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Topology optimization with boundary element method using topological derivative and its gradient

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Abstract. A new methodology of topology optimization of thermal conducting solids using the Boundary Element Method (BEM) and Topological Derivatives (TD) is presented. The TD is evaluated on the boundary and the domain and its gradient is evaluated on the problem's boundary only. Boundary nodes where TD values are low are then moved using the TD's gradient as indicator of direction and the TD's value as indicator of intensity of change. The creation of holes inside the domain is made possible by taking isolines of the TD on the domain. Isolines with low values of the TD are used as the boundary of the new holes. The methodology creates smoother boundaries with reduced number of elements when compared to previously proposed techniques, while maintaining a simple mathematical formulation.

Keywords: Topology Optimization, Topological Derivatives, Topological Sensitivity, Boundary Element Method

1 Introduction

Topology optimization has become a very powerful tool for engineering projects. Many studies have already been developed over the subject and different methods have been presented and gained popularity, the most common ones are:

- Simplified Isotropic Material with Penalization (SIMP); [1][2][3]
- Evolutionary Structural Optimization (ESO); [4][5]
- Level-Set Method (LSM); [6][7][8]
- Topological Sensitivity and Topological Derivatives (TD). [9][10][11][12]

Sigmund and Maute [13] presents a good review on the different methodologies. Each method has its own advantages and disadvantages. The most commonly used nowadays is SIMP, which is a great implementation to be used with the Finite Element Method (FEM). LSM has been gaining special popularity due to it being able to generate smooth boundaries and being easily used with the Boundary Elements Method (BEM). Topological derivatives have also been gaining special interest due to its ease of use with BEM. Marczak [11] proposed a methodology of generation of holes where low values of topological derivatives are found in order to obtain the optimized topology, but it also generated irregular and complex boundaries. Jing et al. [8] recently presented a methodology using the topological sensitivity and LSM with BEM. The present work attempts to do the topology optimization using only the evaluation of topological derivatives and their gradients in order to move boundary nodes and generate new boundaries so that the mathematical formulation is simpler than that of LSM and boundary smoothness is better than only 'punching out' disks of material as proposed by Marczak [11].

2 Mathematical Formulation

The boundary element method formulation for problems governed by the Laplace equation used is given on eq. (1), as presented by Kane [14]:

$$c(\vec{x_d})u(\vec{x_d}) = \int_\Gamma q^* u d\Gamma - \int_\Gamma u^* q d\Gamma, \tag{1}$$

where

$$c(\vec{x_d}) = \begin{cases} 1, & \text{if} \quad \vec{x_d} \in \Omega \\ 1/2, & \text{if} \quad \vec{x_d} \in \Gamma \\ 0, & \text{if} \quad \vec{x_d} \notin \Omega \text{ nor } \Gamma, \end{cases}$$

and where $\vec{x_d}$ is the location of the source point, u is the potential, q is the normal flux at the boundary, u^* is the fundamental solution for the potential. q^* is the fundamental solution for the flux, Γ represents the boundary and Ω represents the domain.

Topological derivatives give the sensitivity of the cost function of a problem's domain given a small hole is introduced at a specific point inside the domain. As a preliminary analysis, only homogeneous Neumann boundary conditions on new boundaries will be used. The TD's formulation for these problems, as presented by Novotny et al. [10], assuming the total potential energy as the cost function is given by eq. (2):

$$D_{\mathrm{T}}(\hat{x}) = k\nabla u \cdot \nabla u - bu, \tag{2}$$

where k is the material's property and b is a constant excitation.

