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Inverse and direct projection schemes for topology optimization using polygonal elements

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Abstract. Projection schemes are used in topology optimization to provide minimum length scale on structural members in topology optimization solutions. On the other hand, inverse schemes provide control over the minimum size of holes in the resultant topology. Literature provide several applications of both schemes associated to regular meshes using quadrilateral finite elements to discretize the design domain. Besides, in the last decade polygonal element has been applied to solve topology optimization problems, providing better representation for the contour of the design domain. This paper applies the concepts of both direct and inverse schemes to unstructured polygonal finite element meshes. Each projection is made via mesh independent functions based upon the minimum length scale for either structural members or holes. A linear and a Heaviside projection function are applied to both schemes. Numerical examples are presented to demonstrate the various features of the projection-based techniques.

Keywords: topology optimization, projection schemes, inverse Heaviside projection, polygonal elements.

1 Introduction

Polygonal finite elements are used in topology optimization to provide solutions with unstructured meshes in non-regular domains [1]. This approach was a natural expansion of the hexagonal elements [2] proposed in order to avoid checkerboard patterns in the optimization solutions. Educational codes for generating unstructured meshes with polygonal elements and for the topology optimization process itself can be found in [3] and [4], respectively. The natural configuration of the meshes, with elements sharing at least one edge, prevents the appearance of checkerboard solutions. However, other numerical instabilities characteristic of the density method, such as mesh dependence, are still present in the solutions.

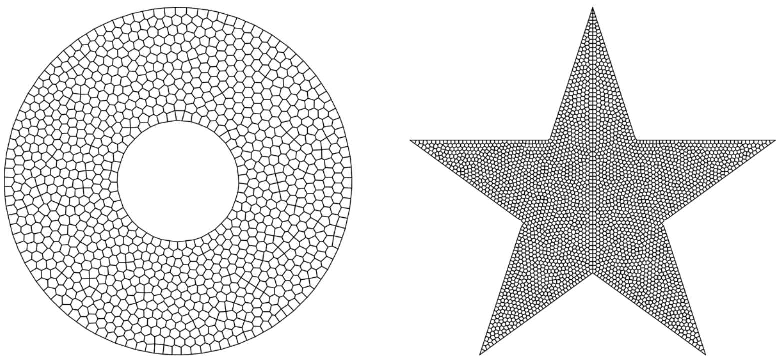
Techniques used to impose manufacturing control to the solutions also seem to provide mesh independency. Guest, Prévost and Belytschko [5] used an approach in which nodal design variables are related to the element densities through projection functions. This kind of approach provide control over the thickness of the resulting structural elements. Almeida, Paulino and Silva [6] introduced an inverse projection scheme providing control over the size of the voids in the solutions. Both works used structured meshes with rectangular 4-node plane elements. However, both approaches can be easily extended to polygonal elements.

Both linear and non-linear functions can be applied to projection schemes. A high percentage of intermediate densities in the final solution is found when using linear projection functions, especially in the inverse scheme. The nonlinear projection functions, in turn, provide solutions with greater physical consistency [5, 6]. Guest, Prévost and Belytschko [5] used a smoothed Heaviside function to obtain solutions with a low percentage of intermediate densities. Almeida, Paulino and Silva [6] used quadratic functions associated with the direct and inverse projection scheme. Due to its more aggressive behavior, the regularized Heaviside function seam to provide solutions closer to the desired solid void pattern. This work investigates a Heaviside function for the inverse projection scheme and its application to unstructured meshes using polygonal elements.

2 Unstructured meshes using polygonal finite elements

Talischi, Paulino and Le [2] used regular hexagonal finite elements to treat checkerboard instability. The formulation was later expanded by Talischi *et al*. [1] working with irregular polygonal finite elements with a variable number of sides. The mesh generation process, described by Talischi *et al*. [3], it is based on Voronoi diagrams, a type of discretization of areas in convex polygonal domains. Once the mesh is obtained, the stiffness matrices of each element can be written with the aid of the Wachpress shape functions [7, 8]. It is necessary to use a generic numerical integration process to obtain the stiffness matrices of generic polygonal finite elements. Sukumar and Tabarraei [7] proposed the discretization of each polygonal element in triangular subdomains. Thus, integration can occur separately in each triangular domain and added later, a process widely discussed in the literature.

