Unifying the Basic Phenomenology of Monotonic Unidimensional Plasticity

Luiz Claudio Oliveira[1], Raquel Santini Leandro Rade[2]

[1]*Dept. of Mechatronics Eng, CEFET-MG – Federal Center for Technological Education from Minas Gerais*
R. Álvares de Azevedo, 400, b. Bela Vista, Divinópolis, 35503-822, Minas Gerais, Brazil
luizco@cefetmg.br, luiz_claudio_o@hotmail.com
[2]*Faculty of Mechanical Eng, UFU - Federal University of Uberlândia*
Rodovia BR-050, km 78 - Bloco 1DCG, Campus do Glória,38410-337, Minas Gerais, Brazil
rslr@ufu.br

Abstract. Stress and strain quantities and related characteristic subdomains of a common monotonic stress-strain diagram are studied. “Component quantities” are explicitly defined, encompassing many components which, although found throughout the literature —ε_e, ε_p, σ–σ_Y, ε–ε_Y—, are not usually comprehensively explored together from one single source. Those quantities are used to form planes ($\sigma_{component}$, $\varepsilon_{component}$) defining characteristic subdomains, which contain an exclusive type of curve to be linearized. An environment is thus constructed, offering a broad context of subdomains where linearized relations can be worked out. As a specific application, using such components and a logarithmic linearization it is possible to unify the unidimensional monotonic solicitation in tension/compression from one single source of equations, comprising classical formulations — Ramberg-Osgood, Hollomon, Swift, Ludwik— and other ones not widely knwon nor used. Numerically (or algebraically), there are two independent sets of elastic plus plastic equations relating the quantities σ, $\sigma_{ep}=\sigma-\sigma_Y$, ε, $\varepsilon_{ep}=\varepsilon-\varepsilon_Y$, $\varepsilon_p=\varepsilon-\varepsilon_e=\varepsilon-(\sigma/E)$, plus accumulated quantities σ_{acc} and ε_{acc} from a previous plastic deformation. Diverse formulations —e.g. logarithmic and/or polynomial ones— can be specified for each subdomain or model. Such final models will comprise series of computer implementable, adjoined numerical expressions. Notice that with this comprehensive context of variables and equations some small inconsistencies in the basic formulation of unidimensional plasticity are revealed, which are qualitatively important for the due description of the phenomenon of plasticity, specially when developping into a more complex equationing.

Keywords: computational mechanics, stress-strain relations, basic phenomenology.

1 Introduction

The tension/compression stress-strain diagram is of fundamental importance as a starting point for a phenomenological macroscopic characterization of mechanical properties. Quantification of the exclusively non-linear curve is usually sought through the simplest means for linearization, aiming simple but accurate basic formulations for describing the relation $\sigma\times\varepsilon$. Such formulations and the diagram typical quantities —e.g. E, σ_Y, $\sigma_{FAILURE}$— make up a phenomenological description of the most basic mechanical behavior of a given material.

Standards for tests resulting the stress-strain curve are found in NBR6152 (ABNT (1992)) or in A370 (ASTM (1993)). For many types of the nonlinearities to be found, Ramberg and Osgood (1943) gave the foundation for a relatively simple, accurate formulation for their description. Their formulation has been retaken by Lemaitre and Chaboche (1985). Lopes and Al-Qureshi (1990) describe the formulations of Swift and Hollomon and show diagrams analogous to some of those used here. Kleemola and Nieminen (1974) compare the formulations of Hollomon, Swift, Ludwik, of close interest here, and also that of Voce. Adams and Beese (1974) divide a simple tensile stress-strain curve into three parts, elastic, non-linear plastic and linear plastic, and suggest a formulation for the non-linear plastic part analogous to one of those developed in this text.

Dieter (1988) discusses the basic points of stress-strain diagrams and some widespread formulations. An account of the details present in the test results and usually neglected in descriptions, specially for unloading-reloading, is given by Dieter (1981). Wulff and alii (1965) present mechanical properties as to their relation to the stress-strain diagram. Helman and Cetlin (1983) is a brief introduction to the stress-strain diagrams and relations used in metal forming. A brief overview of the relations among the phenomenological measured mechanical properties and the material structure can be found in Hertzberg (1983). The systematic study of the $\sigma\times\varepsilon$ diagram had its climax a long time ago, as shown by the date of those references. Nevertheless, some fundamental points of concern have never been dully accounted, what is an important basic lack when developing more evolved formulations. Remark that the separate formulations referred to above have never been worked into a broader general description for the diagram, with consistent reference of quantities among formulations (as performed now), resulting in the predominance of those specialized autonomous formulations —e.g. Ramberg-Osgood, power curve, Swift—.

2 Component quantities and the linearization of the relation $\varepsilon_P(\sigma)$

Assuming the independence of elastic and plastic phenomena:

$$\varepsilon = \varepsilon_e + \varepsilon_p$$

where: $\varepsilon_e = \varepsilon_e(\sigma), \forall\sigma$; $\varepsilon_p = 0,\ \sigma \le \sigma_{E|P},\ \varepsilon_p = \varepsilon_p(\sigma),\ \sigma \ge \sigma_{E|P}$.

