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Dynamic analysis of Euler-Bernoulli beams over elastic foundation by the Boundary Element Method

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Abstract. Continuous beams are engineering structural elements that can be modeled using the Euler-Bernoulli beam theory. This work is concerned with the development of a time-domain Boundary Element Method formulation to solve the problem of continuous beams over elastic foundation. The formulation was developed with the use of the static fundamental solution, generating a formulation of the type Domain-Boundary Element Method (D-BEM). In all the examples included in this work, geometric and material properties of the beams are assumed to be the same. Boundary Element numerical results for displacements, rotations, bending moments and shear forces obtained for uniformly loaded beams, are compared graphically to the results obtained using the Finite Element Method, since the problem of the vibration of a Euler-Bernoulli beam over elastic foundation has no analytical solution. The main purpose of this paper is to prove the viability of the Boundary Element Method for the time-domain analysis of continuous Euler-Bernoulli beams over elastic foundation.

Keywords: Euler-Bernoulli Beams; Beams on Elastic Foundation; Boundary Element Method

1 Introduction

The differential equations of Euler-Bernoulli beam theory can be developed either statically or dynamically. The static formulation describes the beam parameters according to its geometry, constituent materials and the point of the beam to be monitored; regarding this matter, see Beer et al., Hibbeler [1, 2]. The dynamic formulation considers variations as a function of time, such as dynamic load and beam oscillation, see Rao, Graff [3, 4].

To describe the behavior of beams supported on soil, the formulation of beams on elastic base is used, which adds the behavior of the soil as a static variable, as at Hetényi, Brebbia et al. [5, 6], or dynamically , as at Graff [4].

Dynamic analysis of beams by the Boundary Element Method (BEM) is a subject that has instigated research interest. Scuciato et al., Carrer et al. [7, 8] demonstrated that BEM is efficient to generate reliable results. Caring on their research, the main motivation for this paper is to demonstrate that the BEM can also produce accurate results when adding the foundation elastic properties to the governing equation.

The BEM equation for the present work was obtained using the fundamental solution of the static case, considering a distributed load over the beam. In this context, three domain integrals were determined: an integral related to the load, another for the displacement and the last for acceleration. All the domain integrals were computed analytically. For the integrals related to the displacement and acceleration, the domain was discretized employing linear cells. After solving the domain integral that contains the acceleration, it was approximated using the Houbolt Method (Houbolt [9]).

At the end, the viability of BEM was verified by the results obtained for a five-span beam under distributed load. Two types of foundation modulus were considered. The results were compared with the data obtained by the FEM for the same beam.

2 Euler-Bernoulli beams theory

This section begins presenting the two governing equations of the beam elastic line that have been used in this paper:

$$EI\frac{d^4u}{dx^4} = q(x), \tag{1}$$

$$EI\frac{\partial^4u}{\partial x^4} + \kappa u + \rho A\frac{\partial^2 u}{\partial t^2} = q(x,t). \tag{2}$$

Equation (1) is the classic Euler-Bernoulli equation for an uniform static beam (Beer et al. [1]). Equation (2) is the ordinary differential equation which satisfies the vertical deflection $u(x)$ on an elastic foundation under the action of a prescribed vertical load $q(x,t)$, that is represented by a function of position and time (Debnath and Bhatta [10]). E is the Young's modulus, I is the moment of inertia with respect to the neutral axis of the cross-section of area A, admitted constant, ρ is the mass density per unit volume, κ is the foundation modulus of the beam.

