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Solving one-dimensional two-phase flow problems in rigid porous media using L-scheme

Marcelo H. Sabatini[1,2], Marcio A. V. Pinto[3], Michely L. Oliveira[2], Sebastião R. Franco[4]

[1]*Department of Mathematics, State University of Centro-Oeste*
Alameda Élio Antonio Dalla Vecchia, 838, Vila Carli, 85040-167, Guarapuava, Paraná, Brazil
[2]*Graduate Program in Numerical Methods in Engineering, Federal University of Paraná*
Centro Politécnico, Jardim das Américas, 81531-980, Curitiba, Paraná, Brazil
msabatini@unicentro.br, michely-lais@hotmail.com
[3]*Department of Mechanical Engineering, Federal University of Paraná*
Bloco IV do Setor de Tecnologia, Centro Politécnico, Jardim das Américas, 81531-980, Curitiba, Paraná, Brazil
marcio_villela@ufpr.br
[4]*Department of Mathematics, State University of Centro-Oeste*
Rua Professora Maria Roza Zanon de Almeida, 84505-677, Irati, Paraná, Brazil
romero@unicentro.br

Abstract. This work aims to obtain the numerical solution for one-dimensional two-phase flow in rigid porous media. The mathematical model for the problem consists of a system of partial differential equations with a set of algebraic relations using the pressure-pressure formulation based on L-scheme linearization. The finite volume method (FVM) is used to discretize the system of partial differential equations in a uniform grid. The spatial approximation is obtained by the second-order scheme (CDS) and the temporal approximation by the implicit Euler method. Dirichlet boundary conditions are applied. Iterative methods are used to solve the resulting system of algebraic equations. A study on L-scheme was carried out to establish a rule and value of L that guarantees the convergence of this linearization method. The results obtained for the numerical problem are compared with those of a problem with the same characteristics in the literature that uses the pressure-saturation formulation.

Keywords: Finite volumes, L-scheme, Coupled system, Non-linear problem

1 Introduction

Numerical simulations of flows in porous media are famous Engineering problems. Many have been investigated, for instance, in the extraction of oil and natural gas, Hydrology, soil and rock mechanics. In this sense, it is important to understand the flow of fluids in porous media by means of a mathematical model. There are several numerical formulations in the literature to obtain the solution of two-phase flow in rigid porous media based on the pressure-saturation formulation, Bastian [1], Illiano [2], Celia and Binning [3]. However, in this work we use the pressure-pressure formulation, Ataie-Ashtiani and Raeesi-Ardekani [4], Celia and Binning [3], thus the variables of interest are the pressures in each of the two phases. Since the system is non-linear, we used the L-scheme linearization method, and posteriorly, we use the coupled Gauss-Seidel method, Gaspar et al. [5], to solve each system that resulted from the linearization. The goals of this work is to study the pressure-pressure formulation, in order to compare its performance with pressure-saturation formulation and study the L-scheme method.

2 Mathematical and numerical models

The governing equations of two-phase flow in rigid porous media are modeled by a system of differential partial equations that can be written as

$$\frac{\partial(\rho_\alpha \theta_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{q}_\alpha) = F_\alpha, \quad \text{in} \quad \Omega \times \mathcal{T}, \tag{1}$$

where $\Omega \subset \mathbb{R}^+$, $\mathcal{T} = (0, T)$ is a given interval of time, with T being the final time and $\alpha = w, n$ are the phases of the fluid (w is the wetting and n is the non-wetting). We also have that ρ_α is the density, $\theta_\alpha = \phi S_\alpha$ where ϕ is the porosity and S_α is the saturation, $\mathbf{q}_\alpha$ is the vector of the volumetric flow and F_α is the source term, all from phase α. The volumetric flow is given by Darcy's law adapted for multiphase, which is written as

$$\mathbf{q}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha - \rho_\alpha \mathbf{g}), \tag{2}$$

where $\mathbf{K}$ is the intrinsic permeability tensor, $k_{r\alpha}$ is the relative permeability in the porous media, which is considered as a function of saturation S_α, p_α is the pressure, μ_α is the viscosity, all from phase α and $\mathbf{g}$ is the vector of gravitational acceleration. The quantity $\lambda_\alpha = \dfrac{k_{r\alpha}}{\mu_\alpha}$ is called mobility, that is, the ratio of relative permeability function to the phase viscosity, Bastian [1]. The system of equations given by eq. (1) and eq. (2) is complemented by the algebraic relations $S_w + S_n = 1$ and $p_n - p_w = p_c$. These relations imply that the sum of the saturation of the phases must be equal to 1 and that the capillary pressure p_c is defined as the difference between the pressures p_n and p_w. Additionally, from the relation $S_w + S_n = 1$ we have that $\theta_w + \theta_n = \phi$.

