Static and dynamic buckling of deep beams with plate finite elements

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Abstract. The main objective of this paper is to present results on the lateral buckling of beams using finite elements based on Kirchhoff and Mindlin-Reissner Plate theories, merged with membrane elements in order to include the analysis of shells. A MATLAB code was developed to calculate static and dynamic critical loads, buckling modes, frequencies, and vibration modes of thin and thick plates subjected to conservative and non-conservative (also called follower or circulatory) loads using a so-called geometric matrix. In case of displacement-dependent applied forces, it is necessary to implement a matrix that will correct the loads, designated as load matrix. In the case of conservative forces, the load matrix is symmetric, and in the case of non-conservative forces, it is non-symmetric. In the latter case, the critical load usually will correspond to dynamic behavior designated as flutter. Different boundary conditions and loads are considered and several cases of lateral buckling are investigated. Theoretical values when found in the literature or in national and international rules are compared with values determined by Finite Element Method (FEM). The lateral instability of slender beams is very important in practice, because in some situations it may occur prior to ultimate plastic limit state in bending.

Keywords: Lateral buckling; flutter; plates; Finite Element Method; MATLAB online.

1 Introduction

The main objective of this paper is to show the behaviour of lateral buckling of beams using finite elements of membrane and plate. Plate elements can be used to simulate structures such as slabs, raft foundation, bridges and walls. Loss of stability can occur through a combination of bending and torsion, usually by lateral buckling of beams and axial-torsional of columns. In order to obtain critical loads and frequencies, it can be done either by exact solutions using differential equations and integrations, or by approximate solutions using energy method, when the exact solution is unknown or very complex. Besides that, the Finite Element Method, which uses shape function in conjunction with a variational method such as Rayleigh-Ritz method also can be used.

A MATLAB code was developed, using the online platform, to analyse critical loads, frequencies and vibrational modes of plates. The online MATLAB allowed the code to be developed without the need of having the software installed on computer. It is just necessary to have a Web Browser. The sharing of codes, scripts and collaborative works in the online platform provides an easy access to MATLAB in any computer and in anywhere. It can be found in the address https://matlab.mathworks.com/. Cloud synchronization is immediate and it is available, up to the present date, 5 GB of storage for each user.

2 Structural Instability

In relation to the applied load, Ziegler [1] classifies several types of forces as conservative and non-conservative. In this paper, the non-conservative loads used will always be the follower (or circulatory) forces. Classical texts, such as Bolotin [2], Leipholz [3] and Suanno & Silva [4], describe the mathematics that involves the analysis of non-conservative systems, which gained special importance with the growth of the aerospace engineering in the 60s.

There are two types of instability phenomena: divergence (or static) and flutter (or dynamic). The first is when, after the critical load, the structure loses rigidity and thus higher displacements occurs for small load variations. In the flutter, after the critical load, the structure starts vibrating in growing amplitudes. An important fact is that non-conservative systems can lose stability due to both divergence and flutter, whereas conservative systems lose stability only by divergence. A well-known case that caused a structure to collapse due to flutter is the Tacomas Narrows bridge in 1940, where the force of the wind created ever greater oscillations, causing the bridge to collapse. At the time, the phenomenon of flutter was not well understood, and the collapse was attributed to the formation of vortexes or resonance in gusts of wind with frequency close to that of the structure. Only recently has it been accepted that the reason was due to flutter. Recent researches about flutter are about composites (Bashir [5]), wind-tunnels experiments (Tongqing [6]), effect of imperfection and damping in flutter analysis (Hamed [7] and Junior [12]), and many others.

3 Finite Element Method (FEM)

The elements used were the membrane (Fig. 1(a)), a thin plate element called ACM and a thick plate element Mindlin (both represented by Fig. 1(b)).

