XLI CILAMCE
XLI Ibero-Latin American Congress on Computational Methods in Engineering
16-19 November, 2020 FOZ DO IGUAÇU PARANÁ - BRAZIL

# Acoustic wave propagation in non-homogeneous media

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**Abstract.** The propagation of acoustic waves is a subject of great interest in seismic prospecting for oil, where the medium of wave propagation is not homogeneous. In this work, a linear model for the acoustic P-wave propagation in a non-homogeneous media is developed. Starting from the nonlinear equations that governs the fluid motion of a compressible fluid with a point mass source we obtain the linear equation for the acoustic P-wave propagation in heterogeneous medium. This second order partial differential equation is solved numerically by the explicit finite difference method. Two types of boundary conditions were implemented in the numerical model. One of this is pure reflection and the other is of the type "free radiation". The numerical method is second order in the time and space. The time step satisfies the Courant-Friedrichs-Levy (CFL) restriction. A numerical code is written in Fortran language and simulations with a source term of the type "Mexican hat" were performed. The numerical results are compared with those found in the literature and a good agreement was obtained.

**Keywords:** Acoustic Wave, Time Domain, Unbounded Domain, Numerical Stability Condition, Ricker-pressure source.

## 1 Introduction

The linear equation for the propagation of P-wave field generated by an acoustic source in the heterogeneous fluid environment can be deduced from the compressible Navier-Stokes equation [1]. As pointed out by [2] "an adequate description of the acoustic environment is not usually available to feed into the model". We derived this equation in the ocean-acoustic environment and we show below that the a priori information involve two physical properties of the fluid, the density and the adiabatic bulk modulus structures. And, due to the complexity of this environment the wave-P speed is non-uniform in depth.

The P-wave propagation in heterogeneous media has received great attention for the simulation of seismic wavefields in complex media [3], [4], [5] and to avoid side reflections and wraparound longer than the range of times involved in the modeling a different types of absorbing boundary conditions have been proposed [6], [7].

In this work, the second-order P-wave equation has been solved using explicit finite difference scheme that is of second order in time and space. Also, we development of numerical propagation code for making accurate predictions of the P-wave field and we apply the code to possible environmental conditions, frequencies of interest in the applications.

## 2 The acoustic wave equation for non-homogeneous media

In this section, we deduce the acoustic wave equation to study the propagation of a P-wave in a compressible fluid. The starting equations obey the laws of conservation of mass, momentum and energy, and are given respectively by eq. (1), (2), and (3).

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0\,. \tag{1}$$

$$\rho\frac{D\vec{u}}{Dt} + \nabla P = \rho\vec{g} + \nabla \cdot \tau\,. \tag{2}$$

$$\rho T\frac{DS}{Dt} = -\nabla \cdot \vec{q} + \sigma : \nabla\vec{u}\,. \tag{3}$$

where $D()/Dt \ = \partial()/\partial t + \vec{u} \cdot \nabla()$ is the material derivative operator and $\nabla$ is the Nabla or vector differential operator. The scalar physical variables $\rho, P, S,$ and $T$ are specific mass, pressure, entropy, and temperature, respectively. And, the vector variables $\vec{u}, \vec{g}$, and $\vec{q}$ represent the speed, acceleration due to gravity and the flux of heat (Fourier Law $\vec{q} = \lambda\nabla T$). For a Newtonian fluid, the stress tensor is $\sigma = -pI + \tau$ and taking Stokes' hypothesis into account the deviatoric tensor [8] is given by

$$\tau = \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\nabla \cdot \vec{u}I\right). \tag{4}$$

the coefficient $\lambda$ and $\mu$ are fluid proprieties, respectively, thermal conductivity and dynamic viscosity.

Considering as typical scales of time $1/f$ , where $f$ represent the frequency, and pressure $E$, which represents the compressibility coefficient, and as the fluid the water, it is easy to show that the only dominant forces in the Navier-Stokes equations are the inertial and pressure forces. In addition, if the flow is adiabatic and reversible, an isentropic process, there is no heat production. And, equations (1), (2) and (3) are reduced to.

