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A Brief Performance Analysis of Direct Interpolation Technique applied on Bidimensional Advective-Diffusive Problems with Variable Velocity Fields

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Abstract. Research efforts directed to advective-diffusive equation have resurfaced, strongly motivated by the recent use of these mathematical models in pollutant dispersion applications, within the scope of environmental engineering. In the context of the Boundary Element Method (BEM), the treatment of the advective transport term characterizes a technical challenge. The BEM classic formulation, which uses the problem's inherent fundamental solution, is capable of handling high Peclet numbers phenomena, however it is limited to uniform velocity fields. In parallel, the Dual Reciprocity technique, which is more robust and versatile, offers flexibility in describing variable flow velocity fields, nevertheless it remains restricted to representing creeping flows. The recent Direct Interpolation technique provides a balanced alternative to the two previous approaches, as it is able to deal with variable velocity fields, such as Dual Reciprocity, despite that, it maintains stability up to moderate Peclet number. This article aims to expose a preliminary performance of the of Direct Interpolation technique comparing it to the well-established Dual Reciprocity approach, in physical situations with spatial variation of the velocity field, while the effects of advection are gradually increased. Focusing on determining the relative performance of the formulations, in terms of precision and stability, the numerical results are evaluated using benchmark problems with widely known analytical solutions.

Keywords: Boundary Element Method, Direct Interpolation technique, Dual Reciprocity technique, Advective-Diffusive Model, Moderate Peclet Number, Variable Velocity Fields.

1 Introduction

In the context of the Generalized Scalar Field Equation, [1] advection-diffusion problems are relevant for several engineering and science phenomena, with emphasis on applications related to fluid dynamics, thermal convection and mass transfer. In terms of computational thermofluids, the mathematical nature of the convective transport term added to the coupled structure of the governing equations, imposes a series of challenges on the performance of numerical methods, among them, the Boundary Element Method.

The advances consolidated by the Boundary Element Method in dealing with advection-diffusion models are substantial, since when its classic formulation, based on the fundamental solution related to the problem, [2] was proposed. This formulation treats properly physical situations that shows advection dominance[3], however it proves unable to represent variable velocity fields. First, it is necessary to highlight the first attempts to overcome this limitations of the method using variants of the Laplace fundamental solution [4] and also with a symmetrization proposal of the differential operator of the problem, [5] however, both did not obtain the expected success.

After the proposal of the dual reciprocity technique (DRBEM) [6], the boundary element method gains versatility and robustness of application in face of scalar field problems, in general. In the specific case of advective-diffusive problems, the technique is capable of dealing consistently with variable velocity fields, but it presents instability when the effects of advection become dominant, maintaining reasonable results only for small Peclet numbers. [7] [8]

Following the development towards the state of the art, the direct interpolation technique (DIBEM) [9] is proposed, based on an approximation of the entire kernel of the domain integral, unlike the dual reciprocity, which approximates only part of the kernel. The proposal of this technique, which resembles a classic interpolation procedure, has already been applied to relevant two-dimensional models of scalar field such as Poisson [9] and Helmholtz [10] problems, presenting superior results in comparison to dual reciprocity technique. In contrast to the dual reciprocity approach, there is no need to generate auxiliary matrices with DIBEM [6], however this new proposal is more sensitive to internal poles refinement. [11] [12]

More recently, a regularization procedure has been proposed [13] to be used simultaneously with the direct interpolation technique, and has presented satisfactory results on eigenvalue problems generated by Helmholtz Equation. The central concept is based on the elimination of the domain integral singularity through a simple algebraic sum similar to the regularization of Hadamard [14], which, in turn, allows an equality of coordinates between source, field and base interpolation points, significantly simplifying data entry and facilitating codification.

In this article, the performance of the DIBEM technique is tested on two-dimensional advective-diffusive problems with variable velocity field. The work aims to determine the robustness of the recent formulation to treat differential equations with variable coefficients and also to determine the stability of the technique in terms of the magnitude of the advective effects. The performance evaluation of DIBEM is based on comparisons with the classic DRBEM formulation and well-known analytical solutions.

