XLI CILAMCE
XLI Ibero-Latin American Congress on Computational Methods in Engineering
16-19 November, 2020 FOZ DO IGUAÇU PARANÁ - BRAZIL

COMPARING DIFFERENT ADAPTIVE MESH REFINEMENT STRATEGIES APPLIED TO TWO-DIMENSIONAL POTENTIAL PROBLEMS USING BOUNDARY ELEMENT METHOD

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Abstract. The work consists of comparing the performance of different adaptive mesh refinement strategies adaptive refining using the Boundary Elements Method (MEC) based on error estimates made by three error estimators. The three error estimators adopted in this work are: discontinuous estimator, integral estimator and recursive estimator. The parameter adopted in the comparison will be the reduction of the average error calculated in points internal to the geometry so that the calculation of the average error is for the same points in all cases because different discretized geometries are generated from each estimator. The results obtained from the error estimators are compared to the application of uniform refinement to study the efficiency of the refining based on the different error estimators.

Keywords: *Adaptive mesh refinement, Boundary Element Method*

1 INTRODUCTION

The Boundary Element Method is one of the three most popular numerical analysis methods. The terms ”boundary element” or ”boundary integral” have just over 10,000 entries in the Web of Science, [1]. BEM is a technique that needs to discretize only the outline of the problem, which reduces its dimension by one, [2]. The main objective of this work is to compare different methods of adaptive refinement.

Problem analysis using the BEM must consider the development of a mesh based on boundary conditions. If the solution obtained by the method does not have adequate precision, the mesh needs to be refined in order to obtain satisfactory results, [3]. Two types of refinement can be adopted, adaptive or uniform. The first takes into consideration parameters that indicate which elements must be refined while the second performs the refining of all elements of the mesh.

Usually, the mesh refinement process in the discretization phase of the problem is associated with obtaining more precise numerical solutions. However, refining the mesh means increasing the computational effort employed, therefore, the efficient use of this artifice is essential - one must seek the refinement of places that most influence the accuracy of the method.

The adaptive refinement technique aims precisely to perform the refinement efficiently, based on the concept of error estimator, characterized in the form of algorithms responsible for mapping the errors of the solution in a localized way. Thus, it is possible to know which region of the discrete model needs to be refined. Reliable estimation of these errors is essential to ensure a certain level of numerical accuracy and is a key component of adaptive procedures, [4].

The task of indicating where new elements should be created is accomplished by the so-called error indicators (or correction indicators). If these quantities are computed after an initial solution is already available, they are denoted as a posteriori error indicators, [5].

Three error estimators will be considered in this work, the discontinuous, the integral and the recursive. The discontinuous error estimator takes into account the difference of the calculated value of the physical quantity under study between adjacent elements. Through the calculated difference, it is possible to locate in which locations of the discretized mesh the refining should be carried out. The integral error estimator, proposed by [6], uses interpolator polynomials of different orders in the application of the BEM, the results are compared and the difference between the two results is used as an error estimate.

Generally, the values at internal points of the domain with the BEM are determined with the recursive application of the integral equation after all the nodal values in the contour have been calculated. In scalar potential problems, it is shown that the same idea can be used to improve the accuracy of the results in the contour. Instead of the new source points being located within the domain, they are positioned over the contour, with different coordinates than the nodal points. These new points will serve as a reference for an error estimate.

2 BOUNDARY INTEGRAL EQUATION

The Boundary Elements Method (BEM) for potential problems is based on the following eqt. 1 [7]:

$$cT(x_d, y_d) = \int_s Tq^* ds - \int_s T^* q ds, \tag{1}$$

where $c = 1$ for internal points, $c = \frac{1}{2}$ for points on a smooth boundary, T is the temperature and q is the flux.

This formulation discretizes the boundary into descontinous elements. In each boundary-element, the geometry and the state variables are approximated through linear and quadratic interpolations.

3 DISCONTINUOUS ERROR ESTIMATOR

For the discontinuous error estimator, we have to, according to [8]:

$$e_u = \frac{(u_i(\xi = 1) - u_{i+1}(\xi = -1))^2}{|u|^2}. \tag{2}$$

In the eqt. 2, u_i represents the value of the physical quantity of the element under analysis and u_{i+1} represents the value of the adjacent element. In the denominator, u represents the vector that stores the value of the physical quantity at the nodal points.