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{u} = 0\,. \tag{5}$$

$$\rho\frac{D\vec{u}}{Dt} + \nabla P = 0\,. \tag{6}$$

Equations (5) and (6) are not a closed system of equations. Thus, a state equation that relates pressure to specific density is added for closure.

$$\frac{D\rho}{Dt} = \frac{1}{c^2}\frac{DP}{Dt}. \tag{7}$$

Using eq. (7) in eq. (5) and after some arrangement, we have

$$\frac{DP}{Dt} = -\rho c^2\nabla \cdot \vec{u}\,. \tag{8}$$

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla P\,. \tag{9}$$

This set of equation can be solved by any numerical method, like SPH, see Liu [9]. In this work we focus our attention on the linear form of this closed set of equation, then $D()/Dt \ = \partial()/\partial t$. The linear form of eq. (8) and eq. (9) can be combined to give

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{\partial P}{\partial t}\right) = \nabla \cdot \left(\frac{1}{\rho}\nabla P\right). \tag{10}$$

The speed of the wave depends on the various properties of the medium, using the relation $E = \rho c^2$ we can rewrite eq. (10) as

$$\frac{\partial^2 P}{\partial t^2} = E\nabla \cdot \left(\frac{c^2}{E}\nabla P\right). \tag{11}$$

Equation (11) is the linear equation of the acoustic wave used to carry out the propagation of the P-wave in non-homogeneous media. This equation appears in several areas of science, such as geophysics. As we can see quickly, this equation is the classic wave equation for homogeneous media. It is straightforward to include a pressure source term, in two dimensions and with the source term, eq. (11) takes the following form

$$\frac{1}{E}\frac{\partial^2 P}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{c^2}{E}\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{c^2}{E}\frac{\partial P}{\partial y}\right) + \ddot{P}_f. \tag{12}$$

where the source term is defined inside the domain at a specific point $(x_f, y_f)$.

$$\ddot{P}_f(x_f, y_f) = \frac{\partial^2 P_f}{\partial t^2}\,. \tag{13}$$

### 2.1 Boundary conditions

Two types of boundary conditions for the P-wave are implemented at the open boundary. The first type is the reflection condition given by the Neumann boundary condition, eq. (14).

$$\frac{\partial P}{\partial n} = \vec{n} \cdot \nabla P = 0\,. \tag{14}$$

where $\vec{n}$ is the outward normal unit vector.

At the boundary we consider the homogeneous equation with no source term.

$$\frac{\partial^2 P}{\partial t^2} = c^2 \frac{\partial^2 P}{\partial x^2} + c^2 \frac{\partial^2 P}{\partial y^2}\,. \tag{15}$$

The second type of boundary condition is that of free radiation. Following the work done by Israeli and Orszag [10] we applied the radiation boundary condition for the 1D wave equation at and for the 2D wave equation at $\mathrm{x} = \pm X$, given by eq. (16) and eq. (17).

$$\frac{\partial P}{\partial \mathrm{t}} \pm c \frac{\partial P}{\partial x} = 0\,. \tag{16}$$

$$\frac{\partial^2 P}{\partial t^2} \pm c \frac{\partial^2 P}{\partial \mathrm{t}\, \partial \mathrm{x}} - \frac{c^2}{2} \frac{\partial^2 P}{\partial y^2} = 0\,. \tag{17}$$

Then, the set of equations (12), (14), (16) and (17) was used to simulate the P-wave propagation generated by a pressure source in a non-homogeneous medium.

# 3 The numerical scheme

The set of partial differential equations is placed in its discrete form using the explicit finite difference scheme in an evenly spaced grid, $\Delta x$ on the $X$-axis and $\Delta y$ on the $Y$-axis. Capital letters ($I$,$J$) are used to identify the grid points where the variable $P$ is located, otherwise indicate the midpoint between two successive $P$-grid points where the values of variable $\Gamma$ must be specified, see figure (1) below.