The strategy adopted in this work for material removal is to move boundary nodes that have low values of TD in order to reduce the problem's area. The goal is to find the direction in which the TD is less sensitive to change, since that would be where the cost of removing material changes less and, in case of a low value of TD, the boundary is desired to move more significantly. The gradient of the TD gives the direction in which the TD is most sensitive to change. The direction in which the TD is less sensitive to change is then the perpendicular to the gradient of the TD. From this, the direction in which boundary nodes with low values of TD should be moved is perpendicular to the gradient of the TD. On a 2d problem, the gradient of the topological derivative according to the formulation presented on eq. (2) can be obtained by applying the chain rule. Assuming there is no constant excitation (b = 0), since only problems governed by the Laplace equation will be analyzed, the expressions for the TD's gradient are given by eqs. (3) to (5):

$$\nabla D_{\mathrm{T}} = \frac{\partial D_{\mathrm{T}}}{\partial x}\hat{i} + \frac{\partial D_{\mathrm{T}}}{\partial y}\hat{j}, \tag{3}$$

$$\frac{\partial D_{\mathrm{T}}}{\partial x} = k\left(2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x}\right), \tag{4}$$

$$\frac{\partial D_{\mathrm{T}}}{\partial y} = k\left(2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y}\right). \tag{5}$$

In order to use these equations it is necessary to calculate $\partial^2 u/\partial x^2$, $\partial^2 u/\partial y^2$ and $\partial^2 u/\partial x\partial y$ on the boundary, which are not normally calculated by the BEM. The first order gradient of u on the boundary is obtained through the coordinate transformation from the normal and tangential coordinate system. The $\partial u/\partial n$ is obtained from the normal flux q, calculated by eq. (1) and the $\partial u/\partial t$ is obtained from the derivative of the potential u along the boundary, calculated using the derivative of the shape function. The first order derivatives of u inside the domain are obtained by the differentiation of eq. (1). The second order derivatives of u on the boundary are then calculated using a radial basis function network (RBFN) as described by Mai-Duy and Tran-Cong [15].

As already discussed the direction of movement for the boundary is perpendicular to the TD's gradient and, in order to make sure nodes move to the inside of the problem's domain, has an angle with the boundary's normal $\hat{n}$ greater than 180$^{\mathrm{o}}$, given by eq. (6):

$$\vec{v} = \frac{-\frac{\partial D_{\mathrm{T}}}{\partial y}\hat{i} + \frac{\partial D_{\mathrm{T}}}{\partial x}\hat{j}}{\left\|-\frac{\partial D_{\mathrm{T}}}{\partial y}\hat{i} + \frac{\partial D_{\mathrm{T}}}{\partial x}\hat{j}\right\|} \cdot sign\left(\left[\frac{\partial D_{\mathrm{T}}}{\partial y}\hat{i} - \frac{\partial D_{\mathrm{T}}}{\partial x}\hat{j}\right] \cdot \hat{n}\right). \tag{6}$$

The intensity of movement of the boundary in direction $\vec{v}$ is limited to a maximum change α_{max} and inversely proportional to the TD's value on each point, so that points that have low values of TD are moved more significantly. The intensity of movement α_i for each point is given by eq. (7):

$$\alpha_i = \left(\frac{max(D_{\mathrm{T}}) - D_{\mathrm{T}i}}{max(D_{\mathrm{T}}) - min(D_{\mathrm{T}})} \right)^{\lambda} \alpha_{max}. \tag{7}$$

Results shown in this work were found with $\lambda = 4$. Values for α_{max} are different for each problem and are dependent on the geometry. The change of coordinates in each node ($\vec{V}_i$) is given by eq. (8):

$$\vec{V}_i = \alpha_i \vec{v}_i. \tag{8}$$

Since this methodology isn't capable of generating holes, the use of TD's isolines to change the topology on the the first iteration was implemented so that hole generation was possible. In order to do this, TD values on internal points were calculated and the marching squares algorithm was used to find points of isolines. The isovalues for the isolines that are used to generate new boundaries were selected as the ith percentile of TD's values inside the domain. After finding the points that define the isolines, a B-Spline of degree three is used to smooth these points and generate new geometric nodes.

3 Methodology

The flowchart of the methodology used in this work is given on Fig. 1.