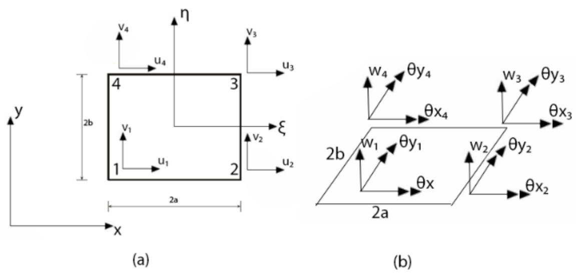

Figure 1. Degrees of freedom of (a) a membrane; (b) an ACM or Mindlin

Membrane element is used when loads are applied in directions which are parallel to the middle plane of the membrane and are uniformly distributed through the thickness. The state of stress is defined by the components σ_x, σ_y, τ_{xy} which are assumed to be independent of z axis. Such state is called plane stress. The ACM is a non-conforming element which uses the Kirchhoff-Love theory of thin plates, which does not consider shear deformation and rotary inertia. On the other hand, Mindlin plate uses the Mindlin-Reissner theory that considers these effects. The plate elements only take in consideration forces that are transverse to the middle plane of the plate. In practise, a plate can be designed as a thin plate when approximately (smallest size of the plate) / thickness > 20. In relation to the commercial software ANSYS, ACM plate corresponds to the SHELL63, and Mindlin, which is the element used by default, corresponds to the SHELL 181.

In relation to matrices, the elastic stiffness matrix is associated with the undeformed configuration of the structure. Its formulation can be found in literature such as Fellippa [8]. The mass matrix has three types: lumped mass, modified diagonal and consistent. Studies by Suanno & Silva showed that the most suitable for dynamic stability is the consistent mass matrix, being the unique adopted in this paper. Its formulation can be found in Petyt [10]. The geometric matrix takes in consideration nonlinear effects associated with large displacements, and it is necessary to calculate buckling critical load. To be calculated, it is necessary to have all the initial stresses of the elements. Its standard formulation is found in Cook [11]. Finally, the load matrix must be implemented when a follower force is applied. For non-conservative forces, the load matrix has a non-symmetrical shape, while for conservative forces it has a symmetrical shape. It is a matrix that only has non-zero terms in the nodes that the forces are being applied to. In Junior [12] it is possible to see how to obtain this matrix.

4 Lateral Buckling of deep beams

The bending moment of a slender beam in its greatest flexural stiffness can cause lateral buckling. For the calculations of the theoretical critical load, a small lateral deflection in the beam is assumed when in the action of the loads. Later, differential equilibrium equations for the deflected beam are obtained and, using the boundary conditions, the result of the buckling loads is reached. Energy methods can also be applied to obtain the same results.

The nominal resistance to lateral buckling depends on factors such as: the unrestrained length of the beam; the boundary conditions; the variation of the bending moment along the beam; the position of the loads; imperfections, which can greatly influence the results, and can generate different failure modes for the structure; among others.

In all the following examples, the warping constant I_w is negligible, because the section is a thin rectangular section.

4.1 Lateral Buckling of beams due to Bending Moment

Considering the case of Fig. 2(a), and using E (Young modulus) = 20e9 N/m^2; I_w (warping constant) = 0 (thin rectangular section) ; L (length) = 1 m; b (height) = 0.30 m and t (thickness) = 0.01 m, the theoretical value is found in Bazant & Cedolin [13].

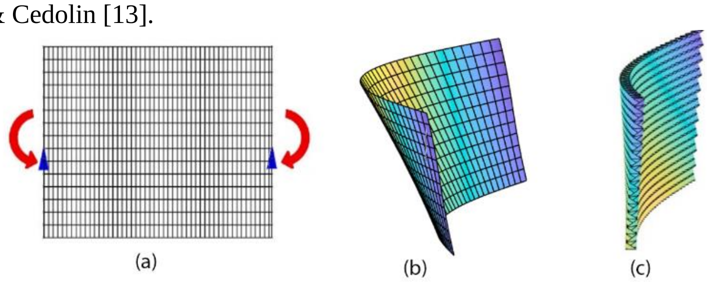

Figure 2 – (a) Beam subjected to uniform bending moment; (b) Buckling Mode in ACM (40x12); (c) Buckling Mode in Mindlin (70x20)

