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Use of an improved fractional derivative model for transient analyses of viscoelastic systems

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Abstract. Vibration control is an important field in the study of structures. One of the ways commonly used to mitigate structural oscillations is passive control using viscoelastic materials. The properties of these materials are strongly dependent on environmental factors, mainly on the temperature and frequency of excitation, so that the models for the characterization of their dynamic behavior must be able to represent this dependence. In this sense, this work presents the modeling of a sandwich beam treated with viscoelastic material and subjected to dynamic loading. For this, a viscoelastic fractional model is presented, the kinematic relations between the layers were obtained, and, applying the variational principles of mechanics the equation of motion of the system was found and evaluated using finite element method. Through this procedure it was possible to represent the transient behavior of the sandwich beam and compare it with the response of the structure with no viscoelastic treatment.

Keywords: viscoelastic material, vibration control, fractional derivative model, finite element method.

1 Introduction

Vibration control is an important field in the study of Engineering and it can be accomplished by the use of active, semi-active or passive control techniques. One of the ways commonly used to mitigate structural vibrations is passive control using viscoelastic materials. Many works have been developed aiming at the application of these materials to the most diverse areas (Filho *et al*. [1], Gonçalves, Rosa and Lima [2], Khoshraftar [3] and Rao [4]). The properties of these materials are strongly dependent on environmental factors, mainly on the temperature and frequency of excitation (Nashif, Jones and Henderson [5]), so that the models for the characterization of their dynamic behavior must be able to represent this dependence.

Many works have been published, especially since the last half of the XX century, aiming at the development of models to represent the behavior of viscoelastic material. In frequency domain the use of a complex modulus, similar to Hooke's law, has shown to be able to present good results, especially when the fractional calculus is applied. In the temporal domain the analysis is more complex and the classical models have shown to be ineffective and/or cumbersome to use. A different approach was proposed by Bagley and Torvik [6], these authors used a fractional derivative model (FDM) to describe the response of viscoelastic materials. The use of fractional calculus in time domain analysis has been shown to be an adequate and very powerful tool that has been used by several researchers (Bagley and Torvik [7], Galucio *et al.* [8], Schmidt and Gaul [9]).

A characteristic of viscoelastic materials that is very well represented by fractional derivative models is the dependence on the stress history, i. e., the response of the material at the present time depends not only on the current load but also on all previous states, which is known as the memory effect. From this memory effect comes damping proprieties of viscoelastic material but it also causes an auto-dependence in stress field equations, which raises computational cost, especially when applied to FE (finite element) models constituted by a big number of DOFs (degrees of freedom). In the Structural Mechanics Laboratory (LMEst) at the Federal University of Uberlândia (UFU), studies have been performed in order to eliminate this auto-dependence though the use of recurrence terms. In this sense, this work aims to apply this model to a sandwich beam treated with viscoelastic material and compare its response with the untreated beam.

2 Sandwich beam model

2.1 Viscoelastic material formulation

Based on a four parameters fractional derivative model for the viscoelastic behavior (Bagley and Torvik [7] and Makris [10]), the stress-strain relations for elongation and shear are described by Eq. (1) and (2), where E_0 and E_∞ are the elongation modulus associated with low and high frequencies, G_0 and G_∞ are the shear modulus associated with low and high frequencies, a_E and a_G are parameters of the model associated with elongation and shear, and α is the fractional derivative order. These parameters are all found by curve fitting. Even though the model requires seven parameters, only four are really necessary, since elongation and shear values are related to each other by the use of the elastic-viscoelastic correspondence principle.

$$\sigma_x + a_E \frac{d^\alpha}{dt^\alpha}\sigma_x = E_0\varepsilon_x + E_\infty \frac{d^\alpha}{dt^\alpha}\varepsilon_x \,. \tag{1}$$

$$\tau_{xz} + a_G \frac{d^\alpha}{dt^\alpha}\tau_{xz} = 2G_0\varepsilon_{xz} + 2G_\infty \frac{d^\alpha}{dt^\alpha}\varepsilon_{xz} \,. \tag{2}$$

