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Analysis of Shear Locking Effect on Reissner Plates Using Meshless Local Petrov-Galerkin Method

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Abstract. The shear locking is an interesting numerical phenomenon which can be found in several formulations, such as those based on Finite Element Method (FEM) and Meshless Local Petrov-Galerkin (MLPG) Method, when they are directly applied to thin plates analyzed through Reissner's theory. It is known that the shear locking effect is caused by using the same interpolation functions for all generalized displacement fields, producing inconsistent results in case of thin plates. In order to avoid this phenomenon, the rotations must be built from the first derivative of the transversal displacement field. In the FEM formulation, this problem is overcome by using reduced-selective integration schemes. However, this alternative can hardly be extended to Meshless Methods due to the non-polynomial characteristic of the approximations. More complex numerical formulations can be considered, however in this paper the variable changing technique is applied, by solving the shear locking effect in a simple and efficient way, without increasing the number of degrees of freedom in plate's problems.

Keywords: Shear locking, Reissner's theory, Meshless Local Petrov-Galerkin Method.

1 Introduction

Bending problems of thin plates are analyzed, applying the Reissner's theory [1] and making use of the first version of the Meshless Local Petrov-Galerkin (MLPG) Method, designed in the literature by MLPG-1 [2]. In this version of the method, the weight function of the Moving Least Squares (MLS) approximation [3] is applied as test function in the problem.

When the Reissner's theory is considered to analyse problems of thin plates using a truly meshless formulation, the numerical phenomenon known as shear locking can occur and thus inconsistent results are obtained.

Therefore, in this paper, an efficient and simple technique based on variable changing is proposed, in order to obtain a formulation for thin plate problems without shear locking effects.

2 Reissner's Theory

Plate is a flat structural elements, submitted to transversal loads and with the thickness much smaller than the others dimensions. In general, a plate can be classified as thin or thick, according to the relation between its thickness (h) and the smallest dimension in the plate's plane (a).

Plates with lower values for the relation h/a, are classified as thin and can be analyzed applying the Kirchhoff's theory [4]. In the other hand, for cases in which the relation h/a reaches higher values, the plates are considered thick, and must be treated considering a thick plate theory, such as Reissner's formulation [5].

The basic difference between the two theories is related to the fact that the effect of shear transversal strains are taken into account in the Reissner's theory, while in Kirchhoff's formulation this effect is disregarded.

2.1 Basic Hypotheses

The basic hypotheses for Reissner's theory are given by: the plate is homogeneous, isotropic and has a linear-elastic behavior; only transversal loading q is considered; the vertical displacements are small compared with the plate's thickness *h*; and the linear variation of the stresses $\sigma_{\alpha\beta}$ along the plate's thickness.

For the Reissner's equations, the following index notation is considered: *i* and *j* ranging from 1 to 3, and *α*, *β* and *γ* ranging from 1 to 2.

Since the load conditions on the plate are $\sigma_{33} = \pm q / 2$ and $\sigma_{\alpha\beta} = 0$, for $x_3 = \pm h / 2$, it is possible to obtain the expressions for the stresses in terms of the internal loads (moments $M_{\alpha\beta}$ and shear forces Q_α) as:

$$\sigma_{\alpha\beta} = \frac{12 M_{\alpha\beta}}{h^3} x_3 , \tag{1}$$

$$\sigma_{\alpha 3} = \frac{3 Q_\alpha}{2h} \left[1 - \left(\frac{2x_3}{h} \right)^2 \right], \tag{2}$$

$$\sigma_{33} = \frac{q x_3}{2h} \left[3 - \left(\frac{2x_3}{h} \right)^2 \right]. \tag{3}$$

The normal stress σ_{33} occurs in the transversal direction of the plate and it is considered disregarded in relation to the other stress components.