To calculate the equation, it is also necessary to know the boundary conditions, which are divided into Essential and Natural. The equations for a generic loading that govern rotation (Θ), bending moment (M) and shear force (Q), respectively, are:

$$u = u(x,t); \Theta(x,t) = \frac{\partial u(x,t)}{\partial x}; M(x,t) = -EI\frac{\partial^2 u(x,t)}{\partial x^2}; Q(x,t) = -EI\frac{\partial^3 u(x,t)}{\partial x^3}. \tag{3}$$

The fundamental solution is given by:

$$u^* = u^*(\xi,x) = \frac{|x-\xi|^3}{12}. \tag{4}$$

From u*, the following functions are defined:

$$\Theta^*(\xi,x) = \frac{\partial u^*(\xi,x)}{\partial x}; M^*(\xi,x) = -EI\frac{\partial^2 u^*(\xi,x)}{\partial x^2}; Q^*(\xi,x) = -EI\frac{\partial^3 u^*(\xi,x)}{\partial x^3}. \tag{5}$$

3 BEM formulation

For a beam defined from $[0$ to $L]$, the weighted residual sentence used for the BEM formulation studied in this paper, is written as:

$$\int_0^L R_D w_D dx + (R_1 w_1)\mid_{x=0} + (R_2 w_2)\mid_{x=0} + (R_3 w_3)\mid_{x=L} + (R_4 w_4)\mid_{x=L} = 0, \tag{6}$$

where w_D is the weighting function for the residual function in the domain, R_D. The residual function in the domain was generated by eq. (2) , resulting in:

$$R_D = \frac{\partial^4 \widetilde{u}}{\partial x^4} + \frac{\kappa \widetilde{u}}{EI} + \frac{\rho A}{EI}\frac{\partial^2 \widetilde{u}}{\partial t^2} - \frac{q(x,t)}{EI}, \tag{7}$$

where ũ refers to approximate values. Terms R_1, R_2, R_3 and R_4 are residuals for the boundary conditions. R_1 and R_2 being the residuals generated for $x = 0$, and R_3 and R_4, for $x = L$. These residues depend on the boundary conditions of each beam. For the pinned-pinned beam, these values are null for displacement and moment.

Terms w_1, w_2, w_3 and w_4, are the weighting functions for the boundary residues. Choosing these functions properly, the number of equations can be reduced, assisting in calculation.

Applying eq. (7), a residual statement and the weighting functions to eq. (6), one obtains the basic BEM equation, written as:

$$\begin{aligned} u(\xi,t) = \frac{1}{EI}\Big[u^*(\xi,x)Q(x,t)\,\big|_{x=0}^{x=L} - \Theta^*(\xi,x)M(x,t)\,\big|_{x=0}^{x=L} + \\ M^*(\xi,x)\Theta(x,t)\,\big|_{x=0}^{x=L} - Q^*(\xi,x)u(x,t)\,\big|_{x=0}^{x=L}\Big] + \\ \frac{1}{EI}\int_0^{\xi} u^*(\xi,x)q(x,t)dx + \frac{1}{EI}\int_{\xi}^{L} u^*(\xi,x)q(x,t)dx - \frac{\kappa}{EI}\int_0^{\xi} u^*(\xi,x)u(x,t)dx - \\ \frac{\kappa}{EI}\int_{\xi}^{L} u^*(\xi,x)u(x,t)dx - \frac{\rho A}{EI}\int_0^{\xi} u^*(\xi,x)\ddot{u}(x,t)dx - \frac{\rho A}{EI}\int_{\xi}^{L} u^*(\xi,x)\ddot{u}(x,t)dx. \end{aligned} \tag{8}$$

Equation (8) can be applied to $\xi = 0$ and to $\xi = L$. However, two boundary conditions are known for each node that constitutes the boundary problem, that is, the problem has four unknowns. Thus, it becomes necessary to obtain two additional equations to solve the problem. These additional equations can be represented by the expression of the rotation, obtained by deriving this equation with respect to ξ. This expression is written as:

