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Reducing the Discretization Error for Global and Local Variables in Poroelasticity Problems

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Abstract. This work analyzes the efficiency of the Repeated Richardson Extrapolation (RRE) to reduce the discretization error (Eh) that results from the numerical resolution of one-dimensional poroelasticity problem. The Finite Difference Method (FDM) was employed with second-order CDS approximation for spatial variables and Crank-Nicolson approximation for temporal variables. The three-point Vanka smoother was used in the iterative process. The multigrid method with W-cycles was used to accelerate the convergence of the iterative process, which involved highly refined grids. The analyze of the results considered as local variables, the displacement and pressure in the central point of the domain, based on localized fixed coordinate coinciding with a node point in all grids considered; and as global variable, the average value of the variable of interest from all node values. It was verified that employing RRE in the problems analyzed results in a significant reduction in Eh.

Keywords: Repeated Richardson Extrapolation; Porous media; Multigrid.

1 Introduction

One of the greatest challenges faced by researchers in the field of Computational Fluid Dynamics (CFD) is the level of accuracy of numerical solutions. Although numerical errors cannot be completely eliminated, it is imperative that they are controlled or minimized in computational numerical simulations. Of all the sources of numerical errors, the discretization error (Eh) is considered the most significant according to Roy and Obeekampf [1]. As ways to reduce this error, Richardson and Gaunt [2], Marchi et al. [3] present as alternatives: a) mesh refinement, which results in an increase in computational cost; b) an increase in the order of accuracy of the approximations, which leads to an increase in the complexity of the numerical model; and c) the use of extrapolation techniques, which is considered a post-processing method of easy implementation and low computational cost. In this context, in the present work we aim to evaluate the efficiency of the Repeated Richardson Extrapolation method (RRE) when applied to a poroelasticity problem, seeking to minimize Eh and consequently increase the level of accuracy of the numerical solution.

Poroelasticity equations mathematically model the interaction between the deformation of a porous elastic material and the fluid flow inside of it. The general three-dimensional theory was formulated by Biot [4] and named after the author. It is currently known as the Biot consolidation model. The analysis and numerical simulation in the Biot consolidation model have become more popular and have been discussed in recent works due to its range of applications, such as in Medicine, Petroleum Engineering, Biomechanics, among other fields of Science and Engineering, Ehlers and Bluhm [5], Franco et al. [6].

The mathematical model we used in this work is the one-dimensional poroelasticity equation, which is described in section 2. The equation is discretized by employing the finite difference method using second order

accuracy approximation, and Dirichlet and Neumann boundary conditions. It was solved by using the three-point Vanka smoother, double and quadruple precision, up to ten extrapolation levels with RRE, and the sufficient number of iterations to achieve the rounding error. We analyzed the following variables: a) displacement and pressure in the center of the domain; and b) the averages of displacements and pressures. The results obtained indicate that the methodology employed in this work is promising in terms of increasing the accuracy of numerical solutions in poroelasticity problems.

2 Mathematical and Numerical Models

The mathematical model used in this study was the one-dimensional poroelasticity equation. Considering the domain space $\Omega = \left(0, \frac{1}{2}\right)$ we have

$$\begin{cases} -\mathrm{E}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial p}{\partial x} = \mathscr{U} \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial u}{\partial x}\right) - K\dfrac{\partial^2 p}{\partial x^2} = \mathscr{P} \end{cases}, \tag{1}$$

where E is Young's modulus, K holds information on physical properties related to the porosity and permeability of the medium and is called hydraulic conductivity, $\mathscr{U}$ is the density of the force applied to the body and $\mathscr{P}$ is the force of injection or extraction of the fluid in the porous medium. The components $u(x,t)$ and $p(x,t)$ respectively represent the displacement and pressure in the spatial direction x. The displacements u are modeled by the first equation, whereas the second equation corresponds to the pressure p.