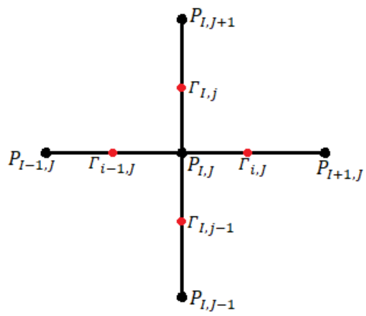


Figure 1. Schematic grid representation of unknown Pressure (the $P$-points) and know information at middle grid points (the $\Gamma$-points).

The finite difference explicit form of eq. (12) is obtained by applying the second-order central difference for time and space [11] resulting in an algebraic equation to calculate the unknown.

$$P_{I,J}^{n+1} = 2P_{I,J}^{n} - P_{I,J}^{n-1} + E_{I,J}\left\{\left(\frac{\Delta t^2 \Gamma_{i,J}}{\Delta x^2}\left(P_{I+1,J}^{n} - P_{I,J}^{n}\right) - \frac{\Delta t^2 \Gamma_{i-1,J}}{\Delta x^2}\left(P_{I,J}^{n} - P_{I-1,J}^{n}\right)\right) + \left(\frac{\Delta t^2 \Gamma_{I,j}}{\Delta y^2}\left(P_{I,J+1}^{n} - P_{I,J}^{n}\right) - \frac{\Delta t^2 \Gamma_{I,j-1}}{\Delta y^2}\left(P_{I,J}^{n} - P_{I,J-1}^{n}\right)\right) + \ddot{P}_f\left(I_f, J_f\right)\Delta t^2\right\}. \tag{18}$$

In eq. (18), $n+1$ indicates the instant of time when the unknowns are being calculated and $\Gamma = \frac{c^2}{E}$. The stability criterion to perform the simulation is given by the relation below that establishes the size of the time step according to the grid information.

$$\Delta t = C_r \frac{\Delta x\, \Delta y}{\Delta s\, c_{max}}. \tag{19}$$

where $c_{max} = \max(c_1, c_2)$ and $C_r \leq 1$, and $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$

The discrete form of eq. (14), eq. (16) and eq. (17) for the wave propagation along the positive X-axis is given by eq. (20), eq. (21) and eq. (22), respectively.

$$P_{nx,J}^{n+1} = 2P_{nx,J}^{n} - P_{nx,J}^{n-1} + 2\left(\frac{c\Delta t}{\Delta x}\right)^2 \left(P_{nx-1,J}^{n} - P_{nx,J}^{n}\right). \tag{20}$$

$$P_{nx,J}^{n+1} = P_{nx,J}^{n} - \frac{c\Delta t}{\Delta x}\left(P_{nx,J}^{n} - P_{nx-1,J}^{n}\right). \tag{21}$$

$$P_{nx,J}^{n+1} = 2P_{nx,J}^{n} - P_{nx,J}^{n-1} - \frac{c\Delta t}{\Delta x}\left(P_{nx,J}^{n} - P_{nx,J}^{n-1} - P_{nx-1,J}^{n} - P_{nx-1,J}^{n-1}\right) + \frac{\Delta t^2}{2\Delta y^2 c^2}\left(P_{nx,J+1}^{n} - 2P_{nx,J}^{n} + P_{nx,J-1}^{n}\right). \tag{22}$$

# 4 Test Cases

The numerical P-wave propagation code was written in the Fortran language. We perform 3 type of tests; in the first test, we explore the reflection boundary condition for the propagation of P-wave field generated by a Ricker-type pressure source, eq. (23), located in the middle of the domain. The purpose of this test is to verify the full symmetry of the results as the wave reflects several times at the rigid boundaries. In the second test, we explore the radiation 1D and 2D boundary conditions for the propagation of P-wave field generated by the sinusoidal pressure source, eq. (24), located out of the middle of the domain [12]. The third test is related to the series of experiments reported in Santos and Figueiró [13].

$$\ddot{P}_f(I_f, J_f) = \left[1 - 2\pi(\pi f_p t)^2\right] e^{-(\pi f_p t)^2}. \tag{23}$$

$$\ddot{P}_f(I_f, J_f) = A\, sin(2\pi f_p t). \tag{24}$$

In the first 2 tests, we considered a homogeneous medium with a compressibility modulus of $E = 2.25 \times 10^9 Nm^{-2}$ and P-wave velocity of $c = 2000\ ms^{-1}$.

