the Finite Difference Method (FDM) (LeVeque [5]). One main concern about such methods relies on the geometric and time discretization that may lead to numerical instabilities or low accuracy and, although the computational resources have largely increased during the past years, high frequency problems demand a finer mesh with element size as small as one twentieth of the considered wave length, causing a major issue due to limit quantity of memory.

A possible approach in determining the proper mesh refinement of a given problem is comparing the numerical simulation with an adequate analytical solution, since it may present results for any given time instant with no need for computing previous iterations. In this sense, Lima et al. [6] proposed an approximate analytical solution for the wave equation for a Dirichlet Boundary Condition (DBC) considering a sine wave as the input function. Although this solution can be used to implement a wide range of frequencies, it is still desirable to determine the analytical response due to any periodic function, such as Gaussian train pulses that are used for ultrasound imaging, rectangular or sawtooth functions, which are the main wave forms in many engineering systems.

This paper presents an analytical solution for the Wave Equation considering an arbitrary periodic input function $u_p(t)$, obtained by approximating $u_p(t)$ by its Fourier series and superimposing the weighted sine and cosine analytical solutions for each fundamental frequency. Results are presented for three different periodic functions: a rectangular pulse sequence; a sawtooth wave function; and a Gaussian train pulse, with an mean-squared-error of order 10^{-3} when compared to FDM simulations.

2 Methodology

In this section, the equations for the analytical wave equation solution and the methodology for compose a periodic signal solution are presented. The code was developed in Python 3.7 Anaconda distribution and the Numpy and Scipy Application Programming Interfaces (APIs) were used for fast vector-matrix manipulations (Lanardo [7]) and the Matplotlib for plotting the results (Johansson [8]).

2.1 One-dimensional Wave Equation Analytical Solution

The formulation for the approximated analytical solution for one-dimensional problem derived by Lima et al. [6] is presented here. The solution is defined within a domain of size $L\ (m)$ and a time range $(0, t_f)\ (s)$. The differential equation and the considered domain are given by Equation 1, where the subscript *s* emphasises that the solution relates to a sine wave DBC:

$$\frac{\partial^2 u_s}{\partial t^2} = c^2 \frac{\partial^2 u_s}{\partial x^2},\ t\ >\ 0,\ x\ \in\ [0, L] \tag{1}$$

where $c\ (m/s)$ is the medium plane wave velocity. The initial conditions for this problem are given by:

$$u_s(x, 0) = 0 \tag{2}$$

$$\frac{\partial u_s(x, 0)}{\partial t} = 0 \tag{3}$$

The DBC is given by Equations 4 and 5.

$$u_s(x = 0, t) = 0 \tag{4}$$

$$u_s(x = L, t) = sin(\omega t), \tag{5}$$

where ω is the input angular frequency in rad/s. The approximate weak solution to this problem is given by Equation 6 where N_p is the number of approximation terms:

$$u_s(x, t) \approx \frac{x}{L} sin(\omega t) + \sum_{n=1}^{N_p} sin(\frac{n\pi t}{L})[C_n sin(\frac{cn\pi t}{L}) + B_n] \tag{6}$$

with the parameters C_n and b_n given by Equations 7 and 8 and the function $B_n(t)$ given by Equation 9.

$$C_n = -\frac{\omega b_n}{cn\pi} \tag{7}$$

$$b_n = (-1)^{n+1} \frac{2L}{n\pi} \tag{8}$$

$$B_n = K_2 sin(\frac{cn\pi t}{L}) + \frac{L\omega^2 sin(\omega t) b_n}{(cn\pi)^2 - L^2\omega^2} \tag{9}$$

Equation 10 presents the parameter K_2:

$$K_2 = -\frac{L^2\omega^3 b_n}{(cn\pi)^3 - cn\pi L^2\omega^2} \tag{10}$$

2.2 Periodic Signal Approximated Solution

It must be noted that, since the Wave Equation and the DBC are linear operations, the superposition principle is valid, thus, it is possible to approximate any periodic function $u_p(t)$ using Fourier series as follows (Kreyszig [9]):

$$\tilde{u}_p(t) \approx a_0 + \sum_{n=1}^{M} [a_n cos(\beta_n t) + b_n sin(\beta_n t)] \tag{11}$$

$$\beta_n = 2\pi n f \tag{12}$$

where the variable *M* is the number of approximation terms.

As can be observed in Equation 11, the analytical solution for a cosine term, $u_c(t)$, is needed in order to properly approximate the function $u_p(t)$ and this can be achieved by shifting the time variable by $\pi/2$, resulting in Equation 13.

$$u_c(t) = u_s(t + \pi/2) \tag{13}$$

The offset term a_0 and the cosine and sine coefficients, a_n and b_n, can be evaluated numerically using, for example, Gaussian quadrature.