Considering an incompressible fluid and disregarding gravity, the system of equations can be simplified as

$$\frac{\partial(\theta_\alpha)}{\partial t} - \lambda_\alpha \nabla \cdot (\mathbf{K}\nabla p_\alpha) = \frac{F_\alpha}{\rho_\alpha}. \tag{3}$$

Our physical domain will be a segment of lenght L and parallel to the x axis. The governing equations are subject to the Dirichlet type boundary condition, thus $p_\alpha(0, t)$ and $p_\alpha(L, t)$, $t > 0$, are prescribed value, where L represents the domain size. To complete the mathematical formulation, an initial condition must also be given, thus $p_\alpha(x, 0) = p^0_\alpha$ where p^0_α is a prescribed value of the variable pressure. We discretized the spatial domain using the finite volume method (FVM), Maliska [6], Ferziger and Perić [7] and central difference scheme (CDS), Fortuna [8]. For the temporal discretization, we used the implicit Euler method, Fortuna [8], Burden and Faires [9]. We denoted n as the time level, m as the number of iterations, $\tau = \frac{T}{N_t}$ where N_t is the number of time steps, and $h = \frac{L}{N_x}$ where L represents the domain size and N_x is the number of volumes in the space. With the temporal discretization, in the wetting phase, eq. (3) can be written as

$$\begin{aligned} &C_w^{n+1,m}\frac{\delta p_n^{n+1,m+1} - \delta p_w^{n+1,m+1}}{\tau} - \frac{\partial}{\partial x}[K_w^{n+1,m}\frac{\partial}{\partial x}(\delta p_w^{n+1,m+1})] \\ &= \frac{\partial}{\partial x}[K_w^{n+1,m}\frac{\partial}{\partial x}(p_w^{n+1,m})] + F_w^{n+1} - \frac{\theta_w^{n+1,m} - \theta_w^n}{\tau}, \end{aligned} \tag{4}$$

and in the non-wetting phase as

$$\begin{aligned} &C_w^{n+1,m}\frac{\delta p_n^{n+1,m+1} - \delta p_w^{n+1,m+1}}{\tau} - \frac{\partial}{\partial x}[K_n^{n+1,m}\frac{\partial}{\partial x}(\delta p_n^{n+1,m+1})] \\ &= \frac{\partial}{\partial x}[K_n^{n+1,m}\frac{\partial}{\partial x}(p_n^{n+1,m})] + F_n^{n+1} - \frac{\theta_n^{n+1,m} - \theta_n^n}{\tau}, \end{aligned} \tag{5}$$

where $C_w^{n+1,m} = \frac{\partial \theta_w}{\partial p_c}$, $K_\alpha^{n+1,m} = \mathbf{K}\frac{k_{r\alpha}}{\mu_\alpha}$ and $\delta p_\alpha^{n+1,m+1} = p_\alpha^{n+1,m+1} - p_\alpha^{n+1,m+1}$.

Considering a uniform grid, we obtain the following system of equations after the spatial discretization

$$(a_w)_i[\delta p_w^{n+1,m+1}]_i + (a_w)_{i+1}[\delta p_w^{n+1,m+1}]_{i+1} + (a_w)_{i-1}[\delta p_w^{n+1,m+1}]_{i-1} + b_i[\delta p_n^{n+1,m+1}]_i = (F_w)_i, \tag{6}$$

$$(a_n)_i[\delta p_n^{n+1,m+1}]_i + (a_n)_{i+1}[\delta p_n^{n+1,m+1}]_{i+1} + (a_n)_{i-1}[\delta p_n^{n+1,m+1}]_{i-1} + b_i[\delta p_w^{n+1,m+1}]_i = (F_n)_i, \tag{7}$$