The theoretical value for this beam is 1.95e+03 N.m. Using the finite element method, in the case of conservative systems, there are two ways to find the critical load, one is by the static criterion, which is the solution to the eigenvalue problem of eq. (1), where $[\boldsymbol{K}]$ is the stiffness matrix, $[\boldsymbol{K}]_{\boldsymbol{G}}$ the geometric stiffness, λ are the eigenvalues (critical loads), and $[\boldsymbol{\Delta u}]$ the eigenvectors (variation of deformation) for each eigenvalue λ. The load matrix must be added to eq. (1) if there is some follower force. Another way to find critical load is by the dynamic criterion, that is, varying the load until the frequency is zero (for conservative cases).

$$\{[\boldsymbol{K}] + \boldsymbol{\lambda}[\boldsymbol{K}]_{\boldsymbol{G}}\} * [\boldsymbol{\Delta u}] = 0. \tag{1}$$

Using the ACM element (40x12 mesh) two moments are found equal to ±1.96e + 03 N.m. The buckling mode is shown in Fig. 2(b). Using the Mindlin element, it also results equal to the theoretical using a 70x20 mesh (Fig. 2(c)), but still cannot generate a continuous buckling mode, requiring a much more refined mesh, requiring considerable computational resources.

It is very important to observe the boundary conditions when calculating the critical moment. The situation most suitable for the theory is when the support is in the centroid line and the degrees of freedom, disregarding the membrane, in w and θ_x are fixed, and θy is free. Fixing only the w also provides the same result. The value of the critical moment found coincides with the values for lateral torsional buckling of the Canadian Standard CAN/CSA-S16-01, the American AISC/LRFD, and the European ENV 1993 standards. That is the critical moment for idealized case of simply supported beam with double symmetry. In these rules, there is also an implementation of a moment gradient factor to consider cases where the moment varies along the beam.

It is important to note that to achieve a simple support in practice is quite difficult. Because of this, it is necessary to introduce semi-rigid connections. These connections often generate a non-linear variation in stiffness, thus preventing the elaboration of a theoretical solution to the problem.

4.2 Lateral Buckling of Cantilever Beam

Figure 3(a) exhibits a case of a cantilever beam acted upon a transverse concentrated load at the end. The critical load of lateral buckling considering non-uniform warping is found in Timoshenko & Gere [14].

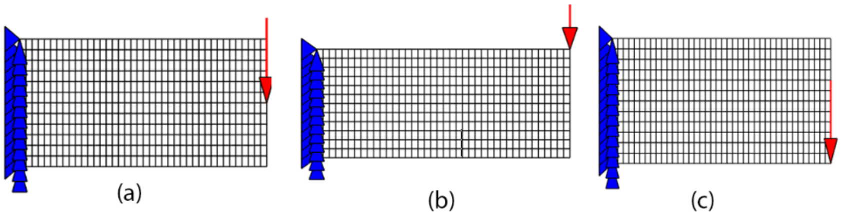

Figure 3 – (a) Mesh of a cantilever beam; (b) Destabilizing load (c) Stabilizing load

The closest result to the theory presented is for the boundary condition where all points on the left side have the degree of freedom w restricted and, in addition, the centroid point of that same side has the degrees of freedom θx and θy restricted (or just θy) as well. Without this last detail, the result tends to other values, and may even result in complex numbers. The result quickly converged to this situation. The theory is 2.49e+03 N, in ACM it is 2.51e+03 N and in Mindlin it is 2.48e + 03 N.

One of the advantages of using the stresses calculated by the membrane finite elements is the possibility to change the position of the load application. In this way, it is noticed that instead of applying a force in the center, if it is applied a force in the upper right of the beam, a destabilizing effect is generated that reduces the critical moment. Likewise, by applying a load to the bottom of the beam, a stabilizing effect is generated, increasing the critical moment. For the case of Fig. 3(b), the theoretical critical load is 2.18e + 03 N and for ACM the result is 2.14e + 03 N. For the case of Figure 3(c), the theoretical value is 2.78e + 03 N, while the ACM value is the same 2.78e + 03 N. Using a suitable boundary condition, it is possible to see the shear stress variation with the change in the load position, as in Fig. 4.