2.2 Equilibrium Equations

By considering the elasticity theory for small strains condition and by taking into account the equilibrium of an infinitesimal plate element, the following equations can be obtained:

$$Q_{\alpha,\alpha} + q = 0 , \tag{4}$$

$$M_{\alpha\beta,\beta} - Q_\alpha = 0 . \tag{5}$$

2.3 Displacements and Strains

For points located on the medium surface of the plate $\left(x_3 = 0 \right)$, the displacements ϕ_α (rotations) and w (transversal displacement) are considered. The ϕ_α and w fields represent the average of the displacements υ_i of points located along the plate's thickness, in the directions of each coordinate axis x_i. Thus, these displacements can be defined as:

$$\phi_\alpha = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \upsilon_\alpha x_3 \, dx_3 , \tag{6}$$

$$w = \frac{3}{2h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \upsilon_3 \left[1 - \left(\frac{2x_3}{h} \right)^2 \right] dx_3 . \tag{7}$$

The expressions of the bending strains $\left(\chi_{\alpha\beta} \right)$ and the transversal shear strains $\left(\psi_\alpha \right)$ for the elastic-linear theory, as a function of the displacements fields ϕ_α and w, can be expressed by the following equations:

$$\chi_{\alpha\beta} = \frac{1}{2}\left(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}\right), \tag{8}$$

$$\psi_\alpha = \phi_\alpha + w_{,\alpha}\,. \tag{9}$$

2.4 Internal Loads

By applying the theory of elasticity for small strains condition, combined with the variational principles, it is possible to obtain the equations of moments $\left(M_{\alpha\beta}\right)$ and shear forces $\left(Q_\alpha\right)$ in terms of the displacements ϕ_α and w:

$$M_{\alpha\beta} = \frac{D\left(1-\nu\right)}{2}\left(\phi_{\alpha,\beta} + \phi_{\beta,\alpha} + \frac{2\nu}{1-\nu}\phi_{\gamma,\gamma}\delta_{\alpha\beta}\right) + \frac{\nu q}{\left(1-\nu\right)\lambda^2}\delta_{\alpha\beta}; \tag{10}$$

$$Q_\alpha = \frac{D\left(1-\nu\right)\lambda^2}{2}\left(\phi_\alpha + w_{,\alpha}\right), \tag{11}$$

where: E is the Young's modulus, ν is the Poisson's ratio, $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ is the flexural modulus and $\lambda = \dfrac{\sqrt{10}}{h}$ is the Reissner's constant.

2.5 Boundary Conditions

For each edge of the plate, three boundary conditions must be satisfied. Thus, considering Γ_1 the part of the plate's boundary Γ, where the displacements ϕ_α and w are prescribed and Γ_2 the region of Γ, where the surface forces p_α and p_3 are given, the following boundary conditions are considered for Γ_1 and Γ_2 $\left(\Gamma = \Gamma_1 \cup \Gamma_2\right)$, respectively:

$$\phi_\alpha = \hat{\phi}_\alpha \quad \text{and} \quad w = \hat{w}\,; \tag{12}$$

$$p_\alpha = \hat{p}_\alpha \quad \text{and} \quad p_3 = \hat{p}_3\,, \tag{13}$$

where: $p_\alpha = M_{\alpha\beta}n_\beta$, $p_3 = Q_\beta n_\beta$, $\hat{p}_\alpha = \hat{M}_{\alpha\beta}n_\beta$, $\hat{p}_3 = \hat{Q}_\beta n_\beta$ and n_β is the unit outward normal vector.

3 Meshless Local Petrov-Galerkin Method for Reissner's Theory

For the numerical simulation of plate bending problems, a truly Meshless Method is considered by applying the MLPG-1, which considers the weight function of the MLS as the test function.

For each field node *i* of the problem, the Weighted Residuals Method is applied on the Eqs. (4) and (5), for a local subdomain Ω_s, and the following equations can be obtained:

$$\int_{\Omega_s}\left(Q_{\alpha,\alpha} + q\right)\, W_i \, d\Omega = 0 \qquad \alpha = 1,\, 2\,; \tag{14}$$

$$\int_{\Omega_s} \left(M_{\alpha\beta,\beta} - Q_\alpha \right) W_i \, d\Omega = 0 \qquad \alpha,\ \beta = 1,\ 2\,, \tag{15}$$

where W_i is the weight function for each node *i*.

By applying the integration by parts on Eqs. (14) and (15), the following set of expressions is obtained:

$$\int_{\Omega_s} Q_\alpha \ W_{i,\alpha} \, d\Omega - \int_{\partial\Omega_s} Q_\alpha n_\alpha \ W_i \, d\Gamma - \int_{\Omega_s} q \ W_i \, d\Omega = 0 \qquad \alpha = 1,\ 2\,, \tag{16}$$

$$\int_{\Omega_s} M_{\alpha\beta} \ W_{i,\beta} \, d\Omega - \int_{\partial\Omega_s} M_{\alpha\beta} n_\beta \ W_i \, d\Gamma + \int_{\Omega_s} Q_\alpha \ W_i \, d\Omega = 0 \qquad \alpha,\ \beta = 1,\ 2\,. \tag{17}$$