In test 3, the six seismic models with increasing degrees of complexity proposed by Santos and Figueiró [13] were considered. The $M_I$ model is of two homogeneous layers separated by a horizontal interface, the $M_{II}$ model is of six homogeneous layers separated by horizontal interfaces, the $M_{III}$ is composed of lateral and vertical variations, the $M_{IV}$ is composed of four layers and has an intrusion, the $M_V$ is three layers with a normal failure, and the $M_{VI}$ model is three layers with one being a basin.

Table 1 shows the values of the physical parameters used in the numerical simulations for the 3 tests performed. For the third case, only the P-wave velocity values for the different media were considered.

Table 1. Parameters used in the three tests for the propagation of the P-wave generated by a pressure source.

| Parameters | Test 1 | Test 2 | Test 3 |
|---|---|---|---|
| $f_p$ | 30 Hz | 30 Hz | 80 Hz |
| Delay | 0.0333 s | 0.0 s | 0.0125 s |
| $L_x$ | 500.0 m | 500.0 m | 1100.0 m |
| $L_y$ | 500.0 m | 500.0 m | 600.0 m |
| $\Delta x = \Delta y$ | 1.0 m | 1.0 m | 1.0 m |
| $N_x$ | 501 | 501 | 1101 |
| $N_y$ | 501 | 501 | 601 |
| $x_f$ | 250.0 m | 0 m | 550.0 m |
| $y_f$ | 250.0 m | 0 m | 2.0 m |
| $t_{max}$ | 0.8325 s | 0.8325 s | 0.5 s |
| $c_{max}$ | 2000 m/s | 2000 m/s | 4000 m/s |
| $c_{min}$ | 2000 m/s | 2000 m/s | 2000 m/s |

# 5 Results

The results for the first test are shown in figure 2. As we can see in fig. 2.a and fig. 2.b, there is a preservation of symmetry in the distribution of the P-wave field throughout the simulation period. With pressure ranges decreasing over time.

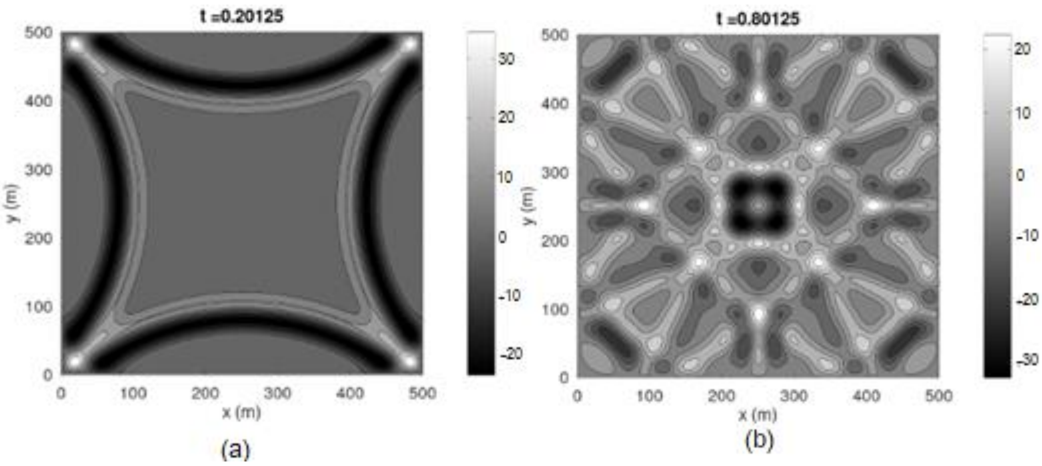


Figure 2. The 30 Hz time-domain solution of the P-wave amplitudes in a homogeneous medium with full reflection boundary condition.