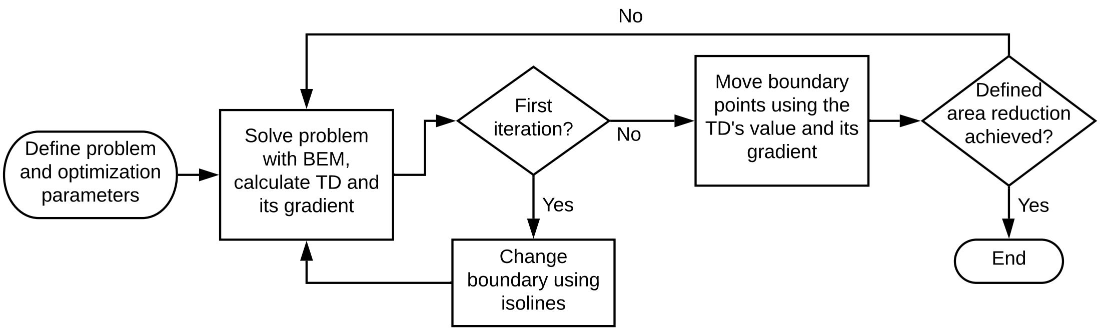
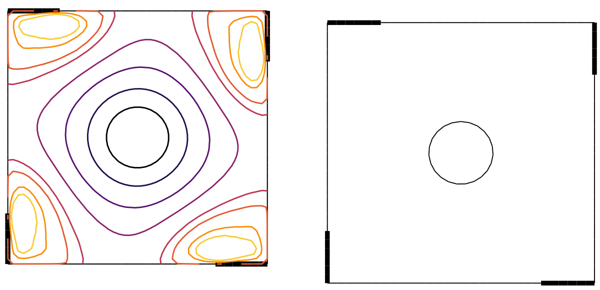


Figure 1. Flowchart of the implemented methodology

On the first iteration, isolines of the TD's surface are used to change the boundaries and make hole generation possible, as seen on Fig. 2. Element density on boundaries generated using isolines is predetermined and the total number of elements can change on this iteration, since elements can be created and deleted.



(a) Isolines of the TD on problem 2 (b) boundaries of problem 2 after first iteration

Figure 2. Topology change using isolines

After first iteration, boundary changes are made with the TD's gradient and value only. In order to do this, the value of α_{max} must first be defined. Each iteration is intended to move nodes according to $\vec{V}$ calculated on eq. (8) so that nodes with lowest values of TD move a distance of α_{max} in direction $\vec{v}$. An example of this strategy is shown on Fig. 3, in which the heat map represents the TD's values inside the problem's domain and the arrows give the node position change if only one step was used. There are overlapping arrows that would be problematic for the resulting geometry, therefore each iteration is divided into a set number of steps in which $\alpha_{maxstep}$ is equal to α_{max}/N_{steps}.

After each step, a smoothing B-spline is used to correct any overlapping and the boundary mesh is regenerated. The B-spline used was of degree 3 and had a number of control points equal to a third of the number of points that the segment being smoothed had. New nodes are then generated using this B-spline and the mesh is updated. The number of new nodes generated is the same as the number of nodes that the segment being smoothed previously had, therefore the number of elements per segment is maintained constant between iterations using this strategy.

After all steps from an iteration, the area of the new mesh is calculated and if the stop criteria is satisfied the optimization is finished, otherwise the problem is solved again and another iteration takes place.

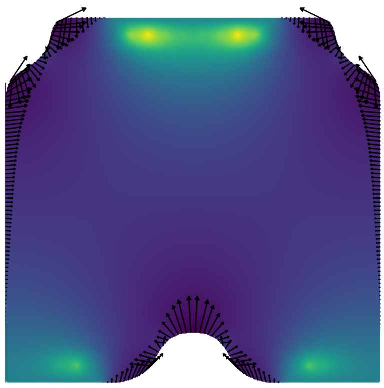
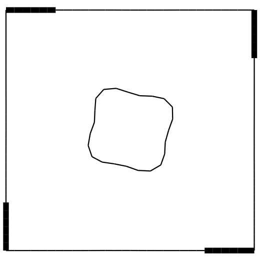
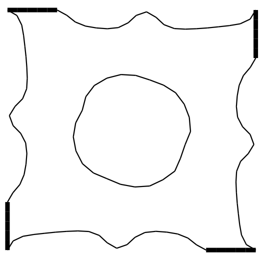


Figure 3. Nodes' position change direction and module if no intermediate steps are adopted

In order to reduce possible distortions as shown on Fig. 4, the nodes to be moved were chosen according to their TD's values. Nodes that presented TD's values below a given percentile were moved. The percentile used to choose nodes to be moved is different for every problem.