2 Mathematical modelling

The mathematical description of an advective-diffusive problem with variable velocities is reasonably more complex than models which consider constant velocities, due to the accounting of the compressibility effects. According to the principles of continuum mechanics [15], in the case of a one-dimensional problem, compressive, in steady state, in a non-homogeneous medium and disregarding the viscous dissipation effects, the following governing equation is originated from the law of energy conservation [16].

$$\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = \rho c_p v \frac{du}{dx} + p\frac{dv}{dx} \tag{1}$$

In Eq. (1), the coefficient $k(x)$ represents a function of thermal conductivity in a non-homogeneous medium, ρ the specific mass of the fluid, v the velocity in the x axis, c_p a thermodynamic property of specific heat and p the flow's pressure field. The unidimensional velocity field has a linear structure, whereas the conductivity function possesses a mathematically convenient structure, designed to accomplish a transformation of variables [17], which is shown in sequence, such as those displayed by Eq. (2), where C is a constant of adjustment and m is a constant associated with the intensity of the advective effects.

$$v = -mx \qquad\qquad k = \rho c_p v x + \frac{\rho v m x^2}{C} \tag{2}$$

The use of an adequate transformation of variables [17] in terms of u', that is inserted in the expression of energy balance in Eq. (1) and after proper algebraic manipulation, as seen on Loeffler and Dan [20], it devises a transformed governing equation, displayed by (3), which holds a classic advective-diffusive structure.

$$\frac{d^2u'}{dx^2} = -\frac{m}{L+mx}\frac{du'}{dx} = v(x)\frac{du'}{dx} \tag{3}$$

The performance of the boundary element method is already known regarding some of its formulations for the classic advective-diffusive model, which is represented by Eq. (3). Massaro [21] has extensively tested, in a very meticulous work, the performance of the Dual Reciprocity formulation (DRM) in two-dimensional advective-diffusive problems. Dan *et al.* [17] tests in his work the performance of the DRM, Quasi-Dual (QDRM) [22] and also a formulation which is based on a harmonic transformation (HTT) in advective-diffusive problems with variable velocity fields. Pinheiro [23] applied the recent direct interpolation (DIBEM) technique in two-dimensional advective-diffusive problems with constant velocity fields. At last, there is still a demand for an evaluation of the behavior of the DIBEM formulation in cases where the velocity field is variable, which is the main motivation for this current article.

3 Direct Interpolation Method

In this section a brief formulation of the Direct Interpolation Method of Boundary Elements (DIBEM) for advective-diffusive problems with variable velocity fields is exposed, which includes the effects of compressibility. In order to achieve a more elegant and algebraically cleaner presentation, the governing Eq. (3) is written in index notation.

$$u_{,ii} = v_i u_{,i} \tag{4}$$

In Eq. (4), the left side represents the diffusive effects (DS), while the right side represents the accounting of the advection, where the velocity field v_i carries the scalar field u. The treatment via BEM on the diffusive side is already widely known in the literature [24] and its inverse integral formulation corresponds to the Eq. (5) below.

$$DS = \quad c(\xi)u(\xi) + \int_\Gamma u(X)q^*(\xi, X)d\Gamma - \int_\Gamma q(X)u^*(\xi, X)d\Gamma \tag{5}$$

The domain integral, which quantifies the advective part (AS) has, a priori, a more challenging treatment, whose mathematical structure is shown below. Each portion of the function in the integrands is kept explicit in order to promote a deeper consistency in exposition.

$$AS = \int_\Omega v_i(X)u_{,i}(X)u^*(\xi; X)d\Omega \tag{6}$$

The application of integration by parts in Eq. (6), in addition to the ensuing use of the Divergence Theorem [25] in order to move the integrals to the boundary $\Gamma(X)$ lead to the following equation.

$$AS = \int_\Gamma n_i(X)v_i(X)u(X)u^*(\xi; X)d\Gamma - \int_\Omega v_i(X)u^*_{,i}(\xi; X)u(X)d\Omega - \int_\Omega v_{i,i}(X)u^*(\xi; X)u(X)d\Omega \tag{7}$$

In (7), the first integral is already written in terms of the boundary $\Gamma(X)$, there still is, however, a domain integral where the velocity fields v_i and another one, which is also written in $\Omega(X)$ whose domain is composed by the divergent of the velocity fields $v_{i,i}$ that is connected to the compressibility effects of the flow [19]. At this point, similarly to the procedures used in an incompressive formulation whose details can be examined in [23], a regularization process is conducted in the second integral of Eq. (7), which is analogous to Hadamard's proposal [14]. This procedure already integrates the DIBEM's standard formulation and can also be viewed in previously stablished works [13].

$$\begin{aligned} AS = \int_\Gamma n_i(X)v_i(X)u(X)u^*(\xi; X)d\Gamma - \int_\Omega v_i(X)u^*_{,i}(\xi; X)\left[u(X) - u(\xi)\right]d\Omega \\ - \int_\Omega v_{i,i}(X)u^*(\xi; X)\left[u(X) - u(\xi)\right]d\Omega - \int_\Omega v_i(X)u^*_{,i}(\xi; X)u(\xi)d\Omega - \int_\Omega v_{i,i}(X)u^*(\xi; X)u(\xi)d\Omega \end{aligned} \tag{8}$$