The simple use of polygonal finite elements associated with the density method prevents the appearance of checkerboard patterns and other numerical instabilities [1, 2]. The wide use of rectangular elements in optimization problems is due to its simplicity. However, these elements are not suitable for non-regular domains, as those illustrated in Fig. 1. Besides, the discretization of non-regular domains, especially the convex ones, is relatively simple in meshes of polygonal elements, requiring no additional efforts between meshing a rectangular or circular domain, for example [3].



a) b)

Figure 1. Non-regular domains discretized in polygonal elements: a) 1000 elements; b) 3000 elements.

3 Projection schemes

Projection schemes in optimization problems are techniques for projecting nodal design variables onto finite elements in order to obtain the densities used in structural analysis. Thus, the density, ρ^e, of each element e is evaluated from a subset (S_e) of the design variables associated with a set of nodes enrolled in a circular domain (Ω_e) of radius r_{min}, whose center coincides with the centroid of the reference element e, as shown in Fig. 2. The influence of nodal variables on the element densities is weighted by the distance between the position of each node ($\mathbf{y}_j$) and the position of the centroid of the element ($\mathbf{y}^e$), according to eq. (1).

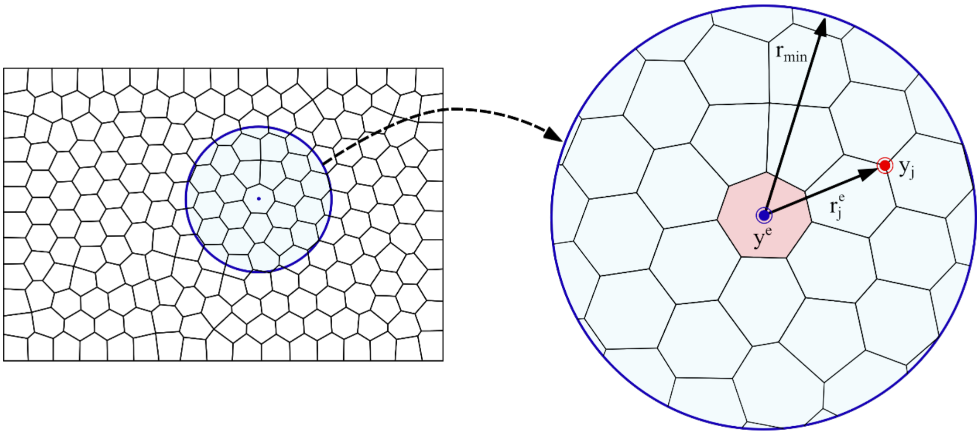


Figure 2. Projection scheme.

$$\rho^e = \frac{\sum_{j \in S_e} x_j \, w(\mathbf{y}_j - \mathbf{y}^e)}{\sum_{j \in S_e} w(\mathbf{y}_j - \mathbf{y}^e)} \tag{1}$$

With:

$$\mathbf{y}_j \in S_e \ \text{ if } \ r_j^e = \left|\mathbf{y}_j - \mathbf{y}^e\right| \leq r_{\min} \tag{2}$$

The influence of nodal design variables on the element densities is computed using a projection function (w). The direct function used by Guest, Prévost and Belytschko [5] is presented in eq. (3). In this case, the nodes closest to the centroid of the element e have a greater influence on the composition of the element density ρ^e. Similarly, Almeida, Paulino and Silva [6] proposed the inverse projection function in eq. (4). In contrast to the direct case, the inverse projection function gives more weight to the elements closer to the border of Ω_e and less to those close to $\mathbf{y}^e$.

$$w_{dir}\left(\mathbf{y}_j - \mathbf{y}^e\right) = \begin{cases} \dfrac{r_{\min} - r_j^e}{r_{\min}} & \text{if } \mathrm{y}_j \in \Omega^e \\ 0 & \text{if } \mathrm{y}_j \notin \Omega^e \end{cases} \tag{3}$$

$$w_{inv}\left(\mathbf{y}_j - \mathbf{y}^e\right) = \begin{cases} \dfrac{r_j^e}{r_{\min}} & \text{if } \mathrm{y}_j \in \Omega^e \\ 0 & \text{if } \mathrm{y}_j \notin \Omega^e \end{cases} \tag{4}$$

Linear projection functions eliminate mesh dependency and other instabilities, such as checkerboard patterns. However, the percentage of intermediate densities is relatively high, especially when large values are adopted for the projection radius. Thus, Guest, Prévost and Belytschko [5] proposed a projection function presented in eq. (5) that penalizes elements with intermediate densities based on a regularization of the Heaviside function. By this proposal, μ^e is the linear projection given in eq. (1).

$$\rho^e = 1 - \mathrm{e}^{-\beta \mu^e\left(\mathbf{x}^{\Omega_e}\right)} + \mu^e\left(\mathbf{x}^{\Omega_e}\right)\mathrm{e}^{-\beta} \tag{5}$$

Where: β controls the softness of the projection curve and $\mathbf{x}^{\Omega e}$ is the subset of design variables in Ω^e.