By the use of Grünwald's formulation for fractional derivatives (Schmidt and Gaul [9]), Eq. (1) becomes Eq. (3), where A, N_l and Δt stand for the Grünwald's coefficients, the time memory and the time step. One finds an analogous equation for shear using Eq. (2).

$$\{\sigma(t)\} = \frac{(E_0 + E_\infty)\{\varepsilon(t)\} + E_\infty \Delta t^{-\alpha}\sum_{j=1}^{N_l} A_{j+1}\{\varepsilon(t - j\Delta t)\} - a_E \Delta t^{-\alpha}\sum_{j=1}^{N_l} A_{j+1}\{\sigma(t - j\Delta t)\}}{1 + a_E \Delta t^{-\alpha}} \,. \tag{3}$$

As mentioned before, the constitutive law for the viscoelastic material depends on the stress and strain history, as it can be noticed in Eq. (3), which makes the analysis more costly. The development of a model based on a recurrence term will eliminate this auto-dependence, as it has been shown by Filho *et al*. [11]. This new approach is based on the analysis of Eq. (3) in successive time steps, in order to find the way that the stresses from the previous times influence the present state. In the n-th step of time, Eq. (3) can be expressed as Eq. (4). Since $n\Delta t$ is an arbitrary time, Eq. (4) is valid for any time t and can be rearranged as Eq. (5), where βs are the recurrence terms expressed by Eq. (6). In order to simplify notation, henceforth the time dependence will be represented as a subscript which means, for example, that $\sigma(t)$ will be represented by σ_t.

$$\begin{aligned}
\{\sigma_{n\Delta t}\} = \beta_1^E\{\varepsilon_{n\Delta t}\} + \frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2\{\varepsilon_{(n-1)\Delta t}\} - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_2\left(\beta_1^E\{\varepsilon_{(n-1)\Delta t}\} + \ldots + \beta_{n-2}^E\{\varepsilon_{2\Delta t}\} + \beta_{n-1}^E\{\varepsilon_{\Delta t}\}\right) + \ldots \\
\ldots + \frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{n-1}\{\varepsilon_{2\Delta t}\} - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{n-1}\left(\beta_1^E\{\varepsilon_{2\Delta t}\} + \beta_2^E\{\varepsilon_{\Delta t}\}\right) \\
+ \frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_n\{\varepsilon_{\Delta t}\} - \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_n\left(\beta_1^E\{\varepsilon_{\Delta t}\}\right)
\end{aligned} \,. \tag{4}$$

$$\sigma_t = \sum_{j=0}^{N_l} \beta_{j+1}^E \varepsilon_{t-j\Delta t} \,. \tag{5}$$

$$\beta_{j+1}^E = \frac{E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{j+1} - \sum_{i=0}^{j} \frac{a_E \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} A_{i+1}\beta_{j+1-i}^E \text{ with } \beta_1^E = \frac{E_0 + E_\infty \Delta t^{-\alpha}}{1 + a_E \Delta t^{-\alpha}} \,. \tag{6}$$

By an analogous procedure, one finds Eq. (7) and (8) for shear.

$$\tau_t = \sum_{j=0}^{N_l} \beta_{j+1}^G \gamma_{t-j\Delta t} \,. \tag{7}$$

$$\beta^G_{j+1} = \frac{2G_\infty \Delta t^{-\alpha}}{1+a_G\Delta t^{-\alpha}} A_{j+1} - \sum_{i=0}^{j} \frac{a_G \Delta t^{-\alpha}}{1+a_G\Delta t^{-\alpha}} A_{i+1}\beta^G_{j+1-i} \text{ with } \beta^G_1 = 2\frac{G_0 + G_\infty \Delta t^{-\alpha}}{1+a_G\Delta t^{-\alpha}}. \tag{8}$$