As of now, a second integration by parts can be performed, this time in the last integral of Eq. (8), aiming to generate two integrals that are expressed in regards to $u(\xi)$. The first one in the right side of Eq. (9) can be taken to the boundary and the second one cancels itself with the last before one in the right side of Eq. (8).

$$\int_\Omega v_{i,i}(X)u^*(\xi; X)u(\xi)d\Omega = u(\xi)\int_\Omega \left[v_i(X)u^*(\xi; X)\right]_{,i}d\Omega - \int_\Omega v_i(X)u^*_{,i}(\xi; X)u(\xi)d\Omega \tag{9}$$

The mathematical structure of Eq. (9) reveals that both regularizations performed generate more than one boundary integral, which leaves the approximations of the regularized terms to the use of radial base functions. The final integral formulation on the advective side of a compressive case can be synthetized as in Eq. (10) below.

$$AS = \int_{\Gamma} n_i(X)v_i(X)u(X)u^*(\xi;X)d\Gamma - u(\xi)\int_{\Gamma} n_i(X)v_i(X)u(X)u^*(\xi;X)d\Gamma$$
$$+\int_{\Omega} v_i(X)u^*_{,i}(\xi;X)\left[u(X)-u(\xi)\right]d\Omega - \int_{\Omega} v_{i,i}(X)u^*(\xi;X)\left[u(X)-u(\xi)\right]d\Omega \quad (10)$$

4 Numerical Simulation

This case study aims to allow for a comparison between the performances of the DIBEM and DRBEM formulations in cases where the velocity field varies spatially. The selected problem is shown as follows with its respective boundary conditions. In order to test DIBEM's new formulation in compressible problems, a square domain with unitary measures was chosen, with the following boundary conditions and hydrodynamics imposed in Figure 1.

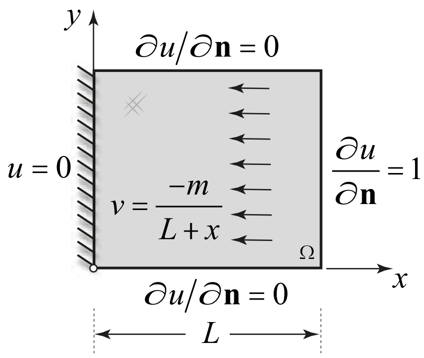

Figure 1. Computational Domain

In this particular case, although the geometry seems to be two-dimensional, given the structure of the boundary conditions, the problem behaves as a one-dimensional. The originally linear velocity field displayed on Eq. (2), it is developed into a field mathematically imposed by the transformation of variables, whose structure is also exposed in Eq. (3) and, for the sake of convenience, in the Figure 1 above.

The approach to the problem is carried out by the DIBEM formulation, which is compared against the well-known DRBEM formulation. First, it is shown below the convergence of the aforementioned formulations in regards to the number of boundary elements (BE) and internal poles (IP). In the left graph, 2 (a), fixa-se uma malha de contorno com 64BE e executa-se o refinamento , a boundary mesh with 64BE is fixed, whereas a gradual refinement of the internal points mesh is performed.