As boundary conditions, $x = 0$ represents free (permeable) drainage with no displacement variation; $x = \frac{1}{2}$ represents right boundary with no displacement or pressure variation, defined by

$$p(0,t) = \mathrm{E}\frac{\partial u(0,t)}{\partial x} = u\left(\frac{1}{2}, t\right) = K\frac{\partial p\left(\frac{1}{2}, t\right)}{\partial x} = 0. \tag{2}$$

Based on the method of manufactured solutions, as in Roy [7], when considering the analytical solution given by

$$u(x,t) = \cos(\pi x)e^{-t} \quad \text{and} \quad p(x,t) = \sin(\pi x)e^{-t}, \tag{3}$$

which satisfies the boundary conditions presented in eq. (2), the forcing terms are given by

$$\mathscr{U} = (\mathrm{E}\pi + 1)\pi\cos(\pi x)e^{-t} \quad \text{and} \quad \mathscr{P} = (1 + K\pi)\pi\sin(\pi x)e^{-t}. \tag{4}$$

For the numerical model, the spatial domain is discretized by the Finite Difference Method (FDM), uniform meshes and Central Difference Scheme (CDS). The temporal approximation and the spatial and temporal connection are made by using the Crank-Nicolson method. In addition, we used a reformulated version of the system of equations exposed by Gaspar et al. [8], which presents an additional smoothing term in the equation corresponding to pressure, making the system more stable without changing its final result. This term is given by

$$-\frac{h^2}{4(\lambda + 2\mu)}\frac{\partial \Delta p}{\partial t}, \tag{5}$$

where the constants λ and μ are Lamé coefficients.

Considering $\mathrm{E} = \lambda + 2\mu$, and adding the smoothing term eq. (5), in eq. (1) corresponding to the pressure, the discretization for internal points, that is, for $i = 2, 3, \ldots, N-1$ results in

$$\begin{cases} -\mathrm{E}\dfrac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \dfrac{p_{i+1}^{n+1} - p_{i-1}^{n+1}}{2h} = \mathcal{U}_i^{n+1} \\ \dfrac{\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h} - \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2h}}{\tau} - \dfrac{h^2}{4\mathrm{E}\tau}\left[\dfrac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2} - \dfrac{p_{i+1}^{n} - 2p_i^{n} + p_{i-1}^{n}}{h^2}\right] = \dfrac{\mathcal{P}_i^{n+1} + \mathcal{P}_i^{n}}{2} + \\ \qquad + \dfrac{K}{2}\left(\dfrac{p_{i+1}^{n+1} - 2p_i^{n+1} + p_{i-1}^{n+1}}{h^2} + \dfrac{p_{i+1}^{n} - 2p_i^{n} + p_{i-1}^{n}}{h^2}\right) \end{cases}, \tag{6}$$

where $i, i-1$ and $i+1$ indicate the spatial discretization, and n and $n+1$ respectively indicate the current and previous time steps. The size of the time step is given by τ and the length of the spatial discretization is given by h. We adapted these equations for the boundaries $i = 1$ and $i = N$. Further details are found in Franco [9].

After the discretizations we obtained systems of equations that were solved by using the three-point Vanka smoother. This method is part of a class of smoothers that perform block smoothing, which are indicated when the solutions of the resulting equation systems have saddle points, as presented by Franco et al. [6], John [10]. In this work, this occurs when $K\Delta p \approx 0$.

2.1 Multigrid Method

The multigrid method was originally proposed by Fedorenko [11], who showed that the speed of convergence with the use of the multigrid technique is better than that of pure iterative methods (without the use of multigrid), which are called singlegrid.

The basic principle of the method is to use a set of grids and alternate smoothing at each grid level and the approximations of these solutions in a coarser grid (with a certain coarsening ratio re) through operators that transfer information from the fine grid to the following coarser grid (restriction operator), and then transfer information from the coarse grid to the following finer grid (prolongation operator), thus reducing the entire spectrum of errors (high and low frequency errors), Briggs et al. [12], Trottenberg et al. [13], Wesseling [14]. The sequence in which the different grids are visited characterizes a multigrid cycle that can be classified as types V, W, F, and others.