In Eqs. (16) and (17), $\partial\Omega_s$ is the boundary of the local subdomain Ω_s, that is composed by: L_s is the internal boundary of $\partial\Omega_s$ which doesn't intersect global boundary of the problem; Γ_{su} is the part of $\partial\Omega_s$ which intersects global boundary, where the essential boundary conditions are prescribed; and Γ_{sq} is the part of $\partial\Omega_s$ which intersects global boundary, where the natural boundary conditions are given. Thus: $\partial\Omega_s = \Gamma_{su} \cup \Gamma_{sq} \cup L_s$, $\Gamma_{su} = \partial\Omega_s \cap \Gamma_u$ and $\Gamma_{sq} = \partial\Omega_s \cap \Gamma_q$.

By selecting a weight function W_i with zero value at internal boundary L_s, the Eqs. (16) and (17) can be re-writing as following:

$$\int_{\Omega_s} Q_\alpha W_{i,\alpha} d\Omega - \int_{\Gamma_{su}} Q_\alpha n_\alpha W_i d\Gamma - \int_{\Gamma_{sq}} \hat{Q}_\alpha n_\alpha W_i d\Gamma - \int_{\Omega_s} q W_i \, d\Omega = 0 \quad \alpha = 1,\ 2\,, \tag{18}$$

$$\int_{\Omega_s} M_{\alpha\beta} W_{i,\beta} d\Omega - \int_{\Gamma_{su}} M_{\alpha\beta} n_\beta W_i d\Gamma - \int_{\Gamma_{sq}} \hat{M}_{\alpha\beta} n_\beta W_i d\Gamma + \int_{\Omega_s} Q_\alpha W_i d\Omega = 0 \qquad \alpha,\ \beta = 1,\ 2\,. \tag{19}$$

In this paper, the Wendland function with continuity C^4 [6] is considered as the weighting W_i and can be defined by the expression below:

$$W_i = \begin{cases} \left[1 - \dfrac{r_i}{R_i}\right]^5 \left[8 \left(\dfrac{r_i}{R_i}\right)^2 - 5 \ \dfrac{r_i}{R_i} + 1\right], & 0 \le r_i \le R_i \\ 0, & r_i \ge R_i \end{cases} \tag{20}$$

where R_i is the radius of the support and r_i is the Euclidean distance between the base and field points. The form of the Wendland C^4 function can be seen in the Fig. 1.

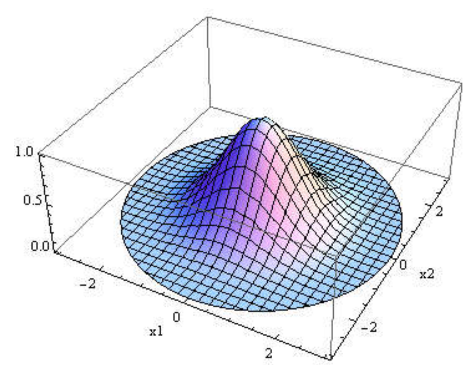

Figure 1. Wendland C^4 function.

The essential boundary conditions are considered for such problems by applying the Penalty Method [7]. In this methodology, the Weighted Residuals Method is firstly considered over the essential conditions and then the generated equations are incorporated in the weak expressions of the plate theory, given by the Eqs. (18) and (19). Thus, applying the Weighted Residuals Method over the essential conditions, the following equations can be obtained:

$$\alpha_p \int_{\Gamma_{su}} \left[w - \hat{w} \right] W_i \, d\Gamma = 0 \,, \tag{21}$$

$$\alpha_p \int_{\Gamma_{su}} \left[\phi_\alpha - \hat{\phi}_\alpha \right] W_i \, d\Gamma = 0, \qquad \alpha = 1, 2 \,. \tag{22}$$

The constant α_p is called the penalty parameter, and it's a high number, that usually varies between $10^4 \times K_{ii,\max}$ and $10^{13} \times K_{ii,\max}$ ($K_{ii,\max}$ is maximum element in the diagonal of the stiffness matrix) [8]. If the penalty parameter is too small, the restrictions of the equations in the Weighted Residuals Method may be violated and, otherwise, if the penalty parameter is a large number, can occur a degradation of the results obtained. Thus, this parameter must be studied for each problem individually.