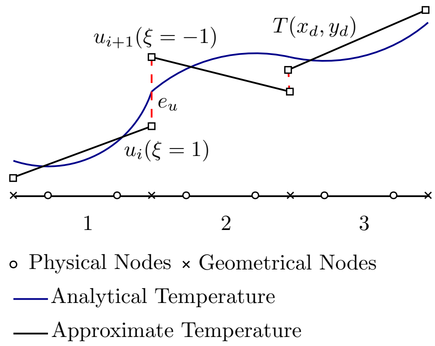


Figure 1. Schematic drawing of the discontinuous error estimator.

Sketched by the figure 1, the discontinuous error estimator calculates the difference of a physical quantity between adjacent elements using the eqt. 2. Each element that discretizes the geometry has an error at each end, and the new nodal points must be added at the ends with the greatest error. In this way, it is possible to locate in which locations of the discretized geometry the error is more accentuated in order to refine locally due to the greater impact to reduce the global error.

4 INTEGRAL ERROR ESTIMATOR

The integral error estimator [9], and is given by:

$$||e_u||^2 = \int_{\Gamma i} (u^0 - \hat{u})^2 d\Gamma. \quad (3)$$

In the eqt. 3, e_u represents the calculated error, u^0 represents the value of the physical quantity calculated at a nodal point using a greater degree in the interpolator polynomial and $\hat{u}$ represents the value calculated using a smaller degree in the interpolator polynomial.

The basis of this error estimator is based on the greater precision in using a higher degree in interpolator polynomials. Although it requires a higher computational cost for its solution, the results obtained can be used as a parameter to compare the values calculated through lower degree interpolator polynomials, the error being the difference between the two results.

5 RECURSIVE ERROR ESTIMATOR

The recursive use of the integral eqt. 1 for numerical values determined for internal variables, adding new nodal points within the domain of a discretized geometry, can be interpreted as a new residual weighting process, thus expecting minimization of the errors obtained through the calculation using the method in its standard form, [10].The placement of the new nodal points is usually performed inside the domain, but in this work the nodal points will be placed on the contour.

The recursive error estimator is quoted in [9], and, in its simplified form, is given by:

$$e_u = T_{rec} - T_{int}. \quad (4)$$

In the eqt. 4, T_{int} is the temperature calculated using the recursive method applying the eqt. 1 at a new nodal point located in a different position from the existing nodes in a discretized geometry and T_{int} is the temperature calculated by interpolating the temperatures of the nodal points present in the element, as shown in the eqt. 5.

$$T_{int} = T_1\Phi_1 + T_2\Phi_2. \quad (5)$$

In the eqt. 5, Φ represents the shape function. Therefore, the error obtained through the eqt. 4 shows the difference between the application of the recursive method by adding new nodal points in the contour of the discretized geometry and applying the integral equation again. The interpolation of the variables is obtained by the standard method at this same point.

As shown in figure 2, a new nodal point is added at a different position from the physical nodes already existing in the geometry. T_1 and T_2 are nodal values obtained from the standard method.

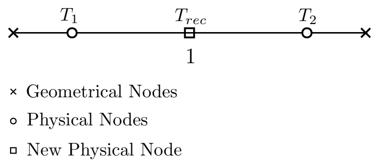
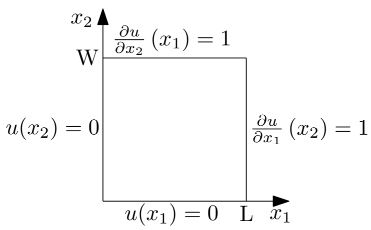


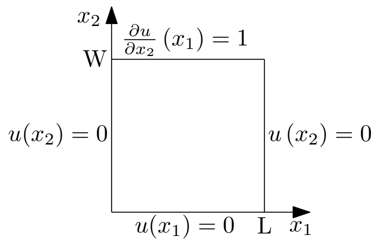
Figure 2. Schematic drawing of the recursive error estimator.

6 NUMERICAL EXPERIMENTS

In order to compare the results of the error estimators with the standard application of the BEM, a uniform refinement was carried out dividing each element into two elements for each iteration. Two case studies will be analysed, addressing heat conduction problems. The schematic drawing of the two problems is shown in the figure 3.



(a) Schematic drawing of the first case study.