where

$$\begin{aligned}(a_\alpha)_i &= -[C_w^{n+1,m}]_i + \frac{\tau}{h^2}([K_\alpha^{n+1,m}]_{i+\frac{1}{2}} + [K_\alpha^{n+1,m}]_{i-\frac{1}{2}}),\\ (a_\alpha)_{i+1} &= -\frac{\tau}{h^2}[K_\alpha^{n+1,m}]_{i+\frac{1}{2}},\\ (a_\alpha)_{i-1} &= -\frac{\tau}{h^2}[K_\alpha^{n+1,m}]_{i-\frac{1}{2}},\\ (b)_i &= [C_w^{n+1,m}]_i,\\ (F_\alpha)_i &= \frac{\tau}{h^2}[K_\alpha^{n+1,m}]_{i+\frac{1}{2}}[p_\alpha^{n+1,m}]_{i+1} - \frac{\tau}{h^2}([K_\alpha^{n+1,m}]_{i+\frac{1}{2}} + [K_\alpha^{n+1,m}]_{i-\frac{1}{2}})[p_\alpha^{n+1,m}]_i +\\ &\quad \frac{\tau}{h^2}[K_\alpha^{n+1,m}]_{i-\frac{1}{2}}[p_\alpha^{n+1,m}]_{i-1} + \tau F_\alpha - [\theta_\alpha^{n+1,m}]_i + [\theta_\alpha^n]_i.\end{aligned}$$

We used L-scheme in the numerical experiments. This linearization defines that $L \geq |C_w|$, Illiano [2], Radu et al. [10]. The linear system resulting from the linearization of the equations above is solved by the coupled Gauss-Seidel method, Gaspar et al. [5].

3 Numerical results

3.1 Code verification

For our tests, we used the problem proposed by Illiano [2]. In that work, Illiano considers the pressure-saturation formulation $\overline{p}$–S_w, where $\overline{p} = \frac{p_w+p_n}{2}$ and proposes the analytical solution $f(x,t) = \overline{p}(x,t) = S_w(x,t) = xt(1-x)$ defined in the domain $D = [0,1]\times[0,1]$, with initial and boundary conditions $f(x,0) = f(0,t) = f(1,t) = 0$.

In this current work, we used the pressure-pressure formulation p_w–p_n, thus, it was necessary to make adjustments to use p_w and p_n instead of $\overline{p}$. By using the definitions of capillary pressure $p_c = p_w - p_n$ and $\overline{p}$ we obtain $p_w = \overline{p} - \frac{1}{2}p_c$, $p_n = \overline{p} + \frac{1}{2}p_c$, where $p_c(S_w) = 1 - \frac{1}{2}S_w^2$. Since $\theta_\alpha = \phi S_\alpha$, we have that $\theta_w = \phi\sqrt{2-2p_c}$ and $\theta_n = \phi - \theta_w$, implying that $C_w = \frac{\partial\theta_w}{\partial p_c} = -\frac{\phi}{\sqrt{2-2p_c}}$, $p_c \neq 1$. These expressions are used to find the source terms

$$F_w = -\frac{1}{2}\rho_w[2\phi(x-1)x + K\lambda_w t(6tx^2 - 6tx + t - 4)], \tag{8}$$

$$F_n = \frac{1}{2}\rho_n[2\phi(x-1)x + K\lambda_n t(6tx^2 - 6tx + t - 4)]. \tag{9}$$

To verify our results, we compared our solutions with the results obtained by Illiano [2] for the saturation S_w. In this verification, data from Table 1 was used. According to Illiano this data was used to obtain easier computations and can be unrealistic but that is not our concern, since we are trying to verify that our code works, thus we are not interested in a simulation of a realistic physical problem.

Figure 1 and Fig. 2 confirm that the results found are in accordance with those presented by Illiano [2].

Table 1. Data used in the verification

λ_w	λ_n	K	ϕ	ρ_w	ρ_n
1	2	1	1	1	2

3.2 Analysis of the L-scheme

A study on the L-scheme was carried out using the problem from the previous section in order to establish a suitable rule and values of L that help the convergence of this linearization method. In Fig. 3 and Fig. 4, $N_x = 10$