By introducing the essential boundary conditions, Eqs. (21) and (22), on Eqs. (18) and (19) and replacing the internal loads (moments and shear forces) by their expressions in terms of the displacements fields ϕ_α and w, Eqs. (10) and (11), the final equations are obtained:

$$\begin{aligned} &\frac{D\left(1-\nu\right)\lambda^2}{2}\left[\int_{\Omega_s}\left(\phi_\alpha + w_{,\alpha}\right)W_{i,\alpha}\, d\Omega - \int_{\Gamma_{su}}\left(\phi_\alpha + w_{,\alpha}\right)n_\alpha W_i \, d\Gamma\right] + \alpha_p \int_{\Gamma_{su}} w \, W_i \, d\Gamma = \\ &= \int_{\Gamma_{sq}} \hat{p}_3 W_i \, d\Gamma + \int_{\Omega_s} q \, W_i \, d\Omega + \alpha_p \int_{\Gamma_{su}} \hat{w} \, W_i \, d\Gamma \qquad \alpha = 1,\ 2, \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{D\left(1-\nu\right)}{2}\left[\int_{\Omega_s}\left(\phi_{\alpha,\beta}+\phi_{\beta,\alpha}+\frac{2\nu}{1-\nu}\phi_{\gamma,\gamma}\delta_{\alpha\beta}\right)W_{i,\beta}\, d\Omega - \int_{\Gamma_{su}}\left(\phi_{\alpha,\beta}+\phi_{\beta,\alpha}+\frac{2\nu}{1-\nu}\phi_{\gamma,\gamma}\delta_{\alpha\beta}\right)n_\beta W_i \, d\Gamma + \right. \\ &\left. +\lambda^2\int_{\Omega_s}\left(\phi_\alpha + w_{,\alpha}\right)W_i \, d\Omega\right] + \alpha_p \int_{\Gamma_{su}} \phi_\alpha W_i \, d\Gamma = \int_{\Gamma_{sq}} \hat{p}_\alpha W_i \, d\Gamma + \frac{\nu}{\left(1-\nu\right)\lambda^2}\delta_{\alpha\beta}\left[-\int_{\Omega_s} q \, W_{i,\beta}\, d\Omega + \right. \\ &\left. +\int_{\Gamma_{su}} q \, n_\beta W_i \, d\Gamma\right] + \alpha_p \int_{\Gamma_{su}} \hat{\phi}_\alpha W_i \, d\Gamma \qquad \alpha,\ \beta,\ \gamma = 1,\ 2. \end{aligned} \tag{24}$$

The Eqs. (23) and (24) must be evaluated at all points considered in the discretization of the problem.

4 Shear Locking

Despite presenting many advantages, the direct application of the MLPG-1 to Reissner's plate theory can result in the same numerical problem which is often found in the analyzes with finite element method (FEM): the shear locking phenomenon.

This phenomenon is caused by using the same interpolation functions for all generalized displacement fields, producing inconsistent results during numerical simulation of thin plates. In order to avoid this effect, the rotations ϕ_α must be built from the first derivative of the transversal displacement field w.

As occur in FEM, in MLPG-1 the degradation of the numerical solution produced by shear locking becomes even more critical as the thickness of the plate decreases. In the FEM formulation, this problem is overcome by applying reduced-selective integration schemes which can hardly be extended to Meshless Methods due to the non-polynomial characteristic of the approximations.

However, in this paper, the variable changing technique is considered, in order to obtain a formulation without the effects caused by shear locking. This technique neither increases the total number of degrees of freedom in plate's problems nor considers any type of complex numerical formulations.

The variable changing technique is constructed by using the Eq. (9), where the rotations ϕ_α are separated, put them as a function of the transversal shear strains components $\left(\psi_\alpha\right)$ and the first derivative of the transversal displacement field $\left(w_{,\alpha}\right)$, i.e.:

$$\phi_\alpha = \psi_\alpha - w_{,\alpha}. \tag{25}$$

The Eq. (25) is introduced into the Eqs. (23) and (24), by replacing the rotations ϕ_α . The transversal shear strains ψ_α become the variables of the problem, together with the transversal displacement field w, keeping the total number of variables in the plate's problem. Thus, the Eqs. (23) and (24) can be re-writing as:

$$\frac{D\left(1-\nu\right)\lambda^2}{2}\left[\int_{\Omega_s}\psi_\alpha W_{i,\alpha}\,d\Omega - \int_{\Gamma_{su}}\psi_\alpha n_\alpha W_i\,d\Gamma\right] + \alpha_p\int_{\Gamma_{su}} w W_i\,d\Gamma = \int_{\Gamma_{sq}}\hat{p}_3 W_i\,d\Gamma + \int_{\Omega_s} q W_i\,d\Omega + \\ + \alpha_p\int_{\Gamma_{su}}\hat{w}\;W_i\,d\Gamma \qquad \alpha = 1,\,2, \tag{26}$$

$$\frac{D\left(1-\nu\right)}{2}\left[\int_{\Omega_s}\left(\psi_{\alpha,\beta} + \psi_{\beta,\alpha} - 2w_{,\alpha\beta} + \frac{2\nu}{1-\nu}\left(\psi_{\gamma,\gamma} - w_{,\gamma\gamma}\right)\delta_{\alpha\beta}\right)W_{i,\beta}\,d\Omega - \int_{\Gamma_{su}}\left(\psi_{\alpha,\beta} + \psi_{\beta,\alpha} - \right.\right. \\ \left.\left. - 2w_{,\alpha\beta} + \frac{2\nu}{1-\nu}\left(\psi_{\gamma,\gamma} - w_{,\gamma\gamma}\right)\delta_{\alpha\beta}\,n_\beta\,W_i\,d\Gamma\right) + \lambda^2\int_{\Omega_s}\psi_\alpha W_i\,d\Omega\right] + \alpha_p\int_{\Gamma_{su}}\left(\psi_\alpha - w_{,\alpha}\right)W_i\,d\Gamma = \\ = \int_{\Gamma_{sq}}\hat{p}_\alpha W_i\,d\Gamma + \frac{\nu}{\left(1-\nu\right)\lambda^2}\delta_{\alpha\beta}\left[-\int_{\Omega_s} q W_{i,\beta}\,d\Omega + \int_{\Gamma_{su}} q\,n_\beta\,W_i\,d\Gamma\right] + \alpha_p\int_{\Gamma_{su}}\left(\hat{\psi}_\alpha - \hat{w}_{,\alpha}\right)W_i\,d\Gamma \\ \alpha,\,\beta,\,\gamma = 1,\,2. \tag{27}$$

After solving the linear system of equations, the rotations fields ϕ_α can be obtained from ψ_α and w, by applying the approximation functions and its derivatives with Eq. (25).

5 Numerical Application

The example consists of a simply supported thin square plate, with dimensions in plane of 4.00 m x 4.00 m, thickness h = 0.15 m and submitted to a uniformly load q = -15.0 kN/m^2, as can be seen in the Fig. 2. The rotations ϕ_α and the transversal displacement w are evaluated on the section A-A, at coordinate $x_2 = 3.00$ m.

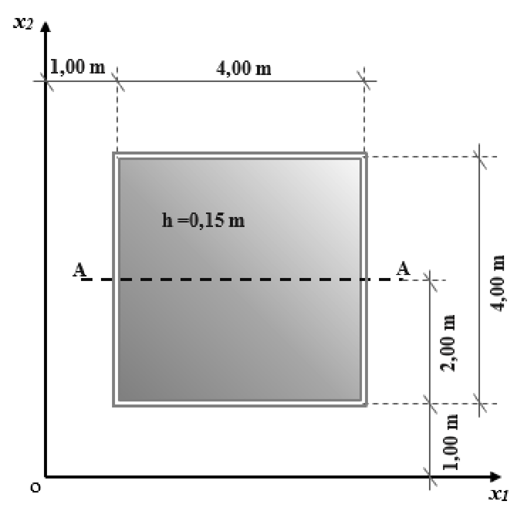

Figure 2. Plate geometry

For this problem, the physical constants are given by: Young's module E = 2.0 x 10^7 kN / m^2 and Poisson's ratio ν = 0.25.

It is considered a set of 293 points, with a uniform space of 0.25 m, being 225 points distributed on the domain of the plate and 68 points located on the plate's boundary. Double nodes are used in the corners of the plate, with 0.0625 m of distance from their original position.

The numerical analyzes are developed by applying the MLS and by using the Wendland C^4 as weighting function. To perform the numerical approximations, monomials with quadratic base ($m_b = 2$) are considered.

Numerical integrations are performed by applying the Gaussian quadrature, with 12 x 12 Gauss's points for domain integration and 6 Gauss's points for boundary integration. The penalty parameter α_p considered in the analyzes is 10^6 x $K_{ii,\max}$.