$$\begin{aligned} \Theta(\xi,t) = \frac{1}{EI}\left[\frac{\partial u^*(\xi,x)}{\partial \xi}Q(x,t)\bigg|_{x=0}^{x=L} - \frac{\partial \Theta^*(\xi,x)}{\partial \xi}M(x,t)\bigg|_{x=0}^{x=L} + \right. \\ \left. \frac{\partial M^*(\xi,x)}{\partial \xi}\Theta(x,t)\bigg|_{x=0}^{x=L} - \frac{\partial Q^*(\xi,x)}{\partial \xi}u(x,t)\bigg|_{x=0}^{x=L}\right] + \\ \frac{1}{EI}\int_0^{\xi} \frac{\partial u^*(\xi,x)}{\partial \xi}q(x,t)dx + \frac{1}{EI}\int_{\xi}^{L} \frac{\partial u^*(\xi,x)}{\partial \xi}q(x,t)dx - \frac{\kappa}{EI}\int_0^{\xi} \frac{\partial u^*(\xi,x)}{\partial \xi}u(x,t)dx - \\ \frac{\kappa}{EI}\int_{\xi}^{L} \frac{\partial u^*(\xi,x)}{\partial \xi}u(x,t)dx - \frac{\rho A}{EI}\int_0^{\xi} \frac{\partial u^*(\xi,x)}{\partial \xi}\ddot{u}(x,t)dx - \frac{\rho A}{EI}\int_{\xi}^{L} \frac{\partial u^*(\xi,x)}{\partial \xi}\ddot{u}(x,t)dx. \end{aligned} \tag{9}$$

Rewriting eqs. (8) and (9) in matrix form, one has:

$$\begin{aligned} \begin{bmatrix} \mathbf{H}^{CC} & \mathbf{P}^{CC} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{P}}^{CC} & \mathbf{0} \\ \mathbf{H}^{DC} & \mathbf{P}^{DC} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{j+1}^{C} \\ \Theta_{j+1}^{C} \\ \mathbf{u}_{j+1}^{D} \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^{CC} & \mathbf{B}^{CC} \\ \overline{\mathbf{G}}^{CC} & \overline{\mathbf{B}}^{CC} \\ \mathbf{G}^{DC} & \mathbf{B}^{DC} \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_{j+1}^{C} \\ \mathbf{M}_{j+1}^{C} \end{Bmatrix} + \begin{Bmatrix} \mathbf{q}_{j+1}^{C} \\ \overline{\mathbf{q}}_{j+1}^{C} \\ \mathbf{q}_{j+1}^{D} \end{Bmatrix} - \\ \frac{\rho A}{EI} \begin{bmatrix} \mathbf{M}^{CC} & \mathbf{0} & \mathbf{M}^{CD} \\ \overline{\mathbf{M}}^{CC} & \mathbf{0} & \overline{\mathbf{M}}^{CD} \\ \mathbf{M}^{DC} & \mathbf{0} & \mathbf{M}^{DD} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_{j+1}^{C} \\ \mathbf{0} \\ \ddot{\mathbf{u}}_{j+1}^{D} \end{Bmatrix} - \frac{\kappa}{EI} \begin{bmatrix} \mathbf{M}^{CC} & \mathbf{0} & \mathbf{M}^{CD} \\ \overline{\mathbf{M}}^{CC} & \mathbf{0} & \overline{\mathbf{M}}^{CD} \\ \mathbf{M}^{DC} & \mathbf{0} & \mathbf{M}^{DD} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{j+1}^{C} \\ \mathbf{0} \\ \mathbf{u}_{j+1}^{D} \end{Bmatrix}, \end{aligned} \tag{10}$$

where upper index denotes the points location on the beam under study. Letter C represents the boundary nodes and letter D, the internal nodes. At the submatrices, the first upper index corresponds to the position of the source point, related to ξ, and the second, the position of the field point, related to x. Lower index are based on the time of the analysis, $t = t_{j+1}$. The submatrices that compose eq. (10) are:

$$\mathbf{H}^{CC} = \overline{\mathbf{P}}^{CC} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}; \mathbf{P}^{CC} = \begin{bmatrix} 0 & \frac{L}{2} \\ -\frac{L}{2} & 0 \end{bmatrix}; \mathbf{H}^{DC} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}; \mathbf{P}^{DC} = \begin{bmatrix} -\frac{\xi_k}{2} & \frac{L-\xi_k}{2} \end{bmatrix}$$
$$\mathbf{u}^C_{j+1} = \begin{Bmatrix} u(0,t_{j+1}) & u(L,t_{j+1}) \end{Bmatrix}^T; \Theta^C_{j+1} = \begin{Bmatrix} \Theta(0,t_{j+1}) & \Theta(L,t_{j+1}) \end{Bmatrix}^T$$
$$\mathbf{u}^D_{j+1} = \begin{Bmatrix} u(\xi_1,t_{j+1}) & u(\xi_2,t_{j+1}) & \dots & u(\xi_{n-1},t_{j+1}) \end{Bmatrix}^T$$
$$\mathbf{G}^{CC} = \frac{1}{EI}\begin{bmatrix} 0 & \frac{L^3}{12} \\ -\frac{L^3}{12} & 0 \end{bmatrix}; \mathbf{B}^{CC} = \overline{\mathbf{G}}^{CC} = \frac{1}{EI}\begin{bmatrix} 0 & -\frac{L^2}{4} \\ -\frac{L^2}{4} & 0 \end{bmatrix}; \overline{\mathbf{B}}^{CC} = \frac{1}{EI}\begin{bmatrix} 0 & \frac{L}{2} \\ \frac{L}{2} & 0 \end{bmatrix} \quad (11)$$
$$\mathbf{G}^{DC} = \begin{bmatrix} -\frac{\xi^3}{12EI} & \frac{(L-\xi_k)^3}{12EI} \end{bmatrix}; \mathbf{B}^{DC} = \begin{bmatrix} -\frac{\xi_k^2}{4EI} & -\frac{(L-\xi_k)^2}{4EI} \end{bmatrix}$$
$$\mathbf{Q}^C_{j+1} = \begin{Bmatrix} Q(0,t_{j+1}) & Q(L,t_{j+1}) \end{Bmatrix}^T; \mathbf{M}^C_{j+1} = \begin{Bmatrix} M(0,t_{j+1}) & M(L,t_{j+1}) \end{Bmatrix}^T$$
$$\mathbf{q}^C_{j+1} = \begin{Bmatrix} \frac{q(x)L^4}{48EI} & \frac{q(x)L^4}{48EI} \end{Bmatrix}^T; \overline{\mathbf{q}}^C_{j+1} = \begin{Bmatrix} -\frac{q(x)L^3}{12EI} & \frac{q(x)L^3}{12EI} \end{Bmatrix}^T; \mathbf{q}^D_{j+1} = \begin{Bmatrix} \frac{q(x)[\xi_k^4+(L-\xi_k)^4]}{48EI} \end{Bmatrix}.$$

Submatrices $\mathbf{M}$ and $\overline{\mathbf{M}}$ are assembled with expressions that result from the domain integral discretization. These expressions can be found at Appi [11].

3.1 Time-marching scheme

The Houbolt method (Houbolt [9]) is used to approximate the acceleration in time in eqs. (8) and (9). The approximation to velocity and acceleration are given by:

$$\dot{u}_{j+1} = \frac{1}{6\Delta t}[11u_{j+1} - 18u_j + 9u_{j-1} - 2u_{j-2}]$$
$$\ddot{u}_{j+1} = \frac{1}{\Delta t^2}[2u_{j+1} - 5u_j + 4u_{j-1} - u_{j-2}]. \quad (12)$$

Substituting eq. (12) in eqs. (8) and (9), one has the final matrix form written below:

$$\left\{\begin{bmatrix} \mathbf{H}^{CC} & \mathbf{P}^{CC} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{P}}^{CC} & \mathbf{0} \\ \mathbf{H}^{DC} & \mathbf{P}^{DC} & \mathbf{I} \end{bmatrix} + \left(\frac{2\rho A}{EI\Delta t^2} + \frac{\kappa}{EI}\right)\begin{bmatrix} \mathbf{M}^{CC} & \mathbf{0} & \mathbf{M}^{CD} \\ \overline{\mathbf{M}}^{CC} & \mathbf{0} & \overline{\mathbf{M}}^{CD} \\ \mathbf{M}^{DC} & \mathbf{0} & \mathbf{M}^{DD} \end{bmatrix}\right\}\begin{Bmatrix} \mathbf{u}^C_{j+1} \\ \mathbf{\Theta}^C_{j+1} \\ \mathbf{u}^D_{j+1} \end{Bmatrix} =$$
$$\begin{bmatrix} \mathbf{G}^{CC} & \mathbf{B}^{CC} \\ \overline{\mathbf{G}}^{CC} & \overline{\mathbf{B}}^{CC} \\ \mathbf{G}^{DC} & \mathbf{B}^{DC} \end{bmatrix}\begin{Bmatrix} \mathbf{Q}^C_{j+1} \\ \mathbf{M}^C_{j+1} \end{Bmatrix} + \begin{Bmatrix} \mathbf{q}^C_{j+1} \\ \overline{\mathbf{q}}^C_{j+1} \\ \mathbf{q}^D_{j+1} \end{Bmatrix} - \frac{\rho A}{EI\Delta t^2}\begin{bmatrix} \mathbf{M}^{CC} & \mathbf{0} & \mathbf{M}^{CD} \\ \overline{\mathbf{M}}^{CC} & \mathbf{0} & \overline{\mathbf{M}}^{CD} \\ \mathbf{M}^{DC} & \mathbf{0} & \mathbf{M}^{DD} \end{bmatrix}\begin{Bmatrix} -5\mathbf{u}^C_j + 4\mathbf{u}^C_{j-1} - \mathbf{u}^C_{j-2} \\ \mathbf{0} \\ -5\mathbf{u}^D_j + 4\mathbf{u}^D_{j-1} - \mathbf{u}^D_{j-2} \end{Bmatrix}. \quad (13)$$

Concisely, matrices presented in eq. (13) can be written as:

$$\mathbf{H}\mathbf{u}_{j+1} = \mathbf{G}\mathbf{b}_{j+1} + \mathbf{q}_{j+1} + \mathbf{M}\mathbf{v}_j. \quad (14)$$

where the vector u_{j+1} contains the displacement and rotation values for the boundary nodes, as well as, the displacement values for the internal nodes. The vector b_{j+1} contains the values associated with the shear force and the bending moment only at the boundary nodes. The vector q_{j+1} is associated with uniformly distributed loading and the vector v_j is composed only of known displacement values, previously calculated.

4 Continuous beams

The entire formulation described above was for a single span beam. An example, for two spans beam, initially, eq. (14) are written independently for each span; and then grouped in an expanded system, as demonstrated bellow:

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{Bmatrix} + \mathbf{w}. \tag{15}$$

Matrices $\mathbf{H}_i$ are $(4 + n_i) \times (4 + n_i)$ matrices where n_i is the number of internal points for each span, and $\mathbf{G}_i$ are $(4 + n_i) \times 4$ matrices. The u_i vector consists of two vectors:

$$\mathbf{u}_i = \begin{Bmatrix} \mathbf{u}_i^B \\ \mathbf{u}_i^D \end{Bmatrix}; \mathbf{u}_i^B = \begin{Bmatrix} u_{i1}^B & u_{i2}^B & \Theta_{i1}^B & \Theta_{i2}^B \end{Bmatrix}^T; \mathbf{u}_i^D = \begin{Bmatrix} u_{i1}^D & u_{i2}^D & \dots & u_{in_i}^D \end{Bmatrix}^T, \tag{16}$$

u_i^B being composed by the values for the displacement and rotation at the boundary of each beam, and the vector u_i^D contains n_i elements, which represent the displacement for each internal point.

The column vector b_i contains the values for bending moment and shear force at the boundary.

$$\mathbf{b}_i = \begin{Bmatrix} Q_{i1} & Q_{i2} & M_{i1} & M_{i2} \end{Bmatrix}^T; \mathbf{w} = \begin{Bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{Bmatrix}. \tag{17}$$

All vectors and matrices mentioned above contain the values at the current time, $t = t_{j+1}$. The vector w is composed by the previous times, $t = t_j$, $t = t_{j-1}$, $t = t_{j-2}$, plus the loading contribution.

The equations are coupled by taking into account the following equations:

$$u_{(i+1)1} = u_{i2}; \Theta_{(i+1)1} = \Theta_{i2}; M_{(i+1)1} = M_{i2}, \tag{18}$$

where the displacement, rotation or bending moment of the first node of the $(i+1)$- th span is equal to the equivalent of the last node of the i - th span.