It is important to note that the larger β is, the closer the function (5) is to the Heaviside function and that $\beta = 0$ conduces to the linear projection scheme described in eq. (1).

This work proposes a Heaviside approach for the inverse projection scheme, using the projection function (5) associated to the inverse projection.

4 Formulation

The formulation of the topology optimization problem is written as a material distribution where void and solid regions are represented by 0 and 1, respectively. The SIMP model is used to relate the element densities to the stiffness parameter, in this case the Young's module, while penalizing the intermediate densities. A small value (ρ_{min}) is introduced as a lower limit for the densities in order to prevent singularity problems in the stiffness matrix caused by null densities, providing a lower limit to de design variable (x_{min}) as shown in eq. (7). Therefore:

$$\begin{aligned} \min \quad & c\left(\boldsymbol{\rho}(\mathbf{x})\right) = \mathbf{f}^{\mathrm{T}}\mathbf{u} \\ \text{s.t.} \quad & \mathbf{K}\left(\boldsymbol{\rho}(\mathbf{x})\right)\mathbf{u} = \mathbf{f} \\ & \sum_{e\in\Omega} \rho^{e}\left(\mathbf{x}^{\Omega_e}\right) v^{e} \le V \\ & x_{\min} \le x_j \le 1 \qquad j = 1..n \end{aligned} \tag{6}$$

Where: c is the structural mean compliance; **x** is the design variables vector; **f** is the force vector; **u** is the displacement vector; **K** is the stiffness matrix; ρ^e is the density associated to element e; Ω is the design domain; v^e is the volume of element e; V is the prescribed material volume; x_{min} is a small positive value given by eq. (7); x_j is the design variable j; n is number of design variables.

$$x_{\min} = \begin{cases} -\dfrac{1}{\beta}\ln\left(1-\rho_{\min}\right) & \text{if } \beta > 0 \\ \rho_{\min} & \text{otherwise} \end{cases} \tag{7}$$

The sensitivity of the objective function is given by eq. (8):

$$\frac{\partial c}{\partial x_j} = -p\left(\rho^{e}\right)^{p-1}\left(\mathbf{u}^{e}\right)^{\mathrm{T}}\mathbf{k}_0^{e}\,\mathbf{u}^{e}\,\frac{\partial \rho^{e}}{\partial x_j} \tag{8}$$

Where: p is penalization factor of the SIMP model; $\mathbf{u}^e$ is the displacement vector of element e; $\mathbf{k}_0^e$ is the basic stiffness matrix of element e, with the Young's module removed.

Petersson and Sigmund [9] proposed a technique to avoid local minimums during the optimization process. The continuation technique consists of a gradual increase in the penalty coefficient p of the SIMP model that governs the mechanical behavior of the structural finite elements. The original work proposed to begin at $p = 1$ with increments of $\Delta p = 0.5$, with maximum value $p = 5$. For each p, the optimization process is carried out until a local convergence.

Guest, Prévost and Belytschko [5] used continuation with both the linear and Heaviside functions. In the case of the Heaviside function, the value of β was kept fixed during the continuation process. However, Li and Khandelwal [10] tested several variations of the continuation method to the nonlinear projection scheme and concluded that maintaining a fixed value for β is not a good strategy. The authors proposed that initially the coefficient p varies from 0 to 3 keeping $\beta = 0$, which is equivalent to the linear projection function. From this point on, β varies adopting $\beta_{max} = 200$.

In the present work, the proposal by Li and Khandelwal [10] for the continuation technique was adapted in order to reduce the processing time. In step 1, the process starts with $p = 1$ and $\beta = 0$, p is increased by $\Delta p = 0.5$ until the maximum value ($p = 3$). In this stage, β remains constant. Step 2 starts after obtaining the linear results. It is fixed $p = 3$ and β is increased according to eq. (9) up to $\beta = 150$. A convergence criterion on displacements was adopted during step 1. During step 2, a convergence criterion is based on variation of the percentage of intermediate densities. Convergence, in both cases, occurs when a tolerance of 0.1% is reached. The final topology is obtained after the percentage of intermediate densities is below 1% or after the end of the optimization with $\beta = 150$.