2.2 Repeated Richardson Extrapolation (RRE)

The method known as Richardson Extrapolation (RE) can be employed when we have an approximation technique that shows a predictive error term, that is, which depends on a real parameter, what happens with h (spacing between the nodal points - or nodes - of the mesh), and which is represented by means of an analytical series according to Burden and Faires [15]. Richardson Extrapolation was initially used to combine approximations in order to generate results with a higher order of accuracy (p_A) considering a few specific parameters. As stated by Oberkampf and Roy [16], this procedure became known as the standard Richardson Extrapolation. Subsequently, general values for such parameters started to be considered and it became known as the generalized Richardson Extrapolation.

The RE is considered a post-processing method that can be used a posteriori in the $\phi(h)$ solutions obtained in different Ω^h meshes, taking into account the refining ratio $(r = h_g/h_{g+1})$, where the sub-indexes $g+1$ and g respectively represent the fine and coarse meshes. Richardson's original equation, presented in Richardson and Gaunt [2], is

$$\phi_\infty = \frac{h_g^2\phi_{g+1} - h_{g+1}^2\phi_g}{h_g^2 - h_{g+1}^2} + E(\phi_\infty), \tag{7}$$

where ϕ_∞ is the estimated analytical solution, and ϕ_{g+1} and ϕ_g are the numerical solutions in the fine and coarse grids, respectively. By generalizing the RE to any asymptotic order (p_0) and r, we have

$$\phi_\infty = \phi_{g+1} + \frac{\phi_{g+1} - \phi_g}{r^{p_0} - 1}, \tag{8}$$

which will be effective if the numerical solutions ϕ_g only have discretization errors, Novak [17].

The RRE consists of the recursive application of RE in order to raise the order of accuracy of the discretization error (Eh). The recursion process is created based on eq. (8), that is, we consider

$$\phi_0(h_g) = \phi(h_g), \qquad g = 1, 2, \ldots, \tag{9}$$

$$\phi_1(h_{g+1}) = \phi_0(h_{g+1}) + \frac{\phi_0(h_{g+1}) - \phi_0(h_g)}{r^{p_0} - 1}, \qquad g = 1, 2, \ldots. \tag{10}$$

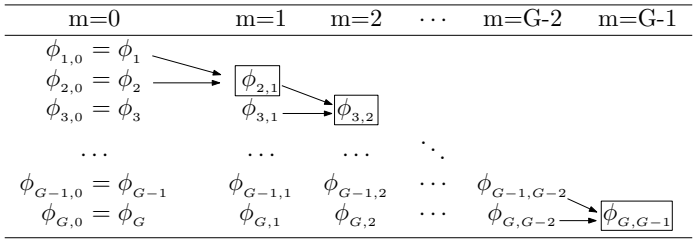
Given the above, considering m as the extrapolation levels, and g indicating the Ω^h, meshes, with m and g being non-null natural numbers, eq. (10) is then represented, as in Marchi et al. [3] by

$$\phi_{g,m} = \phi_{g,m-1} + \frac{\phi_{g,m-1} - \phi_{g-1,m-1}}{r^{p_{m-1}} - 1}, \tag{11}$$

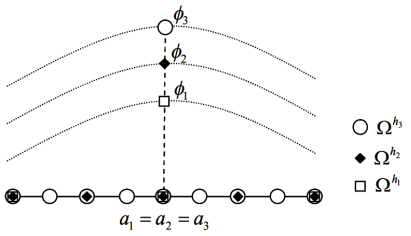
where $m = 1, 2, \ldots$ and $g = m + 1, m + 2 \ldots$.

From a theoretical point of view, eq. (11) can be repeated infinitely, however, for practical applications we must consider a limit value for g, that is, $g = G$, where G is a positive integer that corresponds to the number of meshes adopted. According to Marchi et al. [18], it is assumed that the use of this recursive process eq. (11) provides a progressive increase in the order of accuracy of Eh.