The results for u, Θ, M and Q are determined after eq. (18) are applied to eq. (15) and the system is rearranged. With those results, the values for displacements, cross-section rotations, bending moments and shear forces are computed for internal points.

5 Examples

To validate the formulation, a five-span beam was analyzed. As there is no known analytical solution for the problem proposed in this work, the results were compared with the solution obtained by the Finite Element Method (FEM).

For calculations it was assumed a rectangular concrete beam, which has the following characteristics: $A = 800\ cm^2$, $E = 3 \times 10^{10}\ N/m^2$, $\rho = 2500\ kg/m^3$, each span has length equal to $L = 4.00\ m$. A distributed load of $q = 6.00\ kN/m$ was applied as an instant load. Two values were adopted for the foundation modulus: $\kappa = 0.00\ N/m^3$ and $\kappa = 4.80 \times 10^7\ N/m^3$.

Assuming the use of cells with the same length, the beam was discretized into 20 equal parts for BEM analyses and 80 for FEM discretization. The FEM analysis was calculated with Hermitian cubic polynomials as shape functions. For the time integration, Newmark method was employed, with $\alpha = 1/2$ and $\beta = 1/4$. Both codes were executed with the same time step of $\Delta t = 1.00 \times 10^{-6}$ s, and the total time analysis was 40 milliseconds, resulting in $40,000$ points of data.

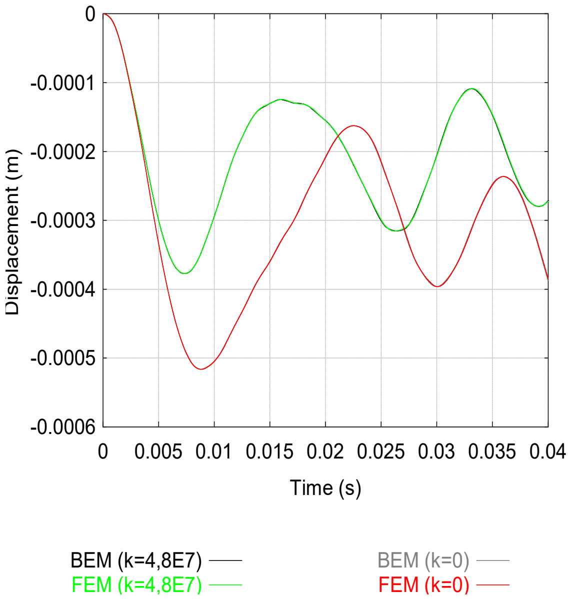

Figure 1. Five-span beam under distributed load: transverse displacement at the central point of the first span (x=2.00 m).

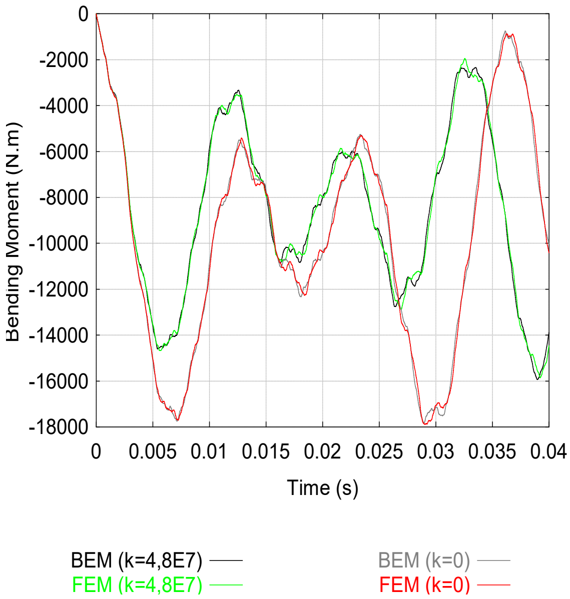

Figure 2. Five-span beam under distributed load: bending moment at support at x = L.

