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PARTICULARITIES IN THE APPLICATION OF THE BOUNDARY ELEMENT METHOD WITH HIGHER ORDER ELEMENTS IN TWO DIMENSIONAL POTENTIAL PROBLEMS

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Abstract. In this work, the Boundary Elements Method (BEM) applied to two-dimensional potential problems with quadratic and cubic isoparametric elements are analyzed. Unlike the constant and linear elements, the modus operandi of the coordinate transformations, the numerical integration procedures and the treatment of singular integrals are not simple, since the Jacobian of the transformation is no longer constant throughout the element and are treated numerically. The objective of this work is to analyses the effect of the self-adaptive integration scheme for solving the Boundary Element Method integrals.

Keywords: Boundary Element Method, Isoparametric Higher Order Elements, Potential Problems, Gauss Quadrature, Self Adaptative Integration

1 Introduction

Numerical integration is one of the topics of greatest relevance in the context of the Boundary Elements Method. This is justified by the fact that the entire representation of the continuum is given only by discretization of its boundary. In principle, since the method reduces the problem to be solved in a dimension, it would be expected that the integrations were simpler. However, although the use of fundamental solution is the main reason for the high accuracy of the method is also a cause of problems related to their integration along the boundary elements, as it has the peculiar characteristic of being singular. It is possible to make the analytical calculation of the integrals of the BEM in certain particular cases; however, it becomes practically impossible to carry it out for elements of higher order, that is, where the variation of the basic variable over the element is not considered constant. Barbosa [1] says that an aggravating factor appears in the case of elements with non-straight geometry, as in the case of quadratic or higher isoparametric elements where the function that performs the transformation of global coordinates in a local strategic system, the Jacobian, is no longer a constant value, but a function of the geometric shape of the element.

2 Laplace's equation

The Laplace Equation is a highly relevant partial elliptical differential equation, since it makes it possible to model the behavior of a scalar variable, the potential $u(x, y)$, in various fields of science, such as astronomy , electromagnetism, fluid mechanics, among other applications. In two dimensions, i.e. in the Euclidean space $\Re 2$, the Laplace equation takes the form, in Cartesian coordinates,

$$\nabla^2 u(x, y) = u_{,ii} = 0 \tag{1}$$

with the natural or essential boundary conditions,

$$u = \bar{u} \qquad em \qquad \Gamma_1 \qquad e \qquad q = \frac{\partial u}{\partial n} = \bar{q} \qquad em \qquad \Gamma_2 \tag{2}$$

where n is the normal outside the boundary, $\Gamma = \Gamma_1 + \Gamma_2$ and the bars indicate known values. More complex boundary conditions, such as combinations of the above, that is:

$$\alpha u + \beta q = \gamma \tag{3}$$

where, α, β and γ are known parameters, which can be easily included. This fact is shown by Brebbia [2], but they will not be considered here for the sake of simplicity.

Now consider that the solution to a Laplace equation is sought in a two-dimensional domain, as represented in Fig. 1.

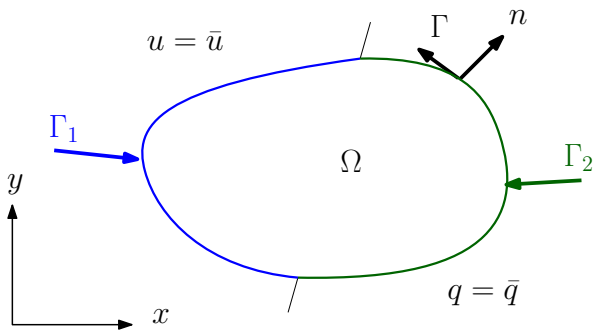

Figure 1. Domain Ω

One of the common approaches is to interpret the errors introduced in the equations eq. (1) and eq.(2) if the exact but unknown values of u and q are replaced by a approximate solution, according to the Weighted Residues Method. This fact is shown by Brebbia [3] together with the application of integration by parts and the Divergence Theorem. Then, the following integral expression results:

$$c(\xi)u(\xi) \quad + \quad \int_\Gamma u(X)q^*(\xi, X)d\Gamma \quad - \quad \int_\Gamma q(X)u^*(\xi, X)d\Gamma \quad = \quad 0 \tag{4}$$