$$\beta_k = \begin{cases} 1.5\beta_{k-1} & \text{if id} > 5\% \\ \beta_{k-1} + 5 & \text{if id} \le 5\% \end{cases} \tag{9}$$

5 Numerical results

All the following examples were discretized in meshes with 10,000 polygonal finite elements. Young Module $E = 1$, Poisson's ratio $\nu = 0.25$ and thickness $t = 1$ were adopted. The problem described in (6) is solved with *optimality criteria method* and the percentage of intermediate densities of the optimized structures (D_i) is evaluated by eq. (10), as proposed by Sigmund [11].

$$D_i = \frac{\sum_{e=1}^{n} 4\rho^e(1-\rho^e)}{n} \times 100\% \tag{10}$$

A computational code was developed in Python 3.8.5 and Julia 1.4.2. The results were plotted using the colormap *jet* available in the matplotlib 3.3.0 library. In order to simplify the text, the following nomenclatures are adopted for the methods used: direct linear projection scheme (PS-DL), inverse linear projection scheme (PS-IL), direct Heaviside projection scheme (PS-DH) and scheme inverse Heaviside projection (PS-IH).

5.1 Example 1

The first example is the cantilever beam shown in Fig. 3a. Although there is no checkerboard pattern in the solution presented in Fig. 3b, obtained with no projection schemes, the topology presents several thin structural elements, which may be a huge problem in the manufacturing process. In the solution is obtained applying any projection scheme the thin elements are eliminated, which delivers a considerably simplified final topology. In all cases a projection radius $r_{min} = 0.04$ was adopted.

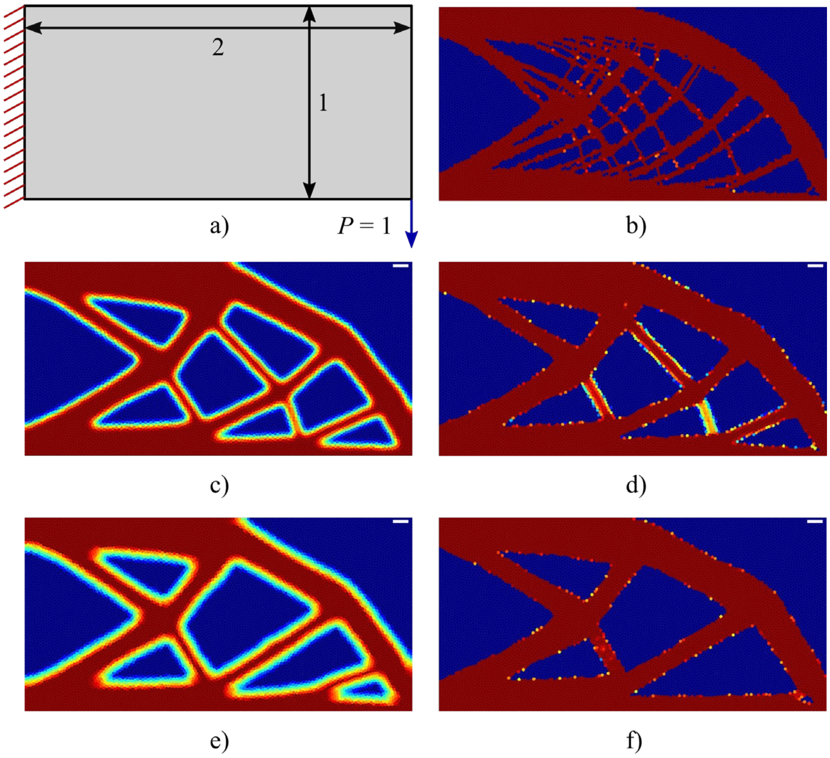


Figure 3: Example 1 – cantilever beam: a) structural domain; b) topology obtained with no filter applied; c) topology obtained applying PS-DL; d) topology obtained applying PS-DH; e) topology obtained applying PS-IL; f) topology obtained applying PS-IH.

The final topologies for PS-DL and PS-IL, shown in Fig.3c and Fig. 3e, are consistent with those found by Almeida, Paulino and Silva [6]. Fig. 3c shows the solution to the problem with the direct linear projection scheme (PS-DL). It is remarkable the high percentage of intermediate densities present in the solution. However, as shown in Fig. 3d and Tab. 1, the application of the Heaviside projection function (PS-DH) reduced this percentage significantly. A similar behavior was observed with the inverse projection scheme. For the linear projections (PS-

DL and PS-IL), the transition range between solid and empty elements observed is larger in the inverse scheme (Fig 3e) than in the direct scheme (Fig 3c). Some intermediate densities still remain in the topology obtained by the direct scheme using the Heaviside projection function (Fig. 3d). The topology obtained by the inverse scheme using the Heaviside projection function (Fig. 3f) no significantly intermediate densities are observed.