(a) pct = 0.5

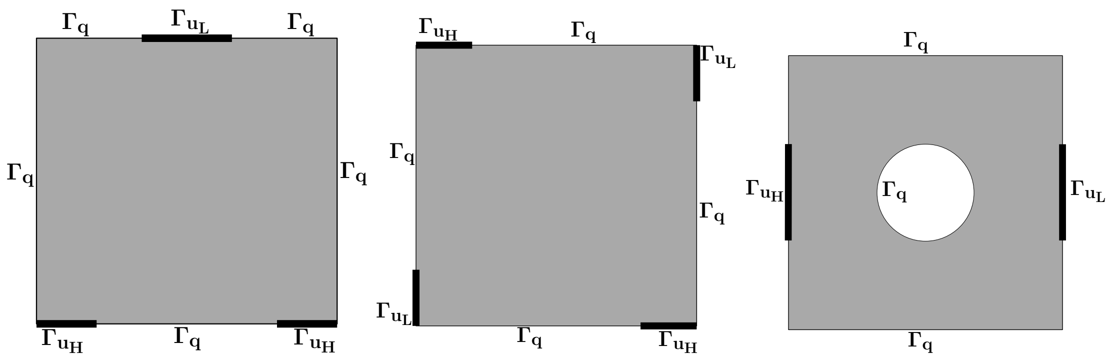


(b) pct = 1

Figure 4. Effect of limiting values of DT

4 Numerical Examples

Three problems were chosen to be analyzed, an inverted V conductor, an asymmetric conductor and a bridge conductor, as can be seen on Fig. 5.



(a) Problem 1 (b) Problem 2 (c) Problem 3

Figure 5. Boundary conditions for each problem analyzed

Boundary conditions on all problems are given on eqs. (9) to (11). All three problems are square domains with side length equal to 1.

$$\bar{u} = 1 \quad \text{for} \quad \Gamma_{u_H}, \tag{9}$$

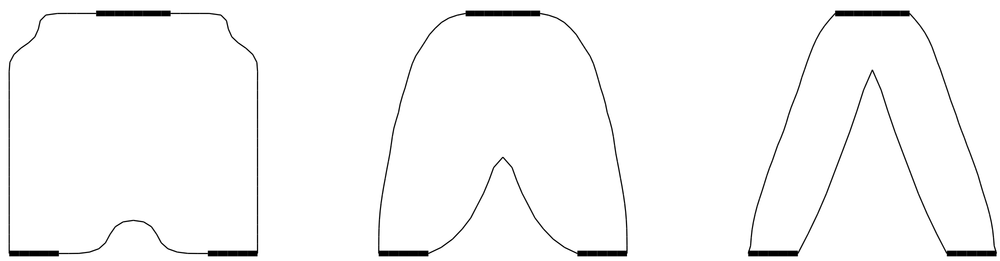
$$\bar{u} = 0 \quad \text{for} \quad \Gamma_{u_L}, \tag{10}$$

$$\bar{q} = 0 \quad \text{for} \quad \Gamma_q. \tag{11}$$

These problems were chosen because their results were already available in the literature, which helps compare the results obtained. Results were compared to the ones obtained by Marczak [11] and Novotny et al. [10].

The following results were found using initially 30 quadratic discontinuous elements per side. On all problems the percentile used for the TD's isovalue for first iteration was the 4th and the number of steps on each iteration for node position change was 6.