In Equation 4 the auxiliary function $u(\xi, X)$ is the fundamental solution and its normal derivative is expressed by $q(\xi, X)$, the argument of these functions considers the source point ξ, which is an arbitrary point in the Ω domain, as these functions depend on the Euclidean distance between two points. The $c(\xi)$ coefficient depends on the smoothness of the boundary, this fact is shown by Brebbia [3]. For a two-dimensional isotropic medium, the fundamental solution of the Laplace equation depends on the Euclidean distance r between a point ξ of application of the concentrated source and any point X of the domain, this fact is shown by Brebbia [2]:

$$u^* = \frac{1}{2\pi} \ln \frac{1}{r} \qquad e \qquad q^* = \frac{1}{2\pi} \frac{1}{r} \frac{\partial r}{\partial n} \tag{5}$$

3 Boundary element method

The formulation of the integral boundary equation for potential problems can be attributed to Jaswon [4] and Symn [5], who already in 1963 presented a numerical method to solve Fredholm-type integral equations. They discretized the boundary of the problems in various segments, or elements, and assumed a constant font density in each one. The discretization of the boundary causes the eq. (4) to take the form:

$$c(\xi)u(\xi) \quad + \quad \sum_{m=1}^{N} \int_{\Gamma_m} u(x)q^*(X, \xi)d\Gamma_m - \quad \sum_{m=1}^{N} \int_{\Gamma_m} q(X)u^*(X, \xi)d\Gamma_m \quad = \quad 0 \tag{6}$$

where N indicates the number of boundary elements that make up the discretization. Writing in matrix form:

$$c(\xi)u(\xi) + \sum_{m=1}^{N} \begin{bmatrix} u_1 & u_2 & \dots & u_M \end{bmatrix} \int_{\Gamma_m} \begin{bmatrix} N_1 \\ N_2 \\ \dots \\ N_M \end{bmatrix} q^* d\Gamma_m - \sum_{m=1}^{N} \begin{bmatrix} q_1 & q_2 & \dots & q_M \end{bmatrix} \int_{\Gamma_m} \begin{bmatrix} N_1 \\ N_2 \\ \dots \\ N_M \end{bmatrix} u^* d\Gamma_m = 0 \tag{7}$$

The equation 7 represents a mixed system involving potential u in nodes, as well as normal derivatives, or flows, q also nodal. M represents the total number of nodes in each element. The number of unknowns of both types is compensated by the prescription of enough known values to generate a system composed of square matrices, in the form:

$$[H]\{u\} - [G]\{q\} = 0 \tag{8}$$

The coefficients of the matrices H and G are also called influence coefficients and the BEM precision is strongly related to the correct obtainment of such coefficients, which are defined:

$$H_{m,j} = \int_{\Gamma_m} N_k q^* J_m d\eta \quad e \quad G_{m,j} = \int_{\Gamma_m} N_k u^* J_m d\eta \tag{9}$$

J_m is the Jacobian of the coordinate transformation relative to the m element. For a smooth curve, the transformation is simple:

$$d\Gamma = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} d\xi = |J| d\xi \tag{10}$$

4 Numerical integration

The coefficients $H_{m,j}$ and $G_{m,j}$ in the previous expressions can be calculated using numerical integration formulas (such as Gaussian Quadrature) for the case $m \neq j$. In the case where m and j are in the same element $(i = j)$, the singularity of the fundamental solution requires a more precise integration scheme. In reality, analytical integration is reserved only for the calculation of the $G_{m,j}$ coefficients. This is because the coefficients $H_{m,j}$, whose integral is more strongly singular, can be calculated by adding the coefficients of the corresponding lines, according to the procedure of obedience to the imposition of a constant potential field, this fact is shown by Brebbia [3], that is, it is taken into account that when a uniform potential field is applied over a finite physical domain, the derivatives of the potential must be null throughout the physical domain:

$$[H]\{u\} = 0 \tag{11}$$

In this way, both the most strongly singular integral $H_{m,j}$ and the value of the coefficient $c(\xi)$ can be calculated indirectly. In the case of the integral $G_{m,j}$, although more weakly singular, its calculation for quadratic and higher order elements is more elaborate, as shown in Barbosa [1].