In order to employ RRE it is necessary to obtain numerical solutions for a given variable of interest in a collection of different meshes. Its schematic representation is shown in Fig. 1(a). In this work the variables of interest used are local variables with the location of their coordinates maintained in all meshes considered and coinciding with a nodal point Fig. 1(b), or still, characterized by a global variable, represented in this work by the mean pressure value (p_m).



(a) Schematic representation of the use of RRE.



(b) Variable with the same nodal position in different meshes.

Figure 1. Schematic representation of the RRE method and types of variables used. Adapted from Martins [19].

Fig. 1(b) shows that ϕ_1, ϕ_2 and ϕ_3 with coordinates a_1, a_2 and a_3 respectively correspond to the numerical solutions obtained in the meshes Ω^{h_1} (coarse), Ω^{h_2} (fine), and Ω^{h_3} (extra-fine).

3 Numerical Errors

For a given variable of interest, the numerical error (E) is defined as the difference between the exact solution analytics (Φ) and its numerical solution (ϕ), Marchi et al. [3], that is,

$$E(\phi) = \Phi - \phi. \tag{12}$$

Numerical errors can be caused by several sources, which are presented in Marchi [20] as: truncation errors (E_T), iteration errors (E_I), rounding errors (E_π) and programming errors (E_{Pr}). When E_I, E_π and E_{Pr} are minimized or even non-existent, E_T is then called, as in Ferzinger and Peric [21], a discretization error (Eh). If Eh is the only source of numerical error, then

$$Eh = E(\phi) = c_0 h^{p_0} + c_1 h^{p_1} + c_2 h^{p_2} + \ldots = \sum_{V=0}^{\infty} c_V h^{p_V}, \tag{13}$$

where the coefficients $c_0, c_1, c_2, \ldots$ are real numbers and can be functions of the dependent variable and its derivatives, but independent of h. The exponents $p_0, p_1, p_2, \ldots$ are the true orders of $E(\phi)$ and its set is represented by $p_V = \{p_0, p_1, p_2, \ldots\}$.

The elements of p_V are positive integers that generally follow the relation $1 \leq p_0 < p_1 < \ldots$ which represents an arithmetic progression of reason $q = p_1 - p_0$. The first term p_0 is called asymptotic order of $E(\phi)$ (or accurary of numerical solution ϕ).

When $h \to 0$ the eq. (13) is reduced to $Eh = c_0 h^{p_0}$. Considering the bilogarithmic graph of Eh *versus* h, we can see that its inclination in relation to the abscissa axis tends to the value of p_0. Therefore, the greater the accuracy value of the solution, the greater the reduction in Eh by refining the mesh.

The values of p_V can be obtained *a priori*, considering the definition of Eh and the use of the Taylor series; or *a posteriori*, as described below using eq. (15). The order of theoretical accuracy of the numerical solution when employing the RRE method, with p_V representing an arithmetic progression for m extrapolation levels, is described in Marchi et al. [3] as:

$$p_V = p_0 + m(p_1 - p_0), \tag{14}$$

which is valid for $g = [1, G]$ and $m = [0, g-1]$. For an *a posteriori* analysis, the values of p_V may be based on the calculation of the effective order of Eh, which when generalized for the RRE method is given, as in Marchi et al. [3] by

$$(pE)_{g,m} = \frac{\log\left(\dfrac{E(\phi_{g-1,m})}{E(\phi_{g,m})}\right)}{\log(r)}, \tag{15}$$

which is valid for $g = [2, G]$ and $m = [0, g-2]$. The E values are calculated by eq. (12) and r is the refining ratio between the meshes. When $h \to 0$, in theory, the value of $(pE)_{g,m}$ should tend to the values of p_V indicated by eq. (14), where m represents the number of extrapolations performed.

4 Numerical Results

The problem analyzed in this work is governed by the system of equations represented by eq.(1) and its boundary conditions are determined in eq.(2). The numerical simulations were performed in Fortran and Intel® Parallel Studio XE 2019 compiler. The computer used featured an Intel® Core™ i7-9700KF processor, 3.60 GHz CPU and 16 GB of RAM.