On Figs. 3 and 4 are presented, respectively, the values obtained for rotation ϕ_1 and transversal displacement w on section A-A. The results are compared with those provided by the standard methods based on BEM and FEM formulation. As the rotation field ϕ_2 presents null values in the section A-A, the results are disregard.

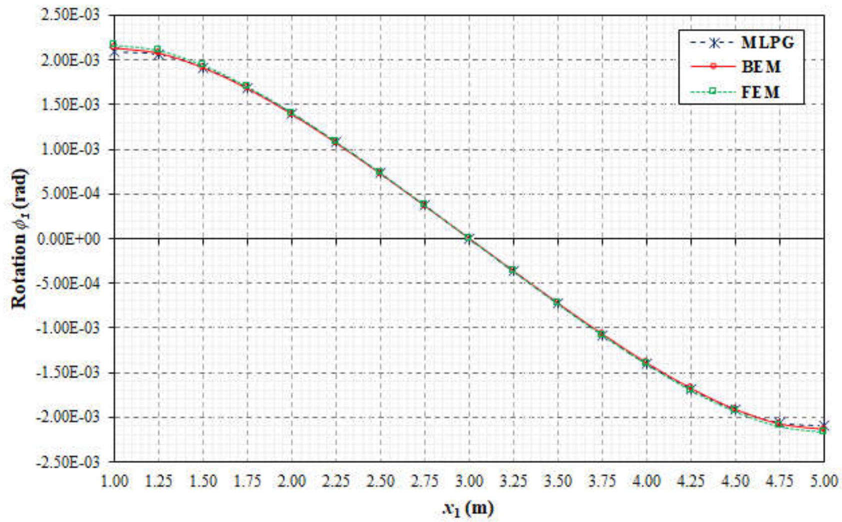

Figure 3. Results for rotation ϕ_1 on section A-A.

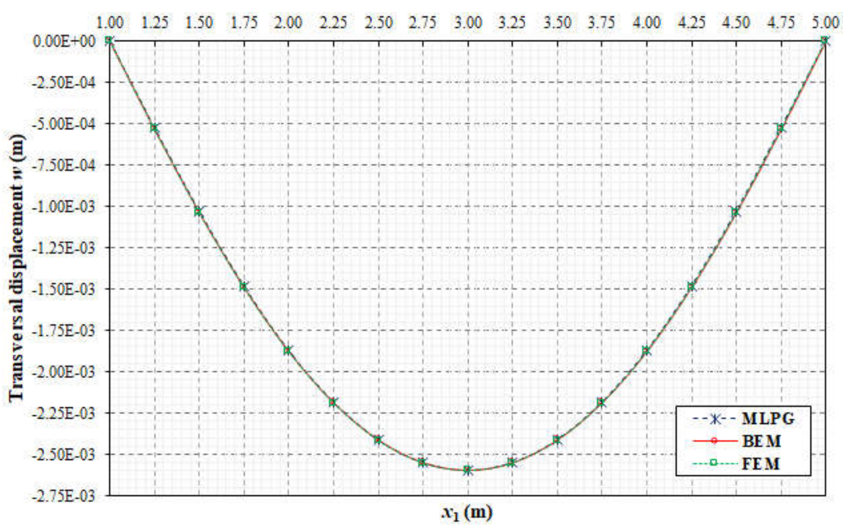

Figure 4. Results for transversal displacement w on section A-A.

6 Conclusions

Although some references, based on the previous experience of the MEF, suggest an increase the degree of the base functions, in order to eliminate the shear locking effect, this procedure proves to be computationally costly, due to the high number of terms that are considered in the base functions. In the same way, the results can also present numerical instability.

A formulation that proved to be efficient and with simple numerical implementation is the variable changing technique. The thin plate problems are solved globally in terms of two shear strains components ψ_α and a transversal displacement field w. The rotations ϕ_α are obtained later by applying the approximation functions.

An important characteristic of this technique is that the total number of degrees of freedom is the same, and it is possible to use the same approximation functions for all displacement fields, which present an advantage in terms of computational requirement.

In the other hand, the equilibrium equations start to involve second derivatives of the transversal displacement field w, which requires special attention with the weight functions considered. Also, during the process of obtaining the rotations ϕ_α through the approximation functions, a very small degradation of these fields can be verified.

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