The Gaussian Quadrature can be used, but it requires a significant number of integration points,fact shown in Barbosa [1]. Such an integration scheme has the advantage of accurately calculating the integrals of polynomial functions of degrees $p = 2NG + 1$. In one dimension, such as that applied in the MEC in two-dimensional problems, it is given by:

$$\int_{-1}^{1} f(x)dx \quad = \quad \sum_{p=1}^{NPG} w_p f(\mu_p) \tag{12}$$

where NPG is the number of integration point, w_p are the weights of Gaussian Quadrature, and μ_p are integration points. Points and weights are tabulated using a normalized range $(-1, 1)$. It should be noted that the core of the BEM integrals is not polynomial and that in the case of the improper integral, it is necessary to calculate what geometrically the area on the curve would represent correctly, to avoid significant numerical errors.

Namely, singular integrals are those that have an unlimited increasing or decreasing behavior in at least one point of the integrand. That is, the integral I is said to be singular, if:

$$I \quad = \quad \int_a^b f(x)dx \quad e \quad \exists \quad c \quad \in \quad [a,b] \quad | \quad \lim_{x \to c} f(x) \quad = \quad \pm\infty \tag{13}$$

4.1 Self adaptive integration

The Self Adaptive Integration scheme proposed by Telles [6], for the calculation of singular integrals, can be understood in a simplified way as the application of a 3^a order transform at the points provided by the Gaussian Quadrature. This transformation moves the points to the vicinity of the singularity, leading to greater precision of

the integral. For cases of one-dimensional integrations, which are the types of integrations used in this work, a summary of the application of this transform is shown below.

Initially, having the integral shown in eq. (12), where the function to be integrated $f(\eta)$ is singular at $\overline{\eta}$:

$$I = \int_{-1}^{1} f(\eta) d\eta \tag{14}$$

η being the natural coordinate of the integral.

It is then adopted a transformation of coordinates from 3^a order, given inaccording to the following relationship:

$$\eta(\gamma) = A\gamma^3 + B\gamma^2 + C\gamma + D \tag{15}$$

γ is the coordinate provided by the integration points of the Gaussian Quadrature and, after the transformation, the coordinate η will be associated with the weight of the Gaussian Quadrature corresponding to the integration point γ. A, B, C and D are parameters dependent on the position of the singularity point η, as explained later in the text.

Such a coordinate transformation remains valid without the need for feature of the domain partition for any position of the singularity point and automatically produces a greater concentration of integration points near the singularity point, this fact is shown by Telles [6]. A priori, parameters A, B, C and D are defined by applying the following conditions:

$$\eta(1) = 0; \quad \eta(-1) = 0; \quad \left.\frac{d\eta}{d\gamma}\right|_{\overline{\eta}} = 0 \quad e \quad \left.\frac{d^2\eta}{d\gamma^2}\right|_{\overline{\eta}} = 0 \tag{16}$$

The exclusive procedure of applying the boundary conditions of eq. (16) for the determination of coefficients A, B, C and D is exclusive, in the context of boundary elements, for cases where the source point coincides with a placement point belonging to the element, that is, the existing singularity occurs in the boundary element, characterizing a singular integral.