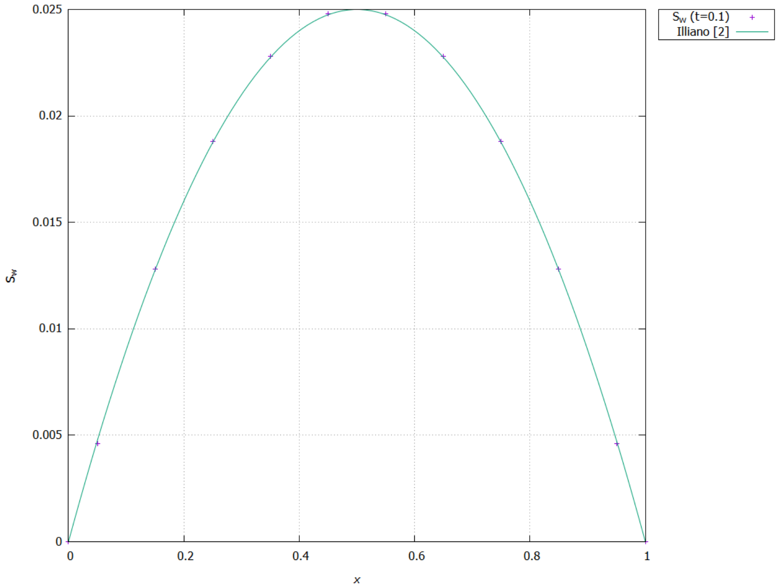

Figure 1. Numerical and analytical solution for saturation for $\tau = h = 0.1$

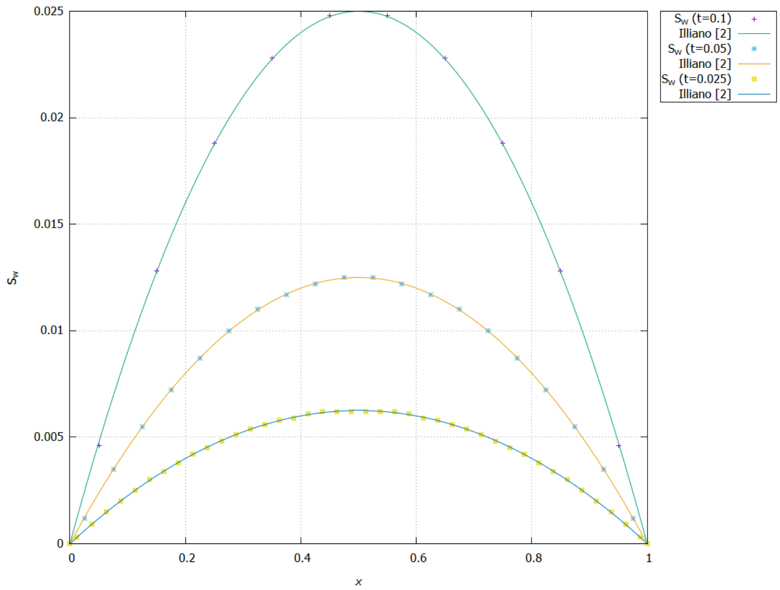

Figure 2. Numerical and analytical solution for saturation for $\tau = h = 0.025$

and $N_x = 20$ respectively, we have $|C_w|$ *versus* x, where $C_w = \frac{\partial \theta_w}{\partial p_c}$. Data from Table 1 was also used in this verification. It is possible to notice that the maximum value of the derivatives $|C_w|$ is at the first time step ($n_t = 1$) and at the first spatial volume. For this reason, we made a geometric adjustment using the set of data from Table 2, which are values of $max|C_w|$ located at the first spatial volume at the first time step as a function of the number of volumes N_x in the grid. Table 2 shows the data to be adjusted.

The best-fitting curve to data was $y(N_x) = 2.3314(N_x)^{1.9658}$ (see Fig. 5). By using a different value of N_x we can predict the $max|C_w|$ and therefore, find the value of L to be used in the linearization scheme. For example, for $N_x = 30$, we obtain $y(30) = 1867.84$. We can affirm that this rule is robust as it meets the convergence criterion of the L-scheme, Illiano [2], Radu et al. [10]. Further studies are needed in order to make the rule more efficient.