On problem 1, the percentile used to choose moving points was the 90th and the stopping criteria was when $A \leq 0.5A_i$ with A_i being the initial area. On problem 2, the percentile used to choose moving points was the 50th and the stopping criteria was on $A \leq 0.8A_i$. For problem 3, the percentile used to choose moving points was the 95th and the stopping criteria was on $A \leq 0.65A_i$. Since problem 3 is symmetric, only half of its domain will be optimized. Results are shown on Figs. 6 to 8.



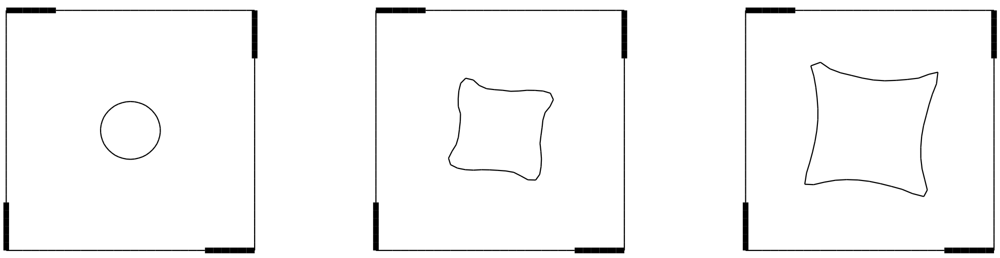
(a) Iteration 1 ($0.94A_i$) (b) Iteration 5 ($0.74A_i$) (c) Iteration 23 ($0.5A_i$)

Figure 6. Optimization of problem 1

The optimized topologies from this methodology were very similar to the ones presented by Marczak [11] and Novotny et al. [10] while being able to maintain smooth boundaries. On problem 1 a higher number of iterations was needed for convergence, possibly caused by the sharp edge generated.

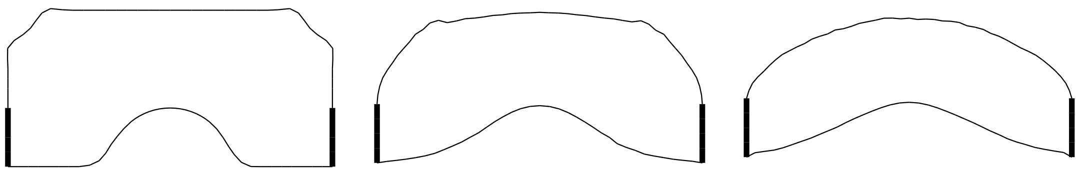
5 Conclusions

This work presented a new methodology of topology optimization intended to couple the boundary element method and topological derivatives, having a simple mathematical formulation while still being able to generate

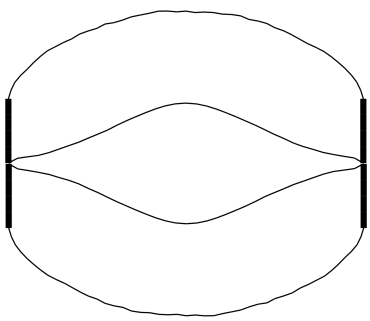


(a) Iteration 1 ($0.95A_i$) (b) Iteration 3 ($0.87A_i$) (c) Iteration 9 ($0.79A_i$)

Figure 7. Optimization of problem 2



(a) Iteration 1 ($0.96A_i$) (b) Iteration 3 ($0.79A_i$) (c) Iteration 6 ($0.64A_i$)



(d) Iteration 6 whole problem

Figure 8. Optimization of problem 3

smooth optimized topologies. The idea is similar to the one presented by Marczak [11], but the topology changing strategy is different in order to obtain smoother boundaries. The optimized topologies obtained on numerical examples are similar to previous literature results while being considerably smoother. Adequate results for the analysed problems were obtained by the proposed methodology. However its implementation is still preliminary and requires further study.

As future work it is intended to extend this methodology for problems governed by other differential equations, to use the presented hole generation strategy on subsequent iterations and to investigate strategies different from using a percentile of the TD for selecting which nodes to be moved, independent of the problem's geometry. It is also intended to use isogeometric boundary element method so no transformation between the smoothing spline and the conventional BEM elements is needed.

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