The later idea is to parameterize the entire mathematical model of the square according to the distance R_{min}. This adaptation is necessary for the case of quasi-singular integrals of any intensity, depending on the distance between the source point, which in this case can be in any generic position, and the interpolating one. The sequence for this so-called Self Adaptive Quadrature begins by determining the shortest distance between the source point ξ and the element, called R_min and thereby calculating the value of the parameter D with eq. (17).

$$D = \frac{2R_{min}}{L} \tag{17}$$

In Equation 17, the parameter L is taken as the distance between the extreme nodes of the element. With D it is possible to determine the $\overline{r}$ adaptation parameter:

$$\begin{aligned} \overline{r} &= 0 && 0.00 \leq D < 0.05 \\ \overline{r} &= 0.85 + 0.24\ln(D) && 0.05 \leq D < 1.30 \\ \overline{r} &= 0.893 + 0.0832\ln(D) && 1.30 \leq D < 3.618 \\ \overline{r} &= 1 && D \geq 3.618 \end{aligned} \tag{18}$$

With the adaptive parameter $\overline{r}$ it is possible to calculate $\overline{\gamma}$, which is the position of the singularity in terms of the transformed variable and then the coefficients A, B, C, D.

$$A = \frac{(1-\overline{r})}{Q} \qquad B = -\frac{3\overline{\gamma}(1-\overline{r})}{Q} \qquad C = \frac{\left(\overline{r} + 3\overline{\gamma}^2\right)}{Q} \qquad D = -B \tag{19}$$

Thus the initial scheme proposed by Telles [6] for singular integrations only, can be generalized to any position of the source bridge ξ, inside or outside the element, by eq. (19) Once the coefficients are calculated, the eq. (14) can be rewritten:

$$I = \int_{-1}^{1} f(\eta(\gamma)) J(\gamma) d\gamma \tag{20}$$

Onde, $J(\gamma)$ is the Jacobian of transformation.

$$J(\gamma) = \frac{d\eta}{d\gamma} = 3A\gamma^2 + 2B\gamma + C \tag{21}$$

From this transform, a Self Adaptive Integration scheme applied to the Boundary Element Method is generated as a function of the distance from the so-called ξ source point in relation to the boundary element on which the integration is carried out. The reader can find further details in the references Telles [6] and Bulcão [7].

5 Methodology

In order to verify the influence of the Self Adaptive Integration on the quality of the numerical solution, Laplace problems of square geometry of unitary side were solved. The value of the potential or the derivative of the potential was calculated where the geometry boundary was unknown. The boundary elements used are quadratic and cubic isoparametric. The domain was considered homogeneous and isotropic, with unitary mechanical properties. The Gaussian Quadratures were solved using 08 Gauss points.

The percentual error at each point j of the boundary was calculated according to eq. (22).

$$\%E(j) = 100\frac{|\hat{v}_j - v_j|}{|v_j|} \tag{22}$$

v_j is the analytical value of the variable (potential or derivative) at the point j, $\hat{v_j}$ is the numeric value.

In this work, to define the shape of the boundary elements, continuous Quadratic (2nd degree polynomials) and Cubic (3rd degree polynomials) functions with isoparametric formulation, that is, the same shape functions used to interpolate the geometry are also used to interpolate physical variables (potential and derivative). In the graphs the formulation with Standard Gaussian Integration is referenced SGI while the formulation coupled to the Telles scheme is referenced SAI.

6 Example 01

Consider the potential problem delimited externally by a square two-dimensional domain Ω on the side L as shown in Fig.(2):

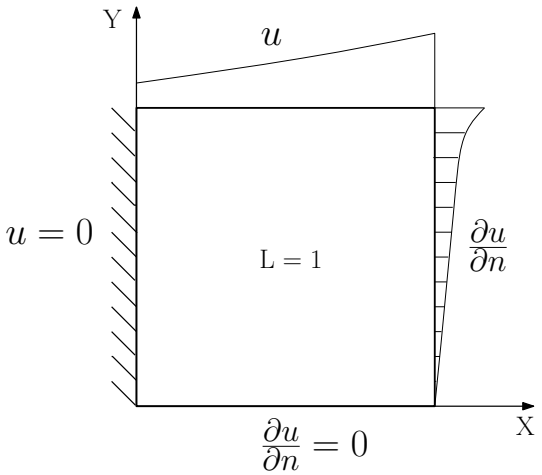

Figure 2. Geometry and boundary conditions of example 1

The analytical solution for the potential derivative is given by eq.(23):