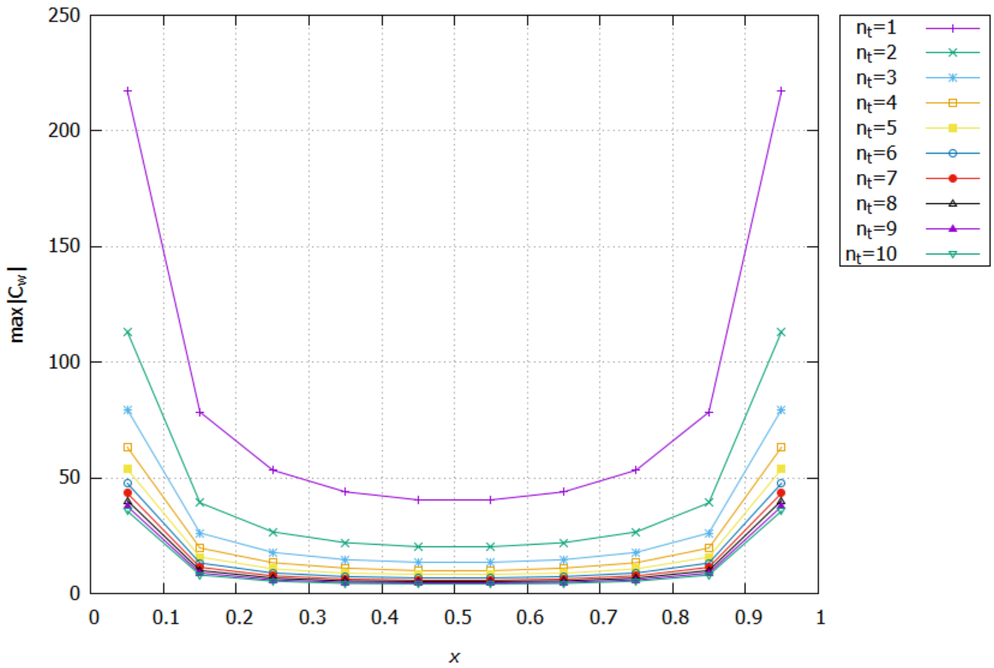

Figure 3. Maximum value of the derivatives $|C_w|$ with $N_x = N_t = 10$ at each time step n_t

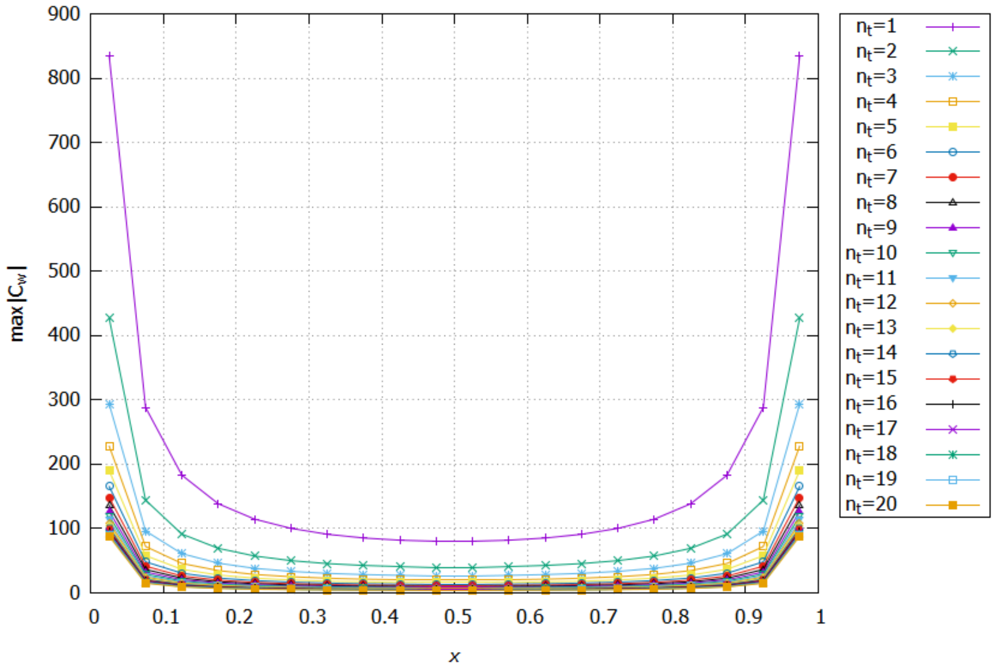

Figure 4. Maximum value of the derivatives $|C_w|$ with $N_x = N_t = 20$ at each time step n_t