Figure (3) shows the results of the propagation of a P-wave generated by a source of sinusoidal pressure. In fig. 3.a we show the result of the 1D wave absorbing boundary condition (ABC) and in fig. 3.b the result of the absorbing boundary condition for a 2D wave. Concentric pressure circles with center at (0,0) are the expected result for this second test. Thus, a substantial gain can be seen quickly when using the 2D-ABC in relation to the 1D-ABC. The circular shape of the pressure isolines is very evident in fig. 3b than in fig. 3.a, in the latter, certain undulations can be seen over the main circle, which is an indication that there is a greater reflection of the P-wave at the open boundary.

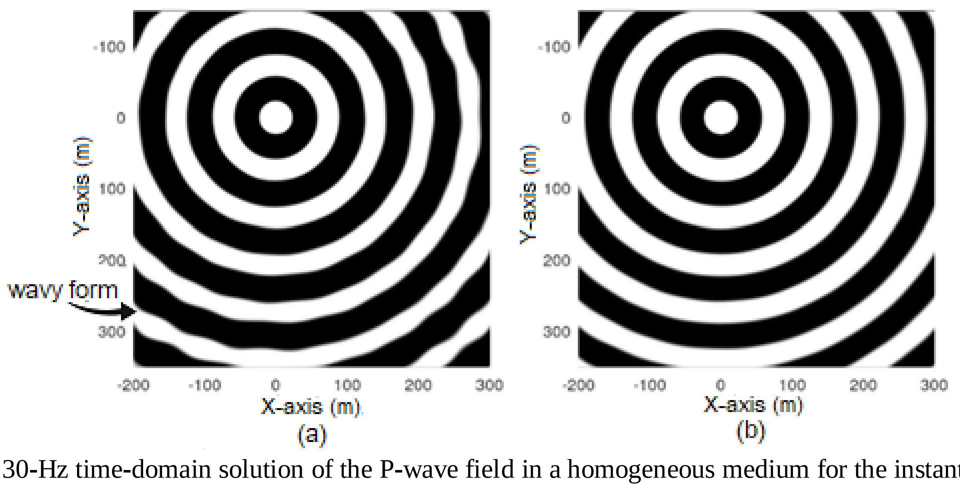


Figure 3. The 30-Hz time-domain solution of the P-wave field in a homogeneous medium for the instant of time $t = 0.8325\ s$. (a) for ABC-1D, and (b) for ABC-2D.

The time domain seismograms for the 6 seismic models of the third test are shown in fig. 4. In the seismogram of fig. 4.a it can be seen that the P-wave generated by the pressure source in the upper layer takes t = 0.2875 s to cover the distance of 550 m with a constant speed of 2000 m/s. Which is almost the same time it takes the reflected wave, through the horizontal interface, to reach the position of the pressure source.

In fig. 4.b we can see that there is a wave reflected by each horizontal interface. The first reflected wave should reach the position of the pressure source at time t = 0.1125 s. and the second wave reflected at time t = 0.1925 s. This can be readily seen in fig. 4.b.