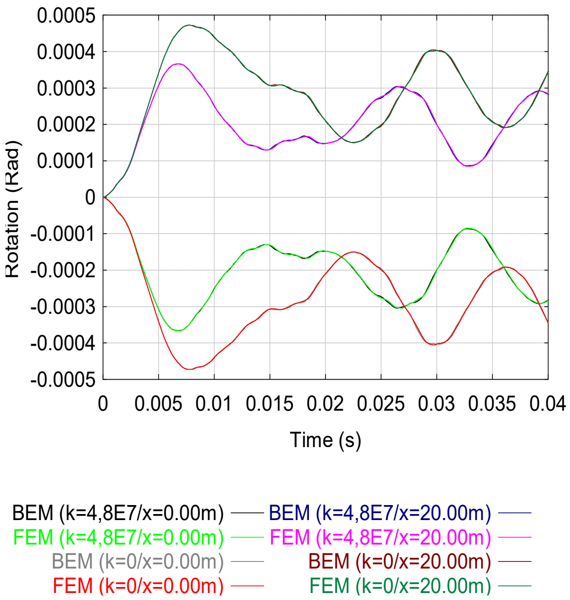

Figure 3. Five-span beam under distributed load: cross-section rotation at supports at x = 0 and x = 5 L.

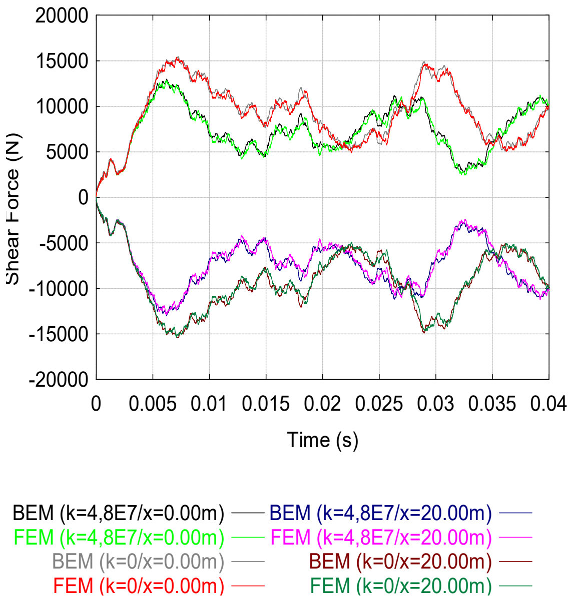

Figure 4. Five-span beam under distributed load: shear force at supports at x = 0 and x = 5 L.

In general, the numerical results for the BEM are consistent with the results presented by FEM, since the results did not differ. Comparing the results obtained for displacement (Fig. 1) and rotation (Fig. 3), a good approximation of these results was observed. Considering the results for the bending moment (Fig. 2) and shear forces (Fig. 4), differences between them are more evident. However, this can be result of the derivative process, since moment is the second derivative of displacement and the shear forces is the third derivative of displacement, both being subject to numerical instability.

Regarding the supports and loading symmetry, with respect to the central point of the third span, along with the uniformity of geometric and physical parameters of the material that constitute the beam, it can be observed that: the displacement for the first span will be equal to the displacement of the k-th span; the displacement of the second span will be equal to the displacement of $(k-1)$-th span, and so on until reaching the central span; the results obtained for rotation and shear force in the first support must be equal in module to the results of the last support under analysis.

The codes being written in different languages (Fortran for the BEM formulation and Python for FEM code) and usage of different computer for each code turn the computational time comparison not possible.

6 Conclusions

Continuous beams are structural elements widely used in engineering, and can be modeled with the Euler-Bernoulli beam theory. Under this theory, three-dimensional problems can be treated as one-dimensional. Hence, this work demonstrated that the BEM can be used to solve this type of problem.

The results presented here reinforce that the formulation developed is capable of producing quite satisfactory results for analysis of continuous beams. Therefore, the initial purpose of the research was achieved. This conclusion can be seen through the analysis of the BEM results compared with those provided by the FEM, where the good convergence of results was observed.

Thus, the continuation of this line of research is expected, exploring different parameters for the same beam, different boundary conditions, since it looks promissory.

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