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Interpolation methods using radial function kernels in non-orthogonal grids

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Abstract. In fluid dynamics, the numerical solution of time-dependent partial differential equations the values for the dependent variables are calculated at discrete points. When the grid used is structured and orthogonal, there are well-established methodology to perform the interpolation of the variable for any point of interest that does not coincide with the points of the grid. This is not simple when it comes to a grid that is unstructured and non-orthogonal. Non-orthogonality introduces a greater number of unknowns than equations, in this case it is necessary to balance the number of unknowns with the number of equations and interpolate the information for the remainder points that will be grouped as source terms. In this work, several bases of radial functions of global and compact support interpolation were tested in orthogonal and non-orthogonal grids. The numerical results were compared with those obtained by analytical solution and good agreement was found. Results are organized in terms of precision and the computational effort required.

Keywords: interpolation, radial function, non-orthogonal grids.

1 Introduction

In fluid dynamics, computer simulations are used to obtain values of interest at discrete points of a mesh or grid. In some situations, it may be desirable to obtain interpolated values at points other than those predefined points in the discretization process. An example of this is the time-dependent iterative methods in non-orthogonal mesh/grids, where non-orthogonality introduces a greater number of unknowns than equations. In this case, interpolation is used to balance the number of unknowns with the number of equations, as in the model shown by J.T.A. Chacaltana, A.M. Frasson, C.F. Loeffler, and A. Bulcão to perform the propagation of acoustic waves in a heterogeneous medium [1].

2 Interpolation

Interpolation is a technique used to estimate the value of the known variable at a given point using the discrete set of known data points. There are several interpolation methods, each classified according to its behavior and precision in different specific application situations. When dealing with computational simulations in the time domain, the use of some interpolation methods can result in a high computational effort, depending on the complexity of the calculation algorithms involved in the method. In this work, we will compare three interpolation methods; the hydrodynamic smoothing particle interpolation (known as SPH), the radial base function interpolation, and the inverse distance weighting interpolation (known as IDW).

2.1 SPH interpolation

This method, developed by M.B. Liu and G.R. Liu [2], defines the approximate value $< f(x) >$ as shown in eq. (1),

$$< f(x) > = \int_{\Omega} f(x')W(x - x', h)dx', \tag{1}$$

where x is the position vector, $W(x - x', h)$ is the weighting function, h is the smoothing length and Ω is the compact support, delimited by the smoothing length. It can be discretized as eq. (2) shows,

$$< f(x) > = \sum_{j=1}^{N} f(x_j)W(x - x_j, h)A_j, \tag{2}$$

where N is the set of known data points contained on the compact support and A_j is the area of the element centered on x_j. The weighting function used on eq. (2) is given by eq. (3). The weighting function is chosen so that its integral in the domain is equal to one.

$$W(x - x', h) = \frac{1}{2}\left(e^{\frac{(x-x\prime)^2}{2\pi(\frac{h}{5})^2}}\right)\left(1 - e^{\frac{5^2}{2}}\right). \tag{3}$$

2.2 Radial basis function interpolation

Developed by Wong, Hon e Golberg (2002) [3], this method can reproduce an unknown function through a known set of N-data points, as shown in eq. (4),

$$< f(x) >= \sum_{j=1}^{N} \alpha_j f(x - x_j), \tag{4}$$

the coefficients α_j are obtained using the known set of N-data points, as shown in eq. (5),

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{bmatrix} W(x_1 - x_1) & \cdots & W(x_1 - x_N) \\ \vdots & \ddots & \vdots \\ W(x_N - x_1) & \cdots & W(x_N - x_N) \end{bmatrix}^{-1} \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix}. \tag{5}$$

The matrix on eq. (5) is symmetric and its main diagonal is null. Since the rows of the matrix are linear independent, the Wronskian is not null, implying that the matrix is inversible. The radial function used on this study is presented in eq. (6),

$$W(r) = \begin{cases} r^2 \ln(r), & r > 0 \\ 0, & r = 0 \end{cases} \tag{6}$$

where $r = |x_n - x_m|$.