2.2 Finite element formulation

Based on the laminated classical theory and the principles of Dynamics and Materials Mechanics, the displacement and deformation fields of each layer of the structure are found. For the viscoelastic sandwich beam a four degree of freedom model is used. Linear interpolation functions are used for u and β - membrane displacement and shear deformation -, and a cubic function is used for w, the transversal displacement. The last DOF, the rotation θ, is related to w by its first derivative. For this analysis it is considered the same w and θ for every layer. The displacement field is therefore described by Eq. (9) and (10), where (1), (2) and (2) refers to base, viscoelastic and constrain layers; N are the shape functions, H are the matrices shown in Eq. (11), (12) and (13); x is the longitudinal coordinate; z is the coordinate in the direction of the thickness of the beam; z_2 is the value of z coordinate measured from the base-viscoelastic layers interface; h_2 is the width of the viscoelastic layer; and q(t) is a vector containing all DOF of the element.

$$w(x,t) = \left[N_w(x)\right]\left\{q(t)\right\}. \tag{9}$$

$$u^{(k)}(x,z,t) = \left[H_u^{(k)}(x,z)\right]\left\{q(t)\right\}, \text{ with k=1, 2, 3}. \tag{10}$$

$$\left[H_u^{(1)}(x,z)\right] = \left[N_u(x)\right] - z\frac{\partial\left[N_w(x)\right]}{\partial x}. \tag{11}$$

$$\left[H_u^{(2)}(x,z)\right] = \left[N_u(x)\right] - z\frac{\partial\left[N_w(x)\right]}{\partial x} + z_2\left[N_\beta(x)\right]. \tag{12}$$

$$\left[H_u^{(3)}(x,z)\right] = \left[N_u(x)\right] - z\frac{\partial\left[N_w(x)\right]}{\partial x} + h_2\left[N_\beta(x)\right]. \tag{13}$$

From the displacement fields, one finds the deformations of each layer by the use of the principles of Continuum Mechanics (Malvern [12]). For an arbitrary element, mass matrix and stiffness matrices for elastic and viscoelastic layers are described by Eq. (14), (15), (16) and (17), where ρ, A, b, li and E stand for density, the area of the transversal section of each layer, the width of the beam, the length of the element and Young's Modulus. It is important to mention that Eq. (16) and (17) are parametrized matrices for elongation and shear of the viscoelastic layer and do not include the recurrence terms yet. Since different recurrence terms are used in stress and strain for each previous time (see Eq. (4)), it is not possible to find a general stiffness matrix.

$$\left[M_e\right] = \sum_{k=1}^{3}\rho_k\left(b\int_{z_k}^{z_k+h_k}\int_0^{li}\left[H_u^{(k)}\right]^T\left[H_u^{(k)}\right]dxdz + A_k\int_0^{li}\left[N_w\right]^T\left[N_w\right]dx\right). \tag{14}$$

$$\left[K_e^{(e)}\right] = b\sum_{k=1,\,3} E_k\int_{z_k}^{z_k+h_k}\int_0^{li}\left[H'^{(k)}_u\right]^T\left[H'^{(k)}_u\right]dxdz. \tag{15}$$

$$\left[K^{*\,(E)}_e\right] = b\int_{z_2}^{z_2+h_2}\int_0^{li}\left[H'^{(2)}_u\right]^T\left[H'^{(2)}_u\right]dxdz. \tag{16}$$

$$\left[K^{*\,(G)}_e\right] = b\int_{z_2}^{z_2+h_2}\int_0^{li}\left[N_\beta\right]^T\left[N_\beta\right]dxdz. \tag{17}$$

Then, global matrices, kinetic and deformation energies are found, and using Lagrange's equations, one finds Eq. (18), which is the equation of motion of the system and it can be solved by numerical methods such as Newmark.

$$\left[M\right]\left\{\ddot{q}(t)\right\} + \left[K^{(e)}\right]\left\{q(t)\right\} + \sum_{i=0}^{N_t}\left(\beta^E_{i+1}\left[K^{*(E)}\right] + \beta^G_{i+1}\left[K^{*(G)}\right]\right)\left\{q(t-i\Delta t)\right\} = \left\{F(t)\right\}. \tag{18}$$

3 Numerical simulations

3.1 Validation

In order to validate the sandwich beam finite element model, a modal analysis was performed at Ansys student version software. The first four natural frequencies were compared with the ones predicted by the present theory. The properties of the cantilever beam are shown in Tab. 1, the natural frequencies are compared in Tab. 2 and the mode shapes are shown in Fig. 2. It can be noticed the good agreement between the formulation presented here and the Ansys' results.