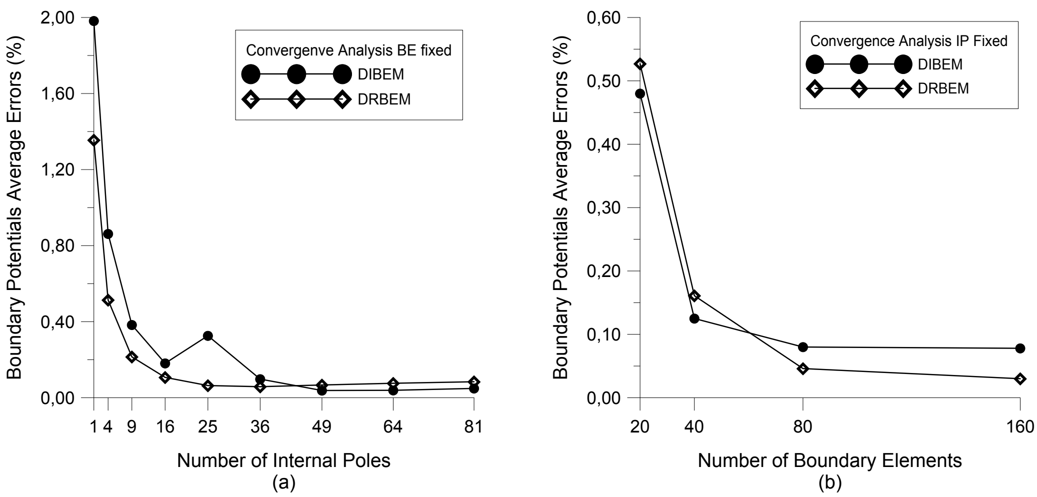

Figure 2. Sensitivity Analysis: (a) Internal Points (b) Boundary Elements

The Figure 2 (a) shows that the DRBEM performs better when fewer poles are used, which is reasonable considering that the DRBEM is less dependent on the amount of poles, due to the fact that only a part of the domain integral's core is interpolated by radial functions. Notwithstanding, the DIBEM by interpolating the entire core of the advective term, it requires a larger amount of poles.

The graph in Figure 2 (a) reveals that in fact both the boundary element formulations tested are, in fact, very sensitive to the refinement of the internal poles in the problem in question. It is worth highlighting that in advective-diffusive problems with constant velocity, the DRM method does not exhibit such fine sensitivity in regards to this mesh parameter, which in this case probably occurs due to the variation in the velocity field. In addition to it, the DIBEM formulation curve displays a peculiar uprising in a mesh with 25PI, which is probably due to an unfavorable spatial distribution of the internal nodes of this mesh. This type of behavior has already been observed in the solution of other cases, such as the calculation of natural frequencies in Helmholtz problems and it is linked to the existence of an optimal positioning of a given amount of internal points, which a priori cannot be quantified. In more refined meshes, the DIBEM demonstrates a slightly smaller boundary potentials average error than the DRBEM.

On the left side, Figure 2 (b) exposes the analysis of the second convergence parameter, which consists in the refinement of the boundary elements mesh, while maintaining the internal nodes mesh in 49IP. The graph reveals that the refinement of the amount of elements in the discretization of the boundary, which promotes a monotonic fall of the errors in more refined meshes in both formulations, with slight lower levels in DRBEM. Once confirmed the convergence of both BEM formulations tested for the problem in question, it is possible to evaluate the behavior of the same parameters in regards to the advective effects of the flow. For this purpose, Peclet's number is gradually increased, while the mesh is fixed on 80BE/81IP, the behavior of the error of the boundary calculated variables is then exhibited on the following Figure 4.

As seen on the solution of Poisson and Helmholtz problems [10], [9], obtaining good results on DIBEM stems from an equilibrium between the amount of boundary nodal points and the amount of poles. Whereas for the DRBEM, as already stated, it is less dependent on the internal interpolation, therefore its results tend to be superior in analysis which a lower amount of poles are fixed and the mesh is more refined.

The next simulations attempt to better examine DIBEM's pronounced sensitivity to the positioning of the internal poles. Considering the two distributions of internal interpolating points, as illustrated by Figure 3.