$$\frac{\partial u}{\partial n} = sinh(x)cos(y)n_x + cosh(x)cos(y)n_y \tag{23}$$

where n_x and n_y are the components of the normal vector in the directions x and y respectively. Figures Fig. (3) and Fig.(4) compile the results of the potential derivative on the left edge $y = 0$ for the quadratic and cubic elements respectively.

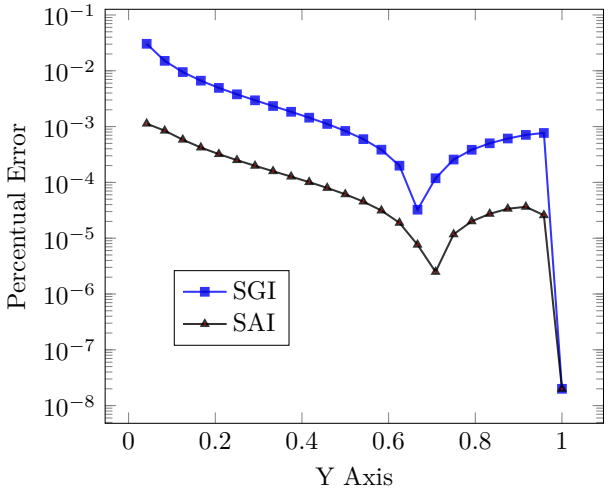

Figure 3. Solutions with quadratic elements

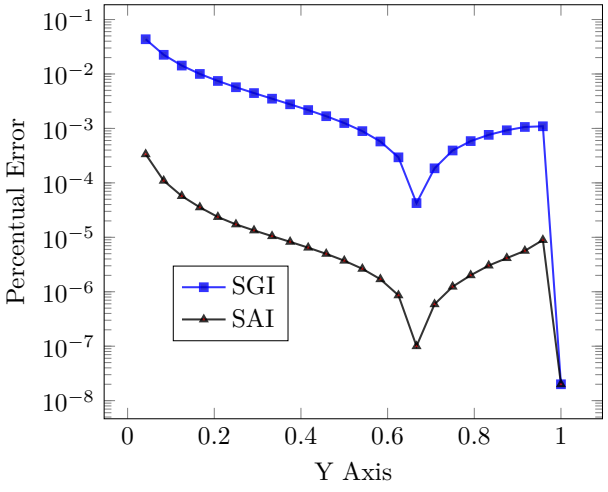

Figure 4. Solutions with cubic elements

A significant reduction in error can be observed at each point in both formulations. The drop reaches 2 orders of magnitude in the cubic elements, showing the effectiveness of the Self Adaptive Integration (SAI) in increasing the accuracy of the BEM in the case under study.The results achieved using just the gauss quadrature does not present a meaningful difference concerning the accuracy; it suggests that the error committed in the solution of the singular coefficients of G matrix is superior that improvement produced by the use of higher order elements

7 Example 02

Figure 5 describes the geometry and boundary conditions for the second example.

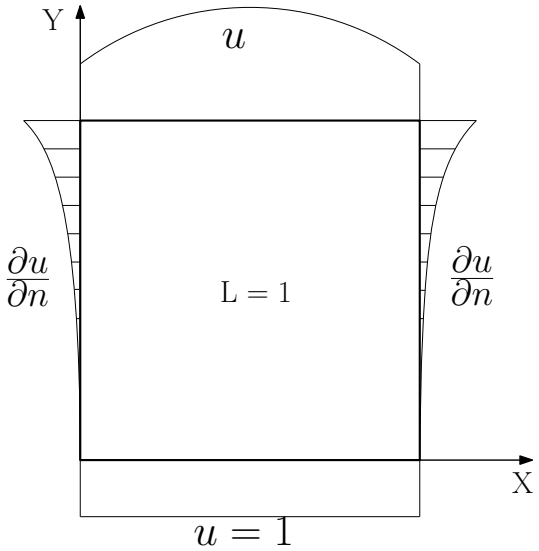

Figure 5. Geometry and boundary conditions of example 1

The analytical solution for the potential is given by eq.(24):