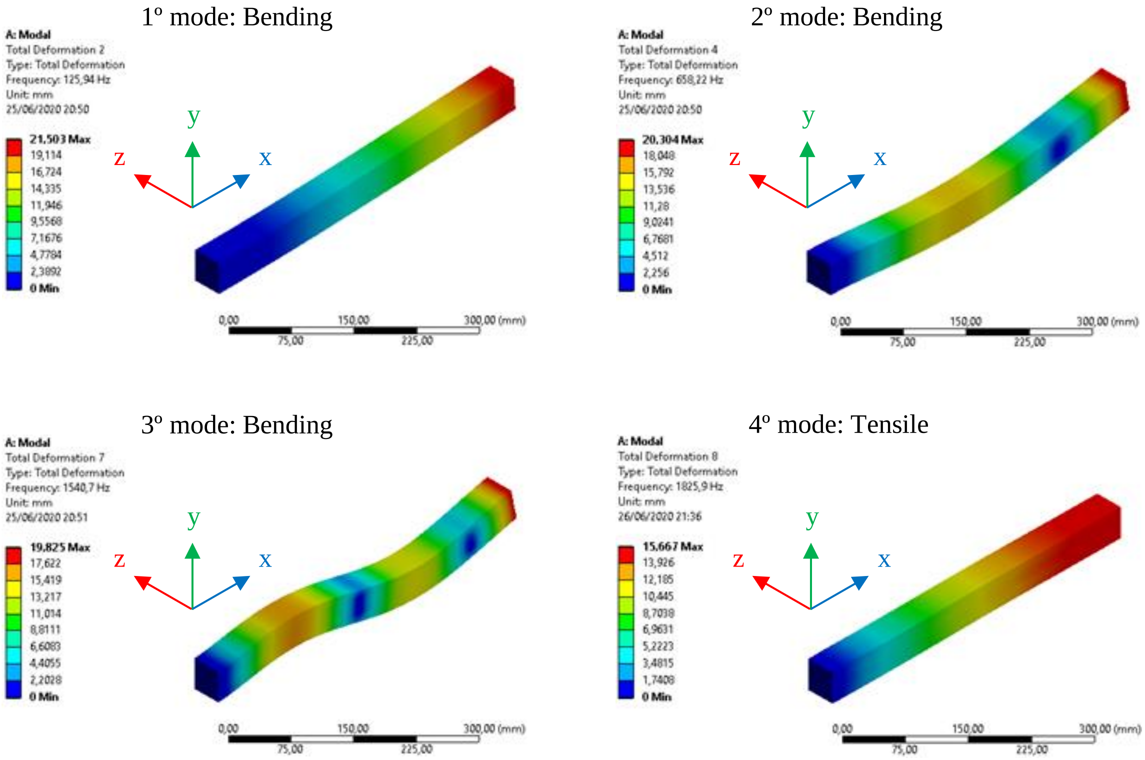

Figure 1. Mode shapes

Table 1. Properties of the sandwich beam

	Steel layer (1)	Lead layer (2)	Steel layer (3)
Length (L)	0.5 m	0.5 m	0.5 m
Width (b)	0.04 m	0.04 m	0.04 m
Thickness (h)	0.015 m	0.02 m	0.01 m
Young's modulus (E)	210 GPa	16 GPa	210 GPa
Shear modulus (G)	80 GPa	5.5 GPa	80 GPa
Density (ρ)	7850 kg/m³	11100 kg/m³	7850 kg/m³