2.3 Inverse distance weighting

Proposed by Shepard (1968) [4], this method is simple to implement and is defined in eq. (7),

$$< f(x) >= \sum_{j=1}^{N} \frac{f(x_j)\dfrac{1}{|x - x_j|^p}}{\sum_{i=1}^{N} \dfrac{1}{|x - x_j|^p}}. \tag{7}$$

Choosing a value greater than 1 for the power parameter (p) means that the weight for points further located will decrease faster. In this work, we use the value of 1 for the power parameter. Only neighboring elements that have the interpolated point as the common vertex were used on this application.

3 Grids

Preliminary tests were carried out on a structured grid (grid type 1, fig. 1a), on an unstructured grid with uniform elements (grid type 2, fig. 2b), and on an unstructured grid with non-uniform elements (grid type 3, Fig. 1c).

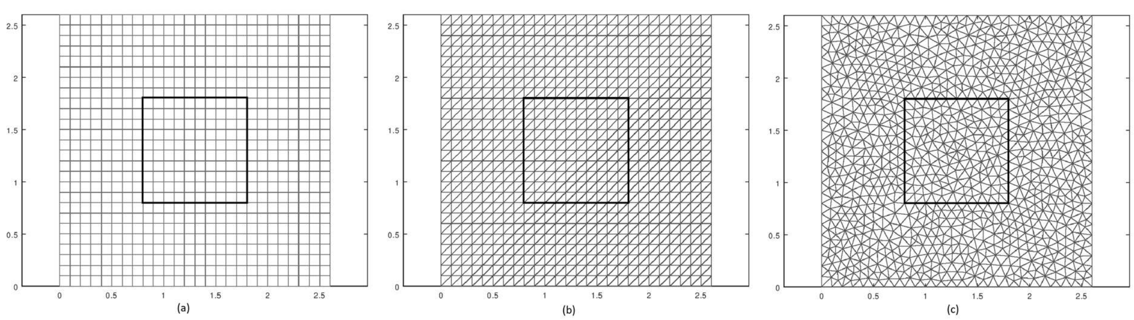

Figure 1. Grids with edges dx=0,1: structured (a), non-structured with uniform elements (b) and non-structured with non-uniform elements (c).

Type 1 and 2 grids were generated in OCTAVE. The type 3 grid was generated in the software GMSH 4.2.2 and were imported into the OCTAVE environment. The differential size of the dx edges is the edge size of elements in the structured grid. In the unstructured grid with uniform elements, it indicates the size of the cathetus of the elements. In the unstructured grid with non-uniform elements it indicates the seeded value for GMSH that defines the size of the edges in the boundary, whereas, for the rest of the domain, the length of the edges oscillates close to this value. To analyze the influence of the grid density effect on the results, nine grids of each type were tested with edges ranging from 0.02 to 0.10. The affective domain to perform the interpolation is a unit-side square, as highlighted in the middle of the grid in fig. 1. Since the SPH interpolation methods use data from points contained in a radius around the interpolated point, the construction of the grid is large enough that the smoothing length of any point within the unit square does not exceed the limits. Frontier treatment techniques will not be discussed in this study. The longest smoothing length tested was h = 8dx, so the total grid size was defined as $(1 + 16dx) \times (1 + 16dx)$.

4 Interpolated functions

Three analytical solutions were selected and applied in the domain of grids to be used as known data for interpolation. Figure 2 shows these functions exemplified in a type 3 grid with $dx = 0{,}05$. The analytical solutions are represented by the eqs. (8), (9), and (10).