(b) Schematic drawing of the second case study.

Figure 3. Schematic drawings of the case studies.

Each method generates a different discretization. Therefore, the analysis of the average error will be performed at 36 internal points so that the comparison is for the same points in all cases. These points are shown in the figure 4.

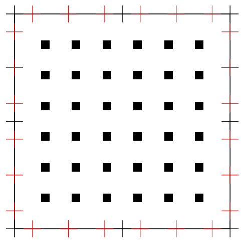


Figure 4. Schematic drawing of the internal points.

The analytical solutions to the problems presented in the figures 3a and 3b are given, respectively, by the eqt. 6 and 7, presented in [11]:

$$u(x_1,x_2) = x_1 + \sum_{n=1}^{\infty} \sin\frac{n\pi x_1}{2L}\left\{\left[\frac{2}{n\pi}\frac{(-1)^{n+1}+1}{\left(\frac{n\pi}{2L}\right)\cosh\left(\frac{n\pi W}{2L}\right)} + \frac{8L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\tanh\left(\frac{n\pi W}{2L}\right)\right]\sinh\left(\frac{n\pi x_2}{2L}\right) - \frac{8L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cosh\left(\frac{n\pi x_2}{2L}\right)\right\} \tag{6}$$

and

$$u(x_1,x_2) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\left(\frac{n\pi x_1}{L}\right)\frac{\sinh\left(\frac{n\pi x_2}{L}\right)}{\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi W}{L}\right)} \tag{7}$$

7 RESULTS

All error estimators are used as a refinement criteria. The elements whose error estimator was greater than the average of all elements were divided in half. This procedure was repeated iteratively in order to study the variation of the average error with the successive repetition of the refining process for the different error estimators.

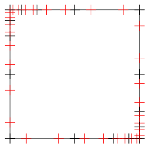
The calculation of the average error is given by:

$$e_{avg} = \frac{|u_{analytical} - u_{calculated}|}{max(u_{analytical})}, \tag{8}$$

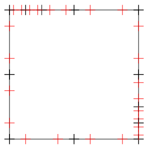
where, $u_{analytical}$ is the physical quantity calculated by the eqt. 6 or 7 and $u_{calculated}$ is the value calculated by the BEM. For all case studies, $W = 1$ and $L = 1$ will be considered.

7.1 RESULTS OF THE FIRST CASE STUDY

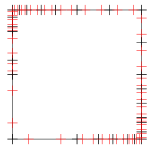
The geometries discretized from the refining process, for the third and fifth iteration, using the different error estimators are shown in the figure 5:



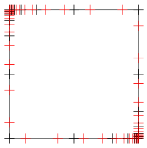
(a) Discontinuous estimator, third refining.



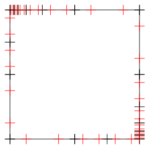
(b) Integral estimator, third refining.



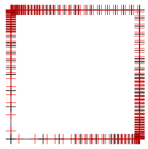
(c) Recursive estimator, third refining.



(d) Discontinuous estimator, fifth refining.



(e) Integral estimator, fifth refining..



(f) Recursive estimator, fifth refining.

Figure 5. Discretized geometries of the first case study.

For this case study, there is a flow discontinuity at the corners. Therefore, a greater error in the vicinity of the corner is expected and, thus, the error estimators concentrate the new nodal points in these locations, as shown in the figures 5a, 5b and 5c.

Through the recursive error estimator, compared to the other two error estimators, there is a greater increase in the number of nodal points used in the geometry discretization from the third to the fifth refining. This is because by calculating the error from the eqt. 4 more elements with errors above 0.5 are identified and refined. Therefore, as observed in the figure 5f elements further away from the corners are also refined whereas in the other two estimators this does not occur.

The comparison of the average error in the internal points between the error estimators using quadratic elements is shown in the figure 6, in which the variation of the average error with the increase of nodal points in the contour can be observed.