In order to illustrate the proposed model, three different functions are presented: (i) a rectangular pulse (Equation 14); (ii) sawtooth wave (Equation 15); (iii) a Gaussian pulse (Equation 16) with $\sigma = 10 \times 10^{-3}$ and $\mu = 50 \times 10^{-3}$ and all three models have a period of $T = 0.15\ (s)$.

$$r(t) = \begin{cases} -1, & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0 \end{cases} \tag{14}$$

$$h(t) = t,\ -0.15 \leq t \leq 0.15 \tag{15}$$

$$g(t) = e^{-(t-\mu)^2/\sigma^2} \tag{16}$$

In order to answer the question *"How and how much the approximation does impact the domain $u_p(t)$ response?"*, three different values of M (Equation 11) are considered: *M = 10, 20, 40* and the mean-squared-error given by Equation 17 is evaluated for the approximated input functions as well.

$$\varepsilon_{RMS} = \sqrt{\frac{1}{K} \sum_{\alpha=1}^{K} (u_\alpha - \tilde{u}(t_\alpha))^2} \tag{17}$$

where u_α is the FDM-TD solution, $\tilde{u}(t_\alpha)$ is the analytical solution evaluated at t_α and K is the number of time samples used.

3 Results

Qualitative (graphical results) and quantitative comparisons (ε_{RMS}) are presented with respect to the FDM-TD implementation as described in Igel [10]. The FDM-TD parameters Δt and Δx were chosen such that the ε_{RMS} between two consecutive simulations was less than 10^{-9} and the same values were adopted for the analytical simulations as well. The geometric domain, time window and plane wave velocity are as follows:

- $L_x = 2.1\ (m)$, $\Delta x = 21.0 \times 10^{-3}\ (m)$
- $t_f = 2.5\ (s)$, $\Delta t = 20.0 \times 10^{-3}\ (s)$
- $c = 10.0\ (m/s)$

The number of approximation terms for Equation 6 is $N_p = 20$ and set fixed for all the simulations. This value was chosen so that the ε_{RMS} between the analytical response for $u_s(x;t)$ and the FDM-TD was lower than 10^{-6}. In order to perform a fair error evaluation, the approximated input function was considered as DBC for FDM-TD.

3.1 Rectangular Pulse

Figure 1 presents the function $r(t)$ along its approximations and Table 1 the respective ε_{RMS} values.

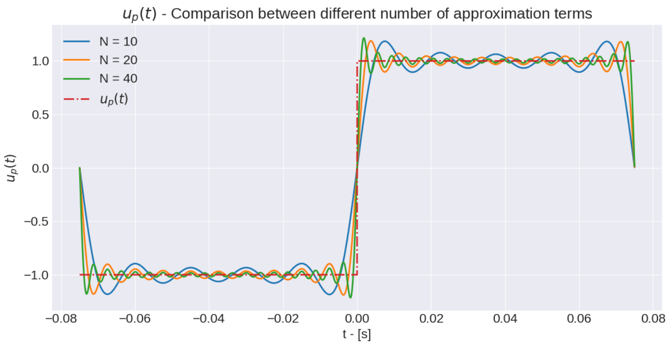

Figure 1. Comparison between different number of approximation terms for a rectangular pulse function

Table 1. ε_{RMS} for rectangular pulse approximation by Fourier series

Fourier terms	ε_{RMS}
10	40.396×10^{-3}
20	20.253×10^{-3}
40	10.180×10^{-3}

Qualitative and quantitative results are presented in Figure 2 and Table 2.

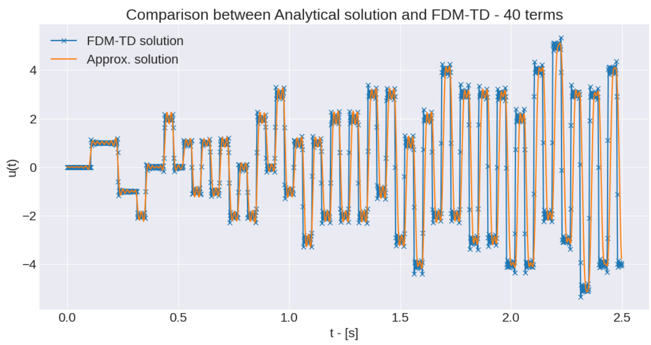

Figure 2. Qualitative result for rectangular wave considering $M = 40$ and evaluated at $x = L_x/2$

Table 2. Quantitative results for rectangular wave evaluated at $x = L_x/2$

Fourier terms	ε_{RMS}
10	48.819×10^{-3}
20	47.655×10^{-3}
40	46.158×10^{-3}

3.2 Sawtooth Wave

Figure 3 presents the function $h(t)$ along its approximations and Table 3 the respective ε_{RMS} values.

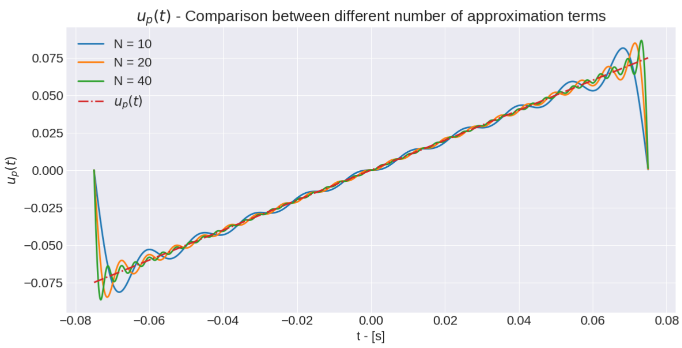

Figure 3. Comparison between different number of approximation terms for a sawtooth wave

Table 3. ε_{RMS} for sawtooth wave approximation by Fourier series

Fourier terms	ε_{RMS}
10	119.880×10^{-6}
20	58.443×10^{-6}
40	28.859×10^{-6}

Qualitative and quantitative results are presented in Figure 4 and Table 4.