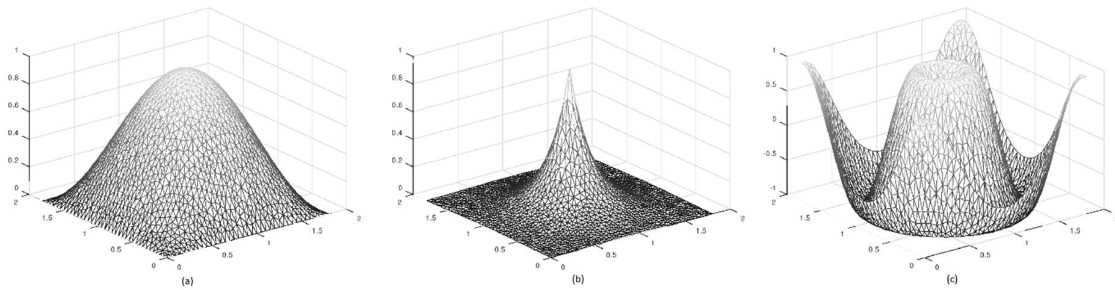

Figure 2. Function 1 with eq. (8) (a), function 2 with eq. (9) (b) and function 3 with eq. (10) (c).

$$f(x) = \sin\left(\frac{\pi x_v}{L}\right)\sin\left(\frac{\pi y_v}{L}\right) \tag{8}$$

$$f(x) = \sin[2\pi(x - x_c)] \tag{9}$$

$$f(x) = \frac{1}{e^{5(x-x_c)}} \tag{10}$$

The position vector is indicated by x, while x_v and y_v are the components of this vector on x and y axis. The position vector of the center of the grid is represented by x_c. The vertexes of the grid were used as unknown points to be interpolated, while the center of each element was used as known data points. To obtain the interpolation error, the interpolated values were compared to those obtained at the same points using the equations presented above.

5 Results

M.B. Liu and G.R. Liu [2] defines the normalization condition of the weighting function (eq. (11)) as a necessary requirement,

$$\int W(x - x', h)\, dx' = 1, \tag{11}$$

Equation 12 is the discrete form of eq. (11),

$$\sum_{j=1}^{N} W(x - x_j, h) A_j \cong 1. \tag{12}$$

To see if normalization of the weighting function is being satisfied, converging to 1, we show in fig. 3 the absolute value of the difference between 1 and the normalization condition. The normalization condition obtained is the average for each point in the valid domain. The error bar indicates the standard deviation of the normalization condition between grids of the same type with different dx.

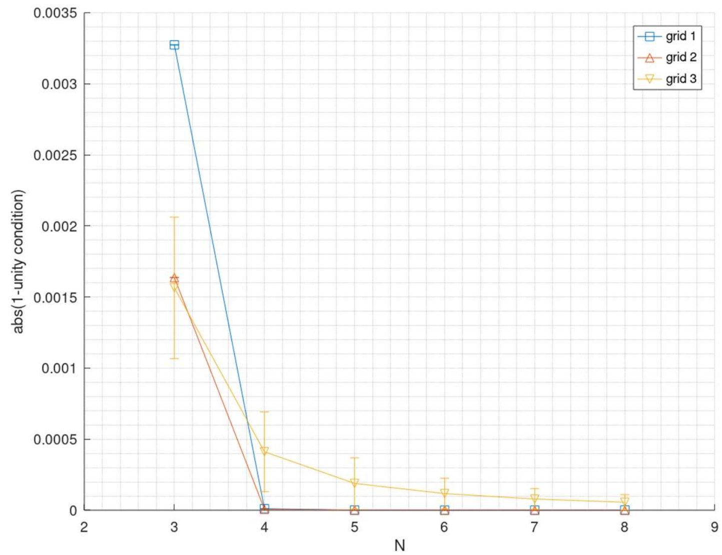

Figure 3. Absolute value of the difference between 1 and the unity condition.

The x axis labeled as N indicates the scalar value used to define the smoothing length ($h = Ndx$). Both the structured (grid 1) and the unstructured with uniform elements (grid 2) converged with the increase in the smoothing length and presented optimal results for h bigger than $4dx$. The unstructured grid with non-uniform elements (grid 3) converged, but resulted on a bigger deviation from 1 than the others two grids even when using $h = 8dx$. Error bars are visible only in grid 3 data points, indicating that convergence in non-orthogonal grids is greatly influenced by change in dx. Figure 4 shows the behavior of the normalization condition for $h = 8dx$, with the x axis showing the dx.