To simulate the problem we adopted: final time $tf = 1\,s$; length of the calculation domain $L = 1/2$; approximation of time variables using the Crank-Nicolson method; multigrid method with W-cycle; three-point Vanka smoother and stopping criterion until reaching rounding error with double or quadruple precision. Thus, the number of significant figures (without extrapolation) is at least 12 and 30, respectively. Therefore, in these figures, we minimized E_π in the solutions. The values used for Young's module and hydraulic conductivity are respectively E $= 10^4$ and $K = 10^{-5}$, which represent real values, Franco et al. [6].

The calculations were performed in different meshes (Ω^h) with coarsening ratio $re = 2$. The coarsest grid has $N = 5$ nodes and the finest grid $N = 4097$ nodes, which corresponds to 11 grid levels. The variables analyzed are: 1) displacement and pressure at the central point, that is, $u_c = u(1/4, 1)$ and $p_c = p(1/4, 1)$ (local variables); and 2) average value of displacement and pressure, that is, u_m and p_m (global variables).

In this work we only present the results of the variable p_m, since the numerical results of the other variables showed similar behaviors. By comparing the discretization error curves without the use of the RRE method (Eh) and with its use (Em), we can see in Fig.2 (with double precision, Real*8, and quadruple precision, Real*16), that the use of RRE is efficient in reducing the discretization error.

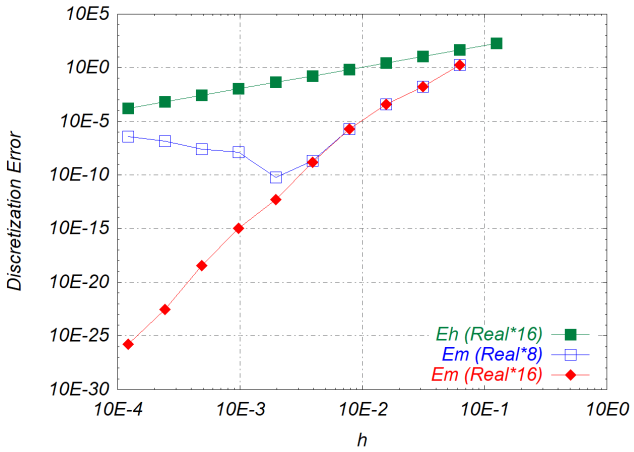


Figure 2. Discretization error for the variable p_m, with and without the use of RRE.

It is also evident that quadruple precision was more significant in reducing Eh than double precision. This is justified due to E_π becoming the main source of numerical error, affecting the efficiency of the RRE method, which in this case occurred after five levels of extrapolation with double precision.

The results presented in Table 1 characterize an example of the effect of RRE on the reduction of Eh, which were evaluated by calculating the ratio $|Eh|/|Em|$. We verified that in the mesh with 65 nodes, when applying four RRE levels, the error was reduced by more than 350 thousand times.

Table 1. Error reduction in three different meshes, variable p_m with quadruple precision.