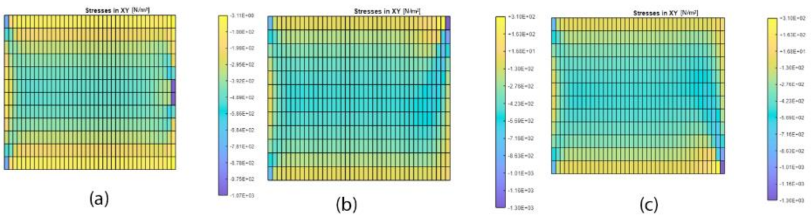

Figure 4 – Stresses in xy for (a) Load at the middle; (b) Load at the upper right; (b) Load at the bottom right

4.3 Lateral Buckling of beam under distributed uniform load (conservative cases)

Figure 5 shows a beam under transverse distributed load in its entire length. The data used are E = 20e9 N / m^2; L = 1 m; b = 0.30 m and t = 0.01 m. The theoretical formula is found in Bazant & Cedolin [13].

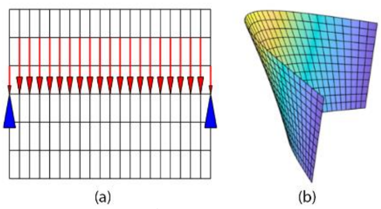

Figure 5 – (a) Simply supported beam under uniformly distributed load; (b) First buckling mode ACM (40x12)

It is again essential the importance of boundary conditions. The theoretical critical load is 1.7923e + 04 N / m. For each type of boundary condition, the result has a different behaviour. For the case where the only restricted degrees of freedom are at both ends of the centroid line and both w, and θx and θy are restricted, for example, we have the result of Fig. 6. It is observed that although the results are relatively close to the theoretical, we see that there is no convergence of the results as the mesh is refined. It is important to note that in all cases of this paper, the warping constant I_w is negligible, as these are thin rectangular sections.

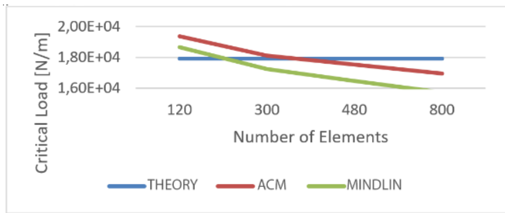

Figure 6 – Critical load when w, θ_X and θ_Y of centroid line are fixed

For the case in which only w in the centroid is fixed, critical loads become complex numbers. Restricting only w and θ_X of the centroid line ends, it is found a value lower than the theory, equal to 1.18e+4 N/m for the ACM. The best case is to consider all points on both sides supported, with only the w restricted (or w and θ_X restricted). In this case, the result converges quickly to the correct one. Thus, the theoretical result is 1.79e+4 N/m, the ACM result at the convergence is 1.80e+4 N/m and for Mindlin 1.77+e4 N/m.

The results obtained using analytical stresses calculated by the theory and the results using the stresses calculated by the FEM were always very similar, even refining the mesh, taking care to not generate stresses similarities with the wrong boundary conditions. It is important to note that for deep beams, the theoretical stresses are not exactly the same as those seen in practice. Therefore, for these beams, the FEM becomes better to capture the stresses of the elements.

4.4 Lateral Buckling of beam under distributed uniform load (non-conservative cases)

Figure 7(a) shows a beam under a distributed load at the top along its length. The data used are the same as in the previous case (E = 20e9 N/m^2; L = 1 m; b = 0.30 m and t = 0.01 m).

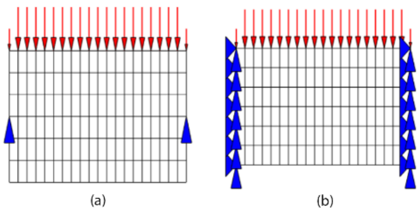

Figure 7 – (a) Simply Supported beam; (b) Clamped beam

In the previous section, it was seen that considering a conservative distributed load on the beam's centroid line (Fig. 5(a)), the critical load is equal to 1.80e+4 N/m (for ACM). However, when changing the load to the upper part of the beam (Fig. 7(a)), there is a destabilizing effect that reduces the critical load to 1.49e+4 N/m. To change the load to a non-conservative type, it is necessary to implement the load matrix. The difference between the load matrix of the Beck Column (distributed load in the x direction) and the load matrix for this case, is that the non-null terms as a function of w are the ones corresponding to the upper longitudinal line of the beam and the rotation of the beam is in θ_X instead of θ_Y.