$$u\left(x,y\right) = \frac{sinh(\pi x)}{sinh(\pi)} sin(\pi x) + 1 \tag{24}$$

The figures Fig. (6) and Fig. (7) compile the results for the quadratic and cubic elements respectively.

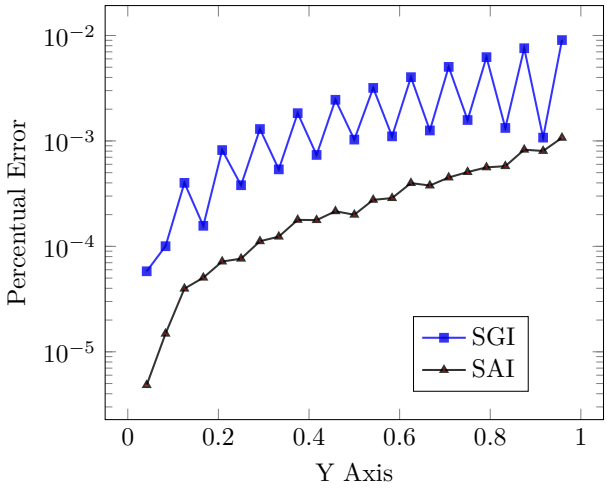

Figure 6. Solutions with quadratic elements

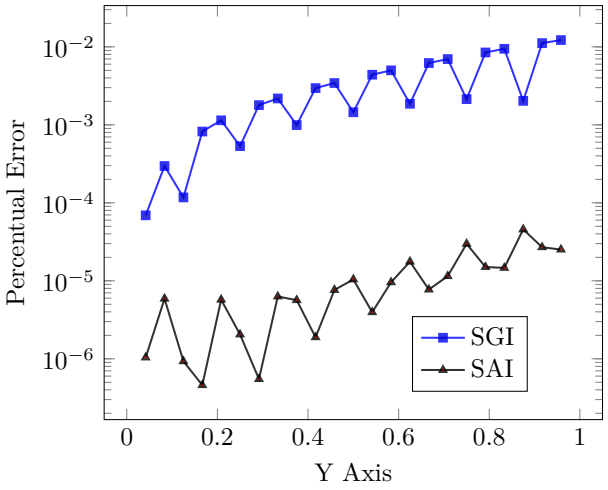

Figure 7. Solutions with cubic elements

Such as in the previous example, the use of higher-order elements without Telles self-adaptive integration scheme was not effective. In this example, particularities related to the prescribed boundary conditions even produced numerical oscillations which are due to error in the numerical calculus of the singular G coefficients.

8 Conclusions

The Boundary Element Method is a robust discrete technique, capable of producing high levels of accuracy far superior to other numerical methods in many applications. However, it requires special care when dealing numerically with its boundary integrals, as they have singular functions at their core, which are improper convergent integrals. When elements of higher-order are used, the special treatment of the singular integrals and the Jacobian involved in the transformation of coordinates is imperative.

One of the most effective tools in this sense is the self-adaptive integration scheme, as this feature allows the numerical calculation of the weakly singular integrals associated with the fundamental solution, which, although they are weakly singular, result in a great numerical error, due to the oscillations that produce in the results. The use of the aforementioned scheme not only eliminates such oscillations but also greatly reduces the level of the numerical error. In addition, it allows the greater potential of approximation with elements of a higher order to effectively translate into better quality results.

For future work it is intended to verify the effectiveness of the methodology presented here in solving problems of greater complexity, both geometric and referring to the imposed load.

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