Table 2. Natural frequencies

Frequency no.	Ansys (Hz)	Present theory (Hz)	Percent error (%)
1	125.94	125.65	0.2
2	658.22	662.19	0.6
3	1540.7	1560.6	1.3
4	1825.9	1824.6	0.1

3.2 Curve fitting - FDM parameters

In order to determine the parameters for the FDM, an optimization using the differential evolution method was performed. Soovere and Drake [13] proposed analytical equations to describe the master's curves of several viscoelastic material. For ISD 112, the material used in this work, figure 3 shows, for the temperature of 27 ºC, the adequacy between Soovere and Drake's equations and the optimization of the complex modulus found by the use of Fourier's transform in Eq. (2).

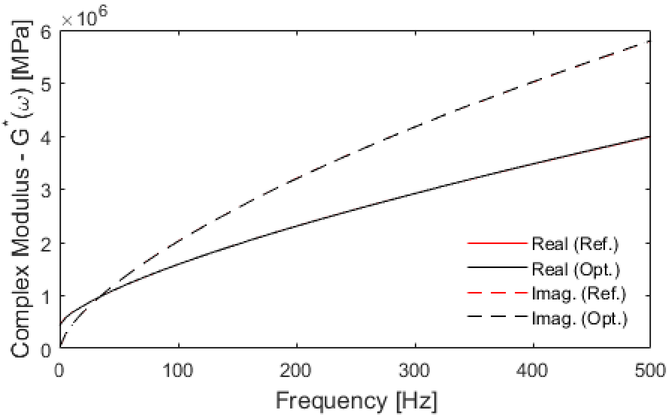

Figure 2. FDM parameters optimization

3.3 Dynamic analysis

For the following simulations, the same cantilever beam geometry and mechanical properties presented in Tab. (1) are used. The only difference is the material of the second layer, which is now made of 3M ISD-112. The properties of this material are shown in Tab. 3 for the temperature of 27 ºC.

Table 3. 3M ISD-112 properties for 27 ºC

ρ (kg/m^3)	G_0 (Pa)	G_∞ (Pa s$^\alpha$)	a_G (s$^\alpha$)	α (-)
1600	423632.8	30177.8	0.00022	0.6766

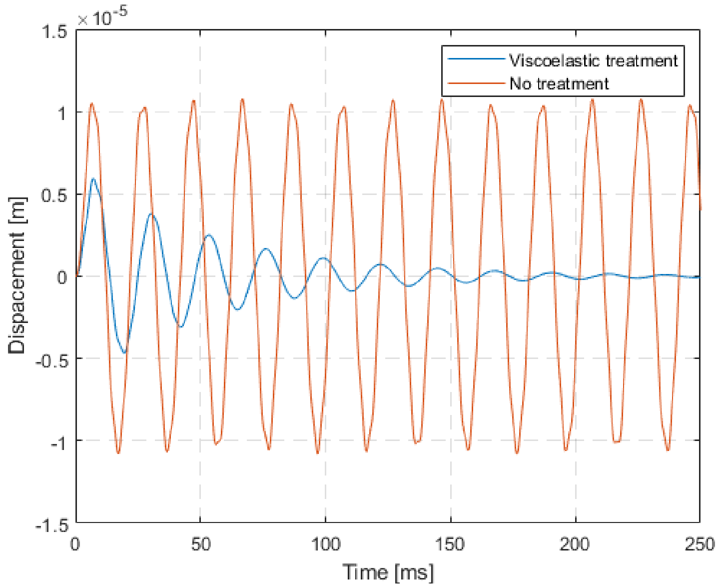

Figure 3. Response to an impulse load

Two dynamic responses are shown in Fig (3) and (4). The beam is first subjected to a triangular unit impulse at the tip and then to a harmonic force at 50 Hz, an excitation frequency very close to the first natural frequency. Both figures show the effectiveness of viscoelastic treatment in vibration control. The untreated beam is theoretically uncapable to dissipate energy and would vibrate indefinitely, even for an impulse load. Near the resonance frequency the response is even worse, the vibration amplitude is always increasing with time and would cause failure of the structure. In both cases the use of viscoelastic material has shown to be a useful technique to mitigate vibration.