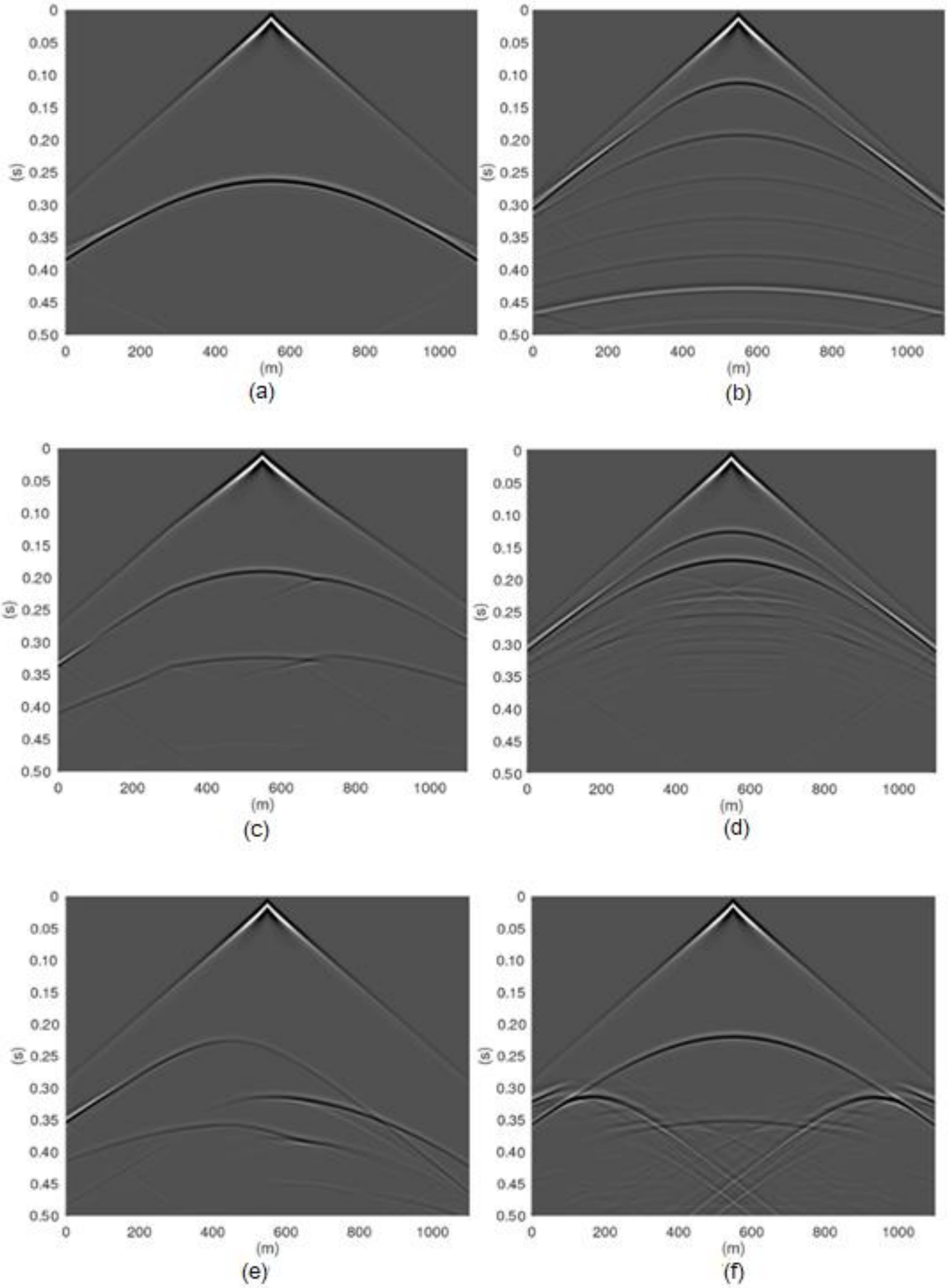

Figure 4. Synthetic seismograms referring to the six seismic models. (a) $M_I$, (b) $M_{II}$, (c) $M_{III}$, (d) $M_{IV}$, (e) $M_V$, and (f) $M_{VI}$

In fig. 4.c we can observe the main shape of two reflected waves and that on these two forms two other reflected wave information can be observed. The main shape of the two reflected waves is due to the fact that the domain was divided into three layers with the speed of the P-wave increasing from top to bottom. And, the other two reflected waves that are found on the main reflected waves are due to the fact that there was another division of the domain with the speed increasing from left to right.

The seismograms shown in fig. 4.d-f capture the main characteristics of the waves reflected by the medium. In particular, the presence of the geological basin present in the seismogram of fig. 4.f

## 6 Conclusions

The numerical model of the linear equation that describes the propagation of the P-wave generated by a pressure source in a non-homogeneous medium was satisfactorily implemented using the finite difference method in the time domain. In principle, the absorbing boundary condition for a plane wave in two dimensions was satisfactory to capture the main characteristics of the medium. Scarce literature was found where two of the three physical variables related by $E = \rho c^2$ are simultaneously reported.

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