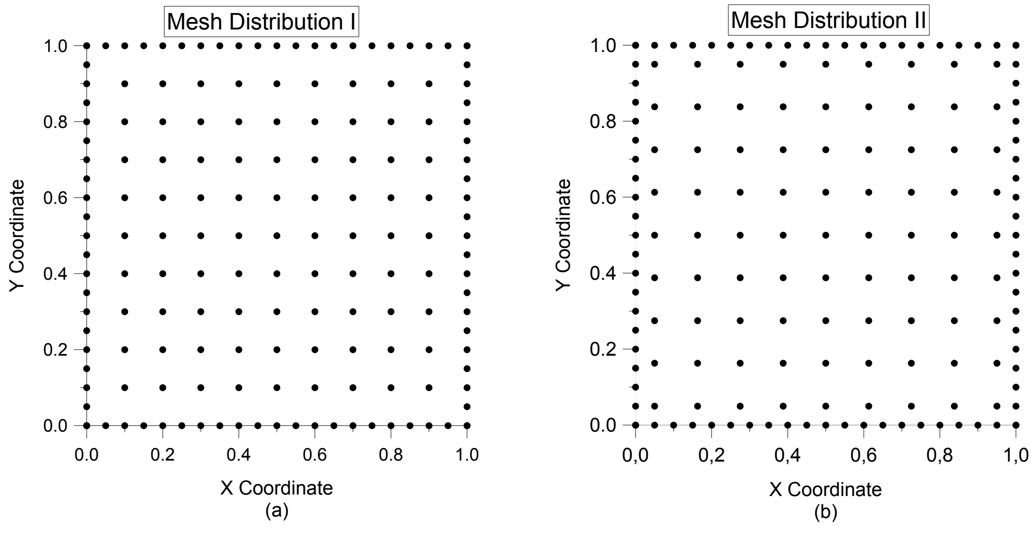

Figure 3. Distribution of Internal Points (a) Type I (b) Type II

In this new test, the parameter m is expanded from one to five and the results are exhibited in the following figure. It is observed that the curves, each one referring to a single mesh, display different performances. For $m = 5$, the configuration returned an error of 0.62%, while for the other one, it generated an error of $0,27\%$. Both error values are low, but the difference is still substantial, especially when considered that DRBEM's results are almost the same. The fact that mesh (b) performed better is given to the softer gradation of the mesh, where there is a higher density of internal interpolating points. A priori, the most indicated disposition of poles is the most uniform as possible, since there is no way to determine the exact optimal position of the poles.

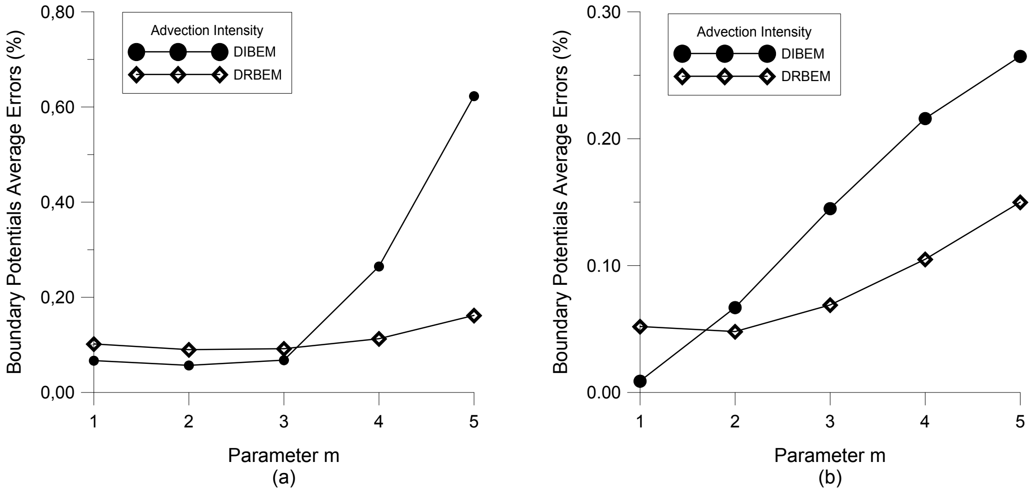

Figure 4. Advection Effects Parametric Analysis - 80BE/81IP

Figure 4 reveals low levels of error for both formulations tested, which can be interpreted as a reasonable analogous behavior. The precision of the DRBEM seems to be slightly superior in this problem when compared to the DIBEM formulation. In this regard, the direct interpolation is a good alternative when analyzing numerical compressive problems with low and moderated Peclet numbers.

5 Conclusions

In general, it is observed a good performance of both techniques tested using the proposed advective-diffusive model with compressive flow. The magnitude of the average errors were kept within a reasonable margin of most engineering applications, even for moderated Peclet numbers, which signals more prominent advection effects in comparison to the diffusive effects. As for the convergence in regards to the mesh parameters, a sensitivity to the internal poles is suggested, which is atypical to the DRBEM technique, given that this technique is not particularly influenced by this parameter in cases with uniform velocity fields. The local variation of the velocity field with space seems to influence the manner in which the Dual Reciprocity technique responds to the refinement of the internal poles of the mesh. Nevertheless, the DIBEM formulation exhibits great sensitivity to the refinement of the internal poles, which is characteristic of its performance in other scalar field problems. Still within the scope of convergence, no substantial response to the difference in the number of boundary elements was observed, both techniques behaved in a very similar manner to the refinement of the mesh.

In the parametric analysis regarding the Peclet number, the DRBEM technique displays slightly superior results, except for a single value, for which DIBEM exhibits better precision. This trend is distinct from the results observed in tests with constant velocity fields, in which the DIBEM performs better. It is worth highlighting that the results obtained from the DIBEM technique in this problem, they vary significantly in relation to the distribution of the internal poles, which requires a deeper and standardized investigation.

At last, it is possible to ascertain that the Direct Interpolation technique is an efficient and precise alternative in treating the proposed variable velocity field problem. The exposed integral formulation extends the scope of application of the DIBEM techniques. In addition to it, there is also a demand for more systemic tests in two and three-dimensional problems with other types of complexities associated.

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