It is important to remember that in the case of non-conservative systems, the eigenvalue problem of eq. (1) provides an imaginary critical load, that is, there would be no critical load for these systems. Therefore, it is necessary to use the dynamic criterion, that is, vary the load and observe the behavior of the frequency. It is done by using eq. (2), in which ω are the eigenvalues (frequency [rad/s]) for each load λ, $[\boldsymbol{K}]_L$ is the matrix load and $[\boldsymbol{M}]$ is the mass matrix. MATLAB makes this calculation extremely easy, just using the *eig* command, without the need for any additional algorithms for non-symmetric matrices.

$$\{[\boldsymbol{K}] + \lambda[\boldsymbol{K}]_G + \lambda[\boldsymbol{K}]_L - \omega^2[\boldsymbol{M}]\} * \{\boldsymbol{\Delta u}\} = 0. \tag{2}$$

When using the dynamic criterion in the non-conservative case of Fig. 7 (a), it is observed that a divergence is obtained instead of a flutter, since the frequency goes to zero when the load reaches the critical value (Fig. 8(a)), which in this case corresponds to approximately 0.65xp_{cre} = 0.97e+04 N/m, where p_{cre} is the critical load of the conservative case, using ACM mesh (20x6). This occurs whether in the boundary condition it is considered that all points on both sides have w and θ_X fixed (situation that best represents the theory of section (4.3) and the eigenvalue problem is $\{[\boldsymbol{K}] + P * [\boldsymbol{K}]_{\boldsymbol{G}} + P * [\boldsymbol{K}]_L - \omega^2[\boldsymbol{M}]\} * \{\boldsymbol{\Delta u}\} = 0$. It is observed that as the load continues to increase, the frequency that had become pure imaginary, returns to a real number and becomes greater than before, and remains in this oscillation as the load increases, as seen in Fig. 8(b). The same is true for the second frequency, as seen in Fig. 8(c). Unlike other cases of divergence, where the frequency remains pure imaginary as the load increases.

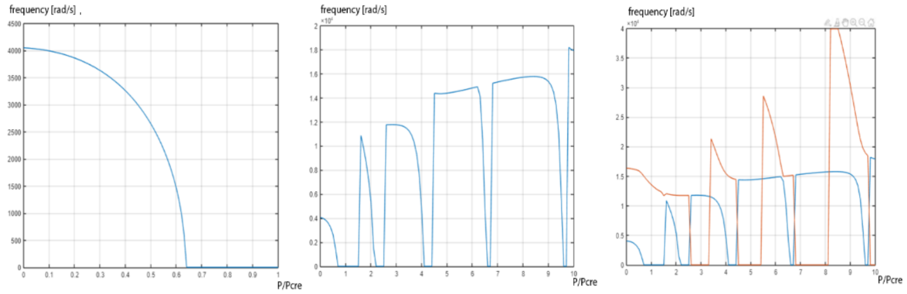

Figure 8 – graphics of load x frequency in which (a) load varies from $P/p_{cre} = 0$ to $P/p_{cre} = 1$; (b) load varies from $P/p_{cre} = 0$, to $P/p_{cre} = 10$ and only the first frequency appears; (c) load varies from $P/p_{cre} = 0$, $to\ P/p_{cre} = 10$ and first and second frequencies appear.