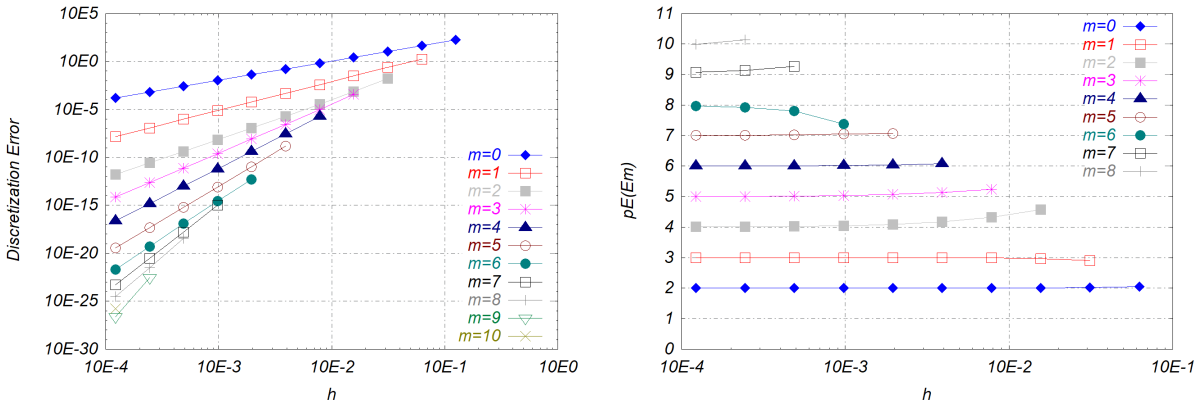
Mesh (h)	65 nodes	513 nodes	4097 nodes
m to Em	4	7	10
$\|Eh\|$	6.734962E-01	1.048402E-02	1.637359E-04
$\|Em\|$	1.825800E-06	1.013794E-15	1.681681E-26
$\|Eh\|/\|Em\|$	3.688772E+05	1.034137E+13	9.736442E+21

Table 2 shows the reduction in the number of nodes of a mesh in order to obtain the discretization error in a same order of magnitude by using the RRE method. For instance, for a given order of magnitude of the error 5.00E-04, with no use of RRE, it is necessary to use a mesh with 4097 nodes, however, by using the RRE method, it requires only 33 nodes, that is, without RRE, the mesh should have 124 times the number of nodes. That is evaluated based on the calculation of the ratio between the number of nodes in the meshes Eh and Em. This provides a reduction in computational effort (less memory and CPU time).

Table 2. Reduction in mesh nodes for fixed errors, variable p_m.

E level	5.00E+00	5.00E-02	5.00E-04
Eh mesh	33 nodes	513 nodes	4097 nodes
$\|Eh\|$	2.705E+00	1.048E-02	1.637E-04
Em mesh	9 nodes	17 nodes	33 nodes
$\|Em\|$	1.783E+00	1.680E-02	3.655E-04
m to Em	1	2	3
Ratio between the number of nodes in Eh and Em	3.66E+00	3.01E+01	1.24E+02

The reduction of Eh for each extrapolation level and their respective orders of accuracy (p_V set) are shown in Fig. 3(a) and 3(b), respectively. The curve indicated by $m = 0$ corresponds to p_m values with no extrapolation.



(a) $Eh \times h$, where m represents the total number of extrapolations. (b) $pE \times h$, where m represents the total number of extrapolations.

Figure 3. Error and order of accuracy $\times$ spacing h between the mesh nodes.

As an example, for $m = 8$, the value of $pE \to 10$ when $h \to 0$, being the highest value found. We can also note that the pE values presented in Fig. 3(b) for each level of extrapolation m represent an arithmetic progression. This confirms what was exposed in subsection 3: the order of theoretical accuracy of the numerical solution when using the RRE method represents an arithmetic progression and can be calculated by eq. (14). The first term in this progression is $p_0 = 2$ (Fig. 3(b)). This corroborates what is described in Strikwerda [22], Fortuna [23], who state that by using second order central difference approximation methods (for spatial domain) and Crank-Nicolson (temporal approximation) the order of accuracy is $O(h^2, \tau^2)$.

5 Conclusion

In this work, we evaluated the efficiency of the Repeated Richardson Extrapolation method applied to the local variables (u_c and p_c) and global variables (u_m and p_m) of a one-dimensional poroelasticity problem, seeking to minimize Eh and increase the accuracy of the numerical solution.

We observed that: 1) the use of RRE proved to be promising in increasing the accuracy of numerical solutions in poroelasticity problems; 2) the more levels of extrapolation are used, the greater the accuracy of the numerical solution; 3) E_π affects the efficiency of the RRE method when it becomes the main source of error, that is, within the limit of the precision established for the calculations; 4) in order to obtain the same order of magnitude as Eh, the use of RRE proved to be highly efficient in reducing the number of nodes in a mesh, thus providing the reduction of computational effort (memory and CPU time).

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