Based on the information in Fig. 3 and Fig. 4, we can create a new rule by establishing a vector of the values of $\mathbf{L}$ where each L_i is the $max|C_w|$ at the time step n_{t_i}. Thus, we will have a vector with the values $\mathbf{L} = (L_1, L_2, ..., L_{N_t})$, where N_t is the total number of time steps. For example, for the grid size $N_x = 10$, this vector would be $\mathbf{L} = (217.22; 113.20; 79.51; 63.22; 53.77; 47.64; 43.35; 40.16; 37.66; 35.62)$. One can notice that the components of this vector have an asymptotic behavior, starting at a large value of L, to posteriorly decrease and tend to a certain value (this behavior is also seen at the first volume in Fig. 3). For this reason, we will use the value of $max|C_w|$ at each time step, which is the value $max|C_w|$ computed at the first spatial volume.

Table 3 shows the column $itmed_L$ (Adjusted), which displays the average number of linearizations, con-

Table 2. $max|C_w|$ at the first spatial volume at the first time step as a function of N_x

N_x	$max\|C_w\|$
10	217.22
20	833.98
40	3267.35
80	12934.04

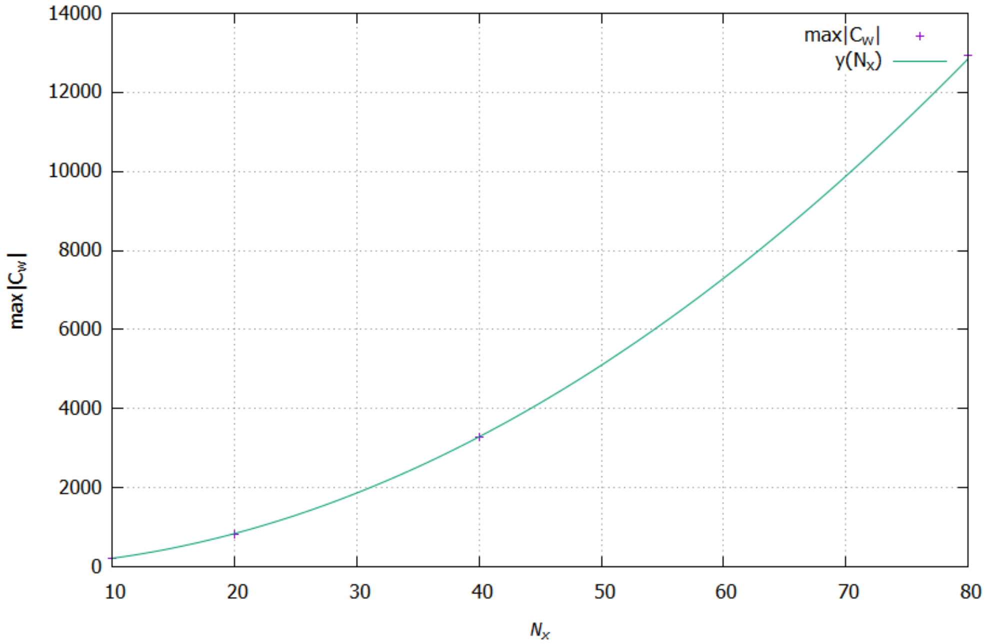

Figure 5. Maximum value of the derivatives $|C_w|$

sidering every time step and using the geometrical adjustment of the data from Table 2. Column $itmed_L$ (**L**) is also shown, displaying the average number of linearizations achieved by using the **L** vector. The table highlights that the method proposed by the new rule to choose L (by using **L**) noticeably reduces the average number of linearizations performed, thus proven to be more efficient.

Table 3. Average number of linearization according to the rules proposed

N_x	$itmed_L$ (Adjusted)	$itmed_L$ (**L**)
10	195.10	80.90
20	610.96	178.05
40	1936.65	392.65
80	5974.46	1169.72

4 Conclusions

Numerical simulations of flows in porous media have been investigated. In this sense, it is important to understand the flow of fluids in porous media by means of a mathematical model. This work presented a model for the simulation of two-phase flow in rigid porous media that uses the L-scheme for linearization and the pressure-pressure formulation. The results our code achieved were verified and are in accordance with those proposed by Illiano [2]. After the verifying the code we carried out a study on L-scheme showing how to choose a value of L. A new way to choose a suitable value of L that is more efficient and guarantees the convergence of the method was also proposed.

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