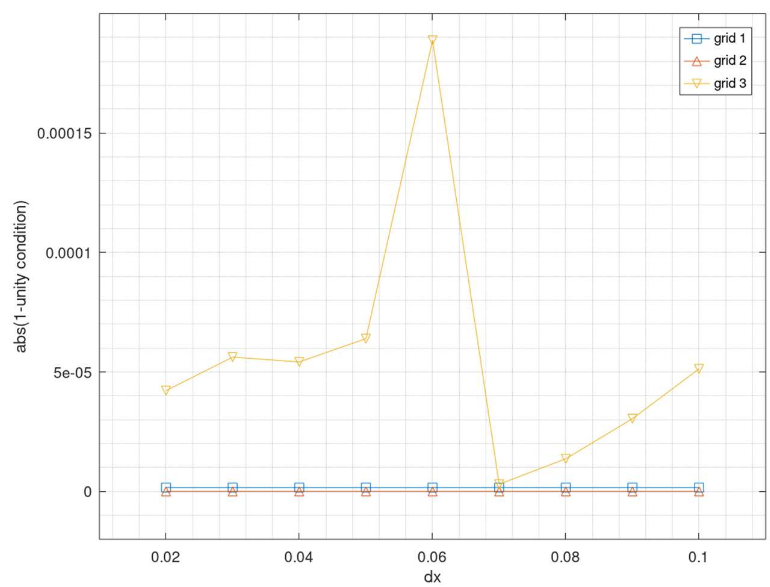

Figure 4. Absolute value of the difference between 1 and the normalization condition for h=8dx.

This influence occurs on every smoothing length tested, as shown in fig. 5,

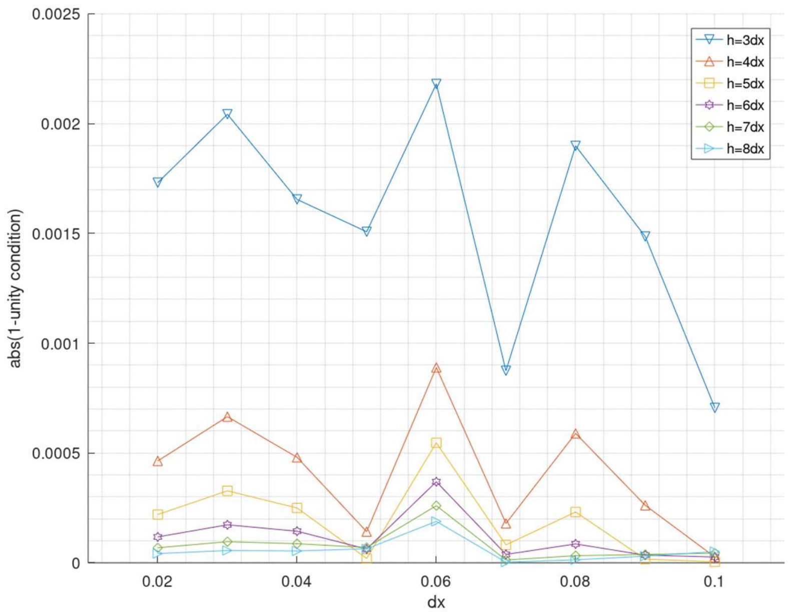

Figure 5. Absolute value of the difference between 1 and the normalization condition for non-structured grids with non-uniform elements.

Figure 6 shows how the absolute error of the tested functions behaves with the increase in the smoothing length $h = Ndx$ in non-orthogonal grids.