Table 1. Intermediate density for example 1

Projection scheme	Projection function	*Di* (%)
No projection	----------------	0.49
Direct	Linear	18.90
Direct	Heaviside	3.12
Inverse	Linear	23.02
Inverse	Heaviside	0.99

5.2 Example 2

In order to show the applicability of the polygonal elements, the solutions for the hook problem analyzed by Talischi *et al*. [4] are presented here applying the inverse projection scheme. The problem geometry is presented in Fig. 4 and the correspondent values are presented in Table 2. In all cases a projection radius r_{min} = 2 was adopted.

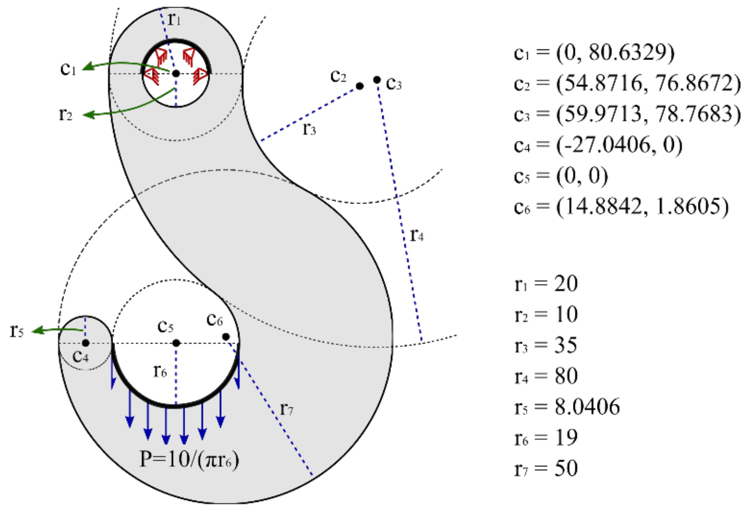


Figure 4: Example 2 – Hook.

As in example 1, the solution without a filter (Fig. 5a) presents lots of thin structural elements. Turning this solution not suitable for practical applications. By applying PS-IL (Fig. 5b) a simpler topology was obtained, but the solution in Fig 5b still presents a high percentage of intermediate densities. By applying PS-IH (Fig. 5c) the simplicity of the final topology was maintained and the percentage of intermediate densities in the final solution was considerably reduced.

Table 2. Intermediate density for example 2

Projection scheme	Projection function	*Di* (%)
No projection	----------------	0.90
Direct	Linear	28.84
Direct	Heaviside	1.30

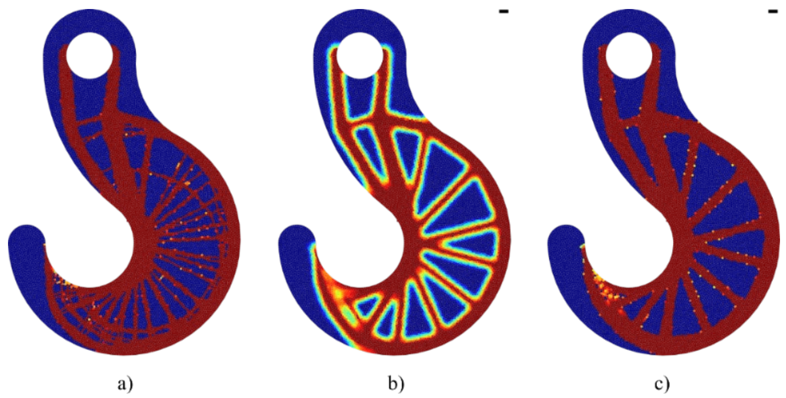


Figure 5: Example 2 – Hook: a) no filter; b) PS-IL; d) PS-IH.

6 Conclusions

The use of polygonal finite elements with projection schemes proved to be an efficient way to solve topology optimization problems in complex domains. The solutions obtained present no mesh dependency nor checkerboard patterns. The dimensions of the structural elements or voids in the final topology can be controlled in order to obtain better solutions from a constructive point of view.

The Heaviside projection function for the inverse projection scheme produced good solutions with the polygonal elements. It is important to note that these results were possible due to the implementation of the modification of the continuity technique, applying variations in p and β. Finally, the use of polygonal finite elements allows the discretization of problems with considerably complex extended domains

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