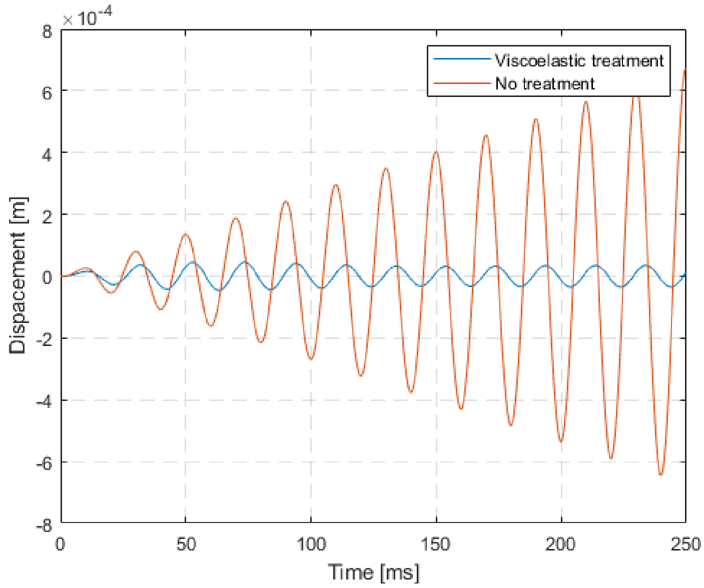

Figure 4. Response to a harmonic loading

To show the vibration control capability of the viscoelastic material, the frequency domain responses for both treated and not treated beams are shown in Fig. 5. The FRF (Frequency response function) is found by the use of Fourier's transform in Eq. (2). It clearly shows the influence of viscoelastic material in vibration control. For the sandwich beam, vibration amplitudes are smaller not only near the resonance frequencies but also at a very wide range of excitation frequencies.

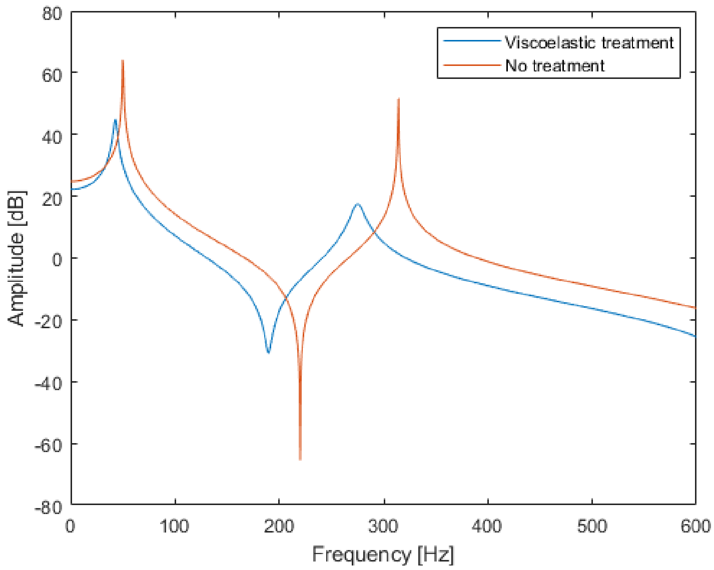

Figure 5. Frequency response function

4 Concluding remarks

This work proposed a new approach to the analysis of viscoelastic treated systems based on recurrence terms. This new recurrence fractional derivative model eliminates the load conditions auto-dependence, which impacts the computational cost. Time domain responses of the treated and untreated beams were analyzed using FDM and showed the capability of the viscoelastic layer in mitigating the vibration of the beam. As further works we suggest the application of model reduction methods to the improved FDM, experimental studies, and a comparison between the classical models and the one proposed in the present work.

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