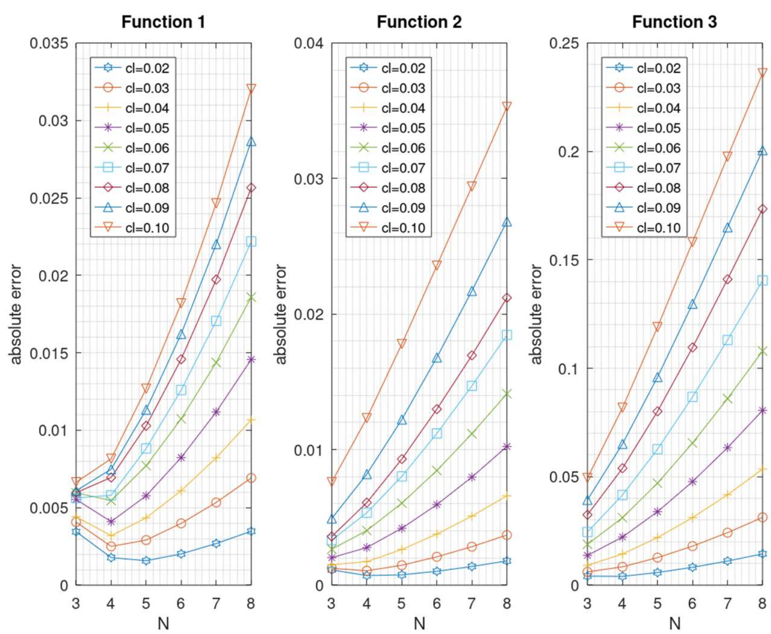

Figure 6. Absolute error for non-structured grids with non-uniform elements.

Although the increase of the smoothing length results in a better convergence to the normalization condition, in interpolation the error increases when data farther from the interpolation vertex is considered (i.e. the smoothing length increases). In all tested case, better error results were found using smoothing length equals to $3dx, 4dx$ or $5dx$. Figure 7 shows the best error obtained using SPH interpolation with the error obtained using radial basis function interpolation and inverse distance weighting (IDW) on the unstructured grid with non-uniform elements.

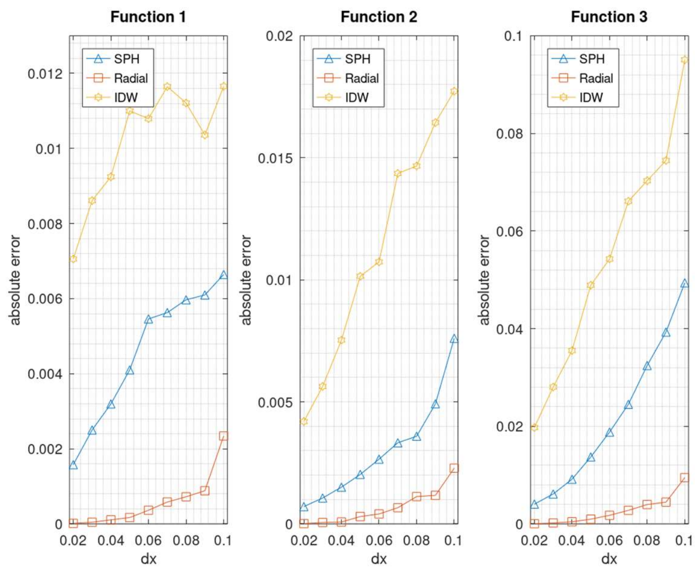

Figure 7. Absolute error for the three interpolation methods tested.

In the interpolation using the radial base function, only the elements in the valid domain were considered as known data. For every dx and interpolated function tested, the better results were obtained using the radial base function interpolation. IDW gave the bigger error in all cases.

6 Conclusions

The radial basis function interpolation method showed the best results for the functions tested in this study. Since it was implemented using only elements on the valid domain, there is no need for boundary treatment in this method, as it would be necessary using SPH. On the downside, it requires more computational effort because it involves the inversion of matrix. Its application in a grid with many elements could even be inviable without any division of the domain, due to the size of the matrix that would need to be inverted. Future tests will be done to compare ways to divide the domain, comparing the error obtained and the computational effort between them. IDW, while easy to implement and as a low cost of computational effort, presented the worst results. Taking this into account, it can still be used in a situation where the computational effort must be minimized, and the error range fits the data range.

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