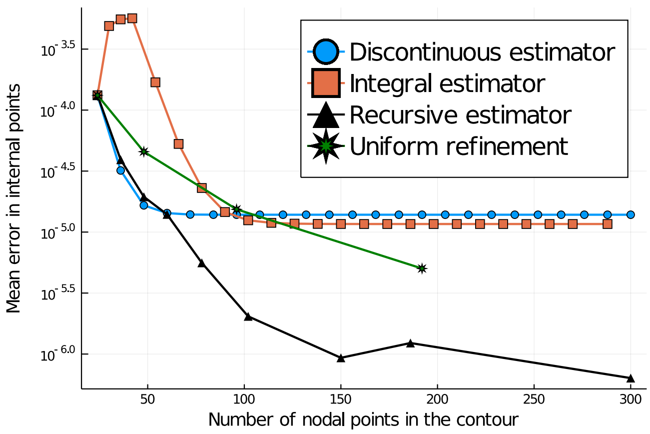
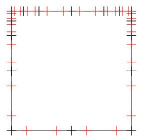


Figure 6. Variation of the average error in the internal points with the increase of the nodal points in the contour for quadratic elements.

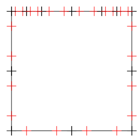
As shown in figure 6, the average error decreases faster for the discontinuous and recursive error estimators compared to the uniform refinement and the integral error estimator, observed in the first iterations. In addition, it is observed that the recursive error estimator was the most efficient in reducing the average error.

7.2 RESULTS OF THE SECOND CASE STUDY

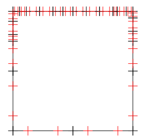
The geometries discretized from the refining process, for the third and fifth iteration, using the different error estimators are shown in the figure 7:



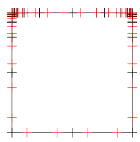
(a) Discontinuous estimator, third refining.



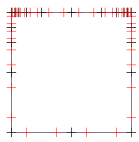
(b) Integral estimator, third refining.



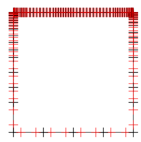
(c) Recursive estimator, third refining.



(d) Discontinuous estimator, fifth refining.



(e) Integral estimator, fifth refining.



(f) Recursive estimator, fifth refining.

Figure 7. Discretized geometries of the second case study.

In figure 7 a concentration of nodal points at the superior corners occurs for all estimators at each refinement is observed. The behavior of error estimators was similar to that observed in the first case study. The comparison of the average error calculated in the internal points, shown in the figure 4, between the error estimators using quadratic elements is shown in the figure 8, in which it can be observed that:

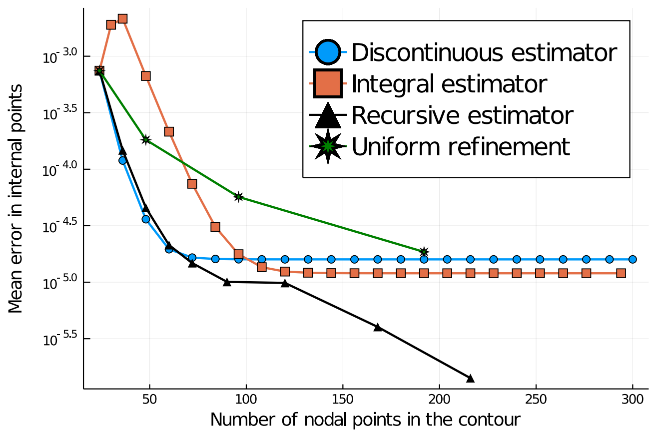


Figure 8. Variation of the average error in the internal points with the increase of the nodal points in the contour for quadratic elements.

The error behavior was similar to the first case, as shown in figure 8. The recursive error estimator was the most efficient in reducing the error and there was a sharp drop in the error in the first iterations for the recursive and discontinuous error estimators.

8 CONCLUSIONS

As seen in the figures 6 and 8 results from refining using the estimators reached an average error of 10^{-5} faster and stabilized while this was not seen in uniform refining.

In the problems analyzed, with a 50-node mesh, the errors obtained were in the order of 10^{-5}, which shows the rapid convergence of BEM for this class of problems. In a future work, problems with more complex geometry and boundary conditions could be analyzed to verify if the behavior of these estimators is maintained.

It can also be highlighted that among the analyzed estimators, the discontinuity has the lowest computational cost since it does not need to solve the problem with two different discretizations, as in the case of the integral estimator, nor apply the eqt. 5 at new points as in the recursive estimator and still obtained better results than the integral and equivalent recursive estimator in the analyzed cases.

Acknowledgements. I am grateful to the Federal University of Espírito Santo (UFES) for the scientific initiation scholarship provided through the integrated scientific initiation program.

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