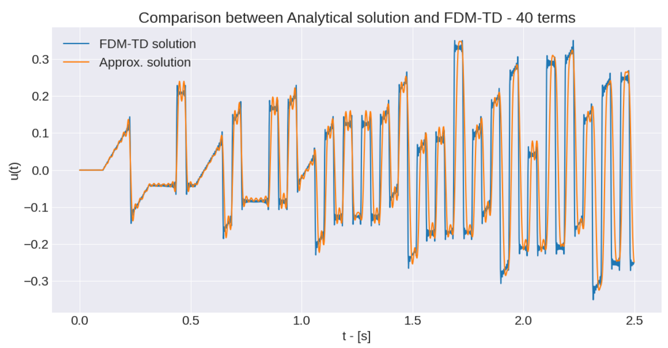

Figure 4. Qualitative result for sawtooth wave considering $M = 40$ and evaluated at $x = L_x/2$

Table 4. Quantitative results for sawtooth wave evaluated at $x = L_x/2$

Fourier terms	ε_{RMS}
10	2.985×10^{-3}
20	3.172×10^{-3}
40	3.090×10^{-3}

3.3 Gaussian Pulse

Figure 5 presents the function $g(t)$ along its approximations and Table 5 the respective ε_{RMS} values.

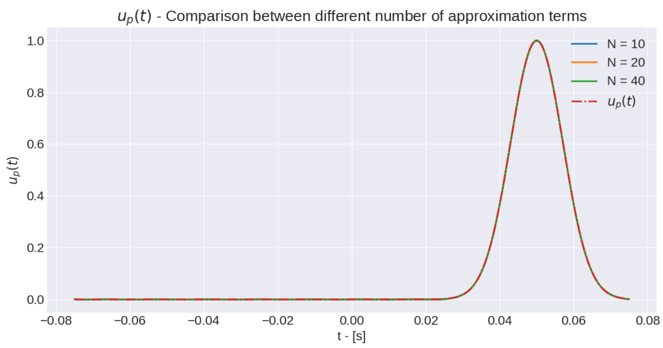

Figure 5. Comparison between different number of approximation terms for a Gaussian pulse

Table 5. ε_{RMS} for Gaussian pulse approximation by Fourier series

Fourier terms	ε_{RMS}
10	8.046×10^{-6}
20	8.892×10^{-9}
40	4.700×10^{-9}

Qualitative and quantitative results are presented in Figure 6 and Table 6.

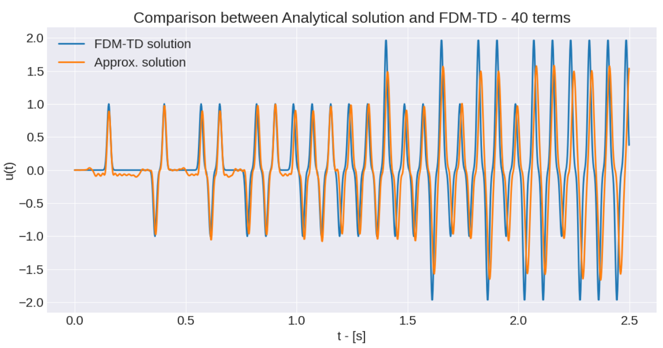

Figure 6. Qualitative result for Gaussian pulse considering $M = 40$ and evaluated at $x = L_x/2$

Table 6. Quantitative results for Gaussian pulse evaluated at $x = L_x/2$

Fourier terms	ε_{RMS}
10	15.002×10^{-3}
20	15.001×10^{-3}
40	14.960×10^{-3}

4 Discussions

As can be observed in Tables 2, 4 and 6 the ε_{RMS} for all simulated wave forms are of order 10^{-3}. The approximation accuracy for $u_p(t)$ (Tables 1, 3 and 5) did not present an expressive influence over the domain response. Figures 2, 4 and 6 show that the analytical response properly represents the response of a periodic signal input function.

5 Conclusions

This paper presented an approximated analytical solution for the Wave Equation considering an arbitrary periodic function as a Dirichlet Boundary Condition. The solution is based on the decomposition of the input wave form into its Fourier series and presented promising results, since it is possible to evaluate the Wave Equation solution for any given pair $(x,t)_i$ without computing previous values and independent of the geometric and time domain refinement, which is a major issue in numerical methods. The next step of this work is to develop the same model for the Neumann Boundary Condition (NBC) and compose more complex models, for example, implementing the Robin Boundary Condition, which is the linear combination of DBC and NBC.

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