To analyze the case of Fig. 7 (b), if the loading is still conservative, when the two sides of the beam are clamped, the static critical load increases from 1.49e+4 N/m (simply supported) to 4.48e+04 N/m using ACM element (40x12). If we transform the load into non-conservative and apply the dynamic criterion using the eigenvalue problem slightly different from the previous, it is $\{[\boldsymbol{K}] + P * [\boldsymbol{K}]_{\boldsymbol{G}} - P * [\boldsymbol{K}]_L - \omega^2[\boldsymbol{M}]\} * \{\boldsymbol{\Delta u}\} = 0$, the flutter is obtained at approximately 1.8 times the corresponding static critical load, that is, equal to 8.06e+4 N/m. The vibration modes for this situation are:

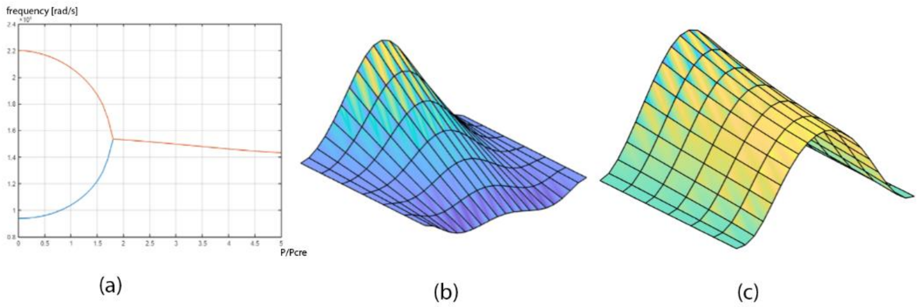

Figure 9 – (a) load x frequency (flutter); (b) Vibration Mode for p=1.8p_{cre}; (c) Vibration Mode for p=3.3p_{cre}

5 Conclusions

Regarding ACM, the good points are that it generally does not need to be very refined. A mesh with few elements has shown to have excellent results, always very close to the theories of each case. The disadvantage of ACM is that it does not consider the effect of shear on the deformation energy or the rotational inertia. In relation to Mindlin, this element proved to be efficient too, especially when the shear effect cannot be negligible. The disadvantage of Mindlin with 4 nodes is that in some cases it had to be very refined, sometimes even more than the computer (or the online platform) could handle.

The finite element method proved to be a powerful tool because it allows to expand to new geometries, include new effects such as non-linearity, visually see how the structures buckle and vibrate, among many other advantages. Writing the code itself gives you the freedom to add effects that in commercial software like ANSYS becomes more difficult. However, using the finite element method is not always easy. Based on the examples of this paper, the importance of boundary conditions for the calculation of critical loads, especially for lateral buckling, is notable. Wrong boundary conditions can cause singularities of membrane stresses, which cause great distortion in the results. In addition, it is necessary to know what degrees of freedoms will be fixed, for example at which points there will be no rotation or displacement, as this also greatly affects the results. It is also important to note that to obtain a theoretical solution, very special boundary conditions are used. Not all boundary conditions allow a solution that can be expressed as a formula. In semi-rigid connections, for example, there is often a non-linear variation in stiffness, thus preventing the elaboration of a theoretical solution to the problem. In addition to the boundary conditions, the load matrix and geometric matrix signals can also affect the buckling result. Imperfections are also another very important topic, being also a valuable suggestion for future work.

Regarding the software used, MATLAB proved to be a very powerful tool, with a very large library and simple commands that greatly simplify the solution of problems. In the case of the non-symmetric load matrix, for example, with MATLAB, it is only necessary to use the eig command to calculate eigenvalues and eigenvectors, without the need for other calculation algorithms.

In addition to the version installed on the computer, the MATLAB online platform was used most of time. It proved to be quite efficient in the sense that it can be used on any computer and anywhere, just having any Web Browser installed. Code processing takes place in the cloud, so MATLAB online works very well even on computers with little computational resources. In addition, it is possible to open more than one window at the same time, so that it is possible to edit and process more than one code simultaneously and open graphics and matrices in different windows, all without leaving the Web Browser. The disadvantage of using the platform is that it was not possible to refine the meshes too much, as it automatically disconnected after some time. Another problem was the matrix size limit. When the meshes were highly refined, the matrices passed the limit supported by the platform. However, all the cases in this article could be solved perfectly using the MATLAB online platform. Regarding the available storage and the synchronization of the online scripts, there was no problem. The 5 GB available was more than sufficient and the synchronization is done automatically, without loss of files.

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