2.1 Decomposition of the total quantities ε and σ

In order to retain just an *exclusively non-linear* behavior curve in the plastic domain, let's adopt the following procedure: from the *total quantities*, ε and σ, it will be taken away either a variable quantity, the (*linear) elastic component quantity* ε_e, or some fixed quantities, the parameters $\varepsilon_{E|P}$ and $\sigma_{E|P}$. Then there will be left either a *nonlinear, plastic component quantity,* ε_p, or *nonlinear, elastoplastic component quantities*, ε_{ep} and σ_{ep}, as shown in the sequence. First, working with the deformation component quantities, let's define:

$\varepsilon_{E|P} = \varepsilon_e(\sigma{=}\sigma_{E|P}) = \sigma_{E|P} / E$; then:

$$\varepsilon_p = \varepsilon - \varepsilon_e = \varepsilon - (\sigma / E), \quad \forall\sigma,\ \forall\varepsilon; \tag{1}$$

$$\varepsilon_{ep} = \varepsilon - \varepsilon_{E|P}, \quad \varepsilon \ge \varepsilon_{E|P}. \tag{2}$$

In a similar way, the following stress component quantity can be defined:

$$\sigma_{ep} = \sigma - \sigma_{E|P},\ \sigma \ge \sigma_{E|P}\ . \tag{3}$$

$\varepsilon_{E|P}$ is the maximum deformation attained in the elastic domain; it is directly related to $\sigma_{E|P}$ (defined previously, item 2). ε_{ep} is an *elastoplastic component of deformation*, obtained taking away the elastic limit deformation $\varepsilon_{E|P}$ from the total deformation ε in the range $\varepsilon \ge \varepsilon_{E|P}$. An *elastoplastic component of stress* σ_{ep} has been obtained in an analogous algebraic way to ε_{ep}, thus the same index notation and nomenclature. ε_e and ε_p are exactly the commonly used elastic and plastic deformations.

2.2 Logarithmic linearization

Experimental data worked out into planes ($\ln(\sigma-\sigma_{taken})$, $\ln(\varepsilon-\varepsilon_{taken})$) will be described by the relation::

$$\ln(\sigma-\sigma_{taken}) = \ln K + (1/M)\ln(\varepsilon-\varepsilon_{taken}), \qquad \sigma > \sigma_{taken},\ \varepsilon > \varepsilon_{taken} \tag{4}$$

where ε_{taken} is one of the component quantities ε_e or $\varepsilon_{E|P}$ and σ_{taken} is $\sigma_{E|P}$, quantities "taken away" as in (1) to (3)

The equation above can be rearranged to obtain the *general forms of the plastic relation* $\varepsilon_p(\sigma)$ *and of its "inverse"* $\sigma(\varepsilon_p)$:

$$\varepsilon_{reduc} = (\sigma_{reduc} / K)^M, \qquad \sigma_{reduc} = K\,\varepsilon_{reduc}^{1/M}$$

where: $\sigma_{reduc} = \sigma - \sigma_{taken}$, $\sigma_{taken} = \sigma_{E|P}$ and $\varepsilon_{reduc} = \varepsilon - \varepsilon_{taken}$, $\varepsilon_{taken} = \varepsilon_e$ or $\varepsilon_{E|P}$; $\sigma \ge \sigma_{E|P}$, $\varepsilon \ge \varepsilon_{E|P}$.

2.3 Reintroducing total quantities

The previously defined quantities can be enhanced into a so-called set of “constitutive” quantities, which other than the “reduced” components (item 3.2) will also include the total stress and total strain. Altogether those constitutive quantities are: σ, σ_{ep}, ε, ε_{ep} and ε_p. These constitutive quantities will be used to form planes (σ_c, ε_c), where either σ_c or ε_c at least must be a “reduced” component, in order to get an exclusively non-linear curve. The exclusively non-linear curves in those planes are then linearized, using the most suitable means (logarithmic linearization, polynomials, etc.). Stress-strain diagrams with such constitutive quantities are shown in Fig. 1.

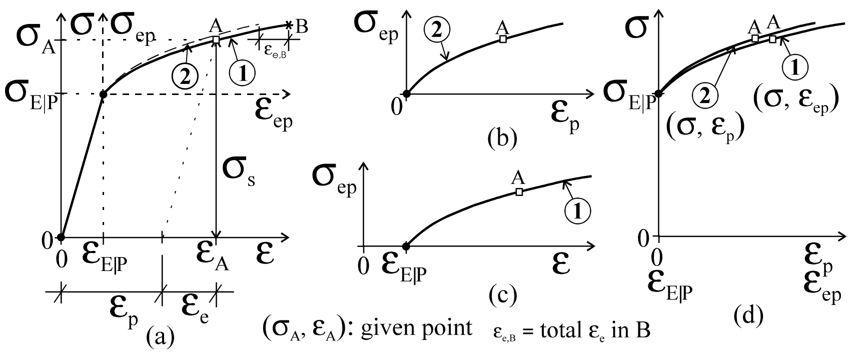

Fig.1: Elastic-elastoplastic behavior: curve ①: $\varepsilon_e+\varepsilon_p$; curve ②: ε_p (translated to $\varepsilon_{E|P}$ in (a)). Characteristic subdomains: a) (σ, ε) and (σ_{ep}, ε_{ep}), b) (σ_{ep}, ε_p), c) (σ_{ep}, ε), d) (σ, ε_{ep}) and (σ, ε_p).

3 Stress/strain plastic behavior models

Model 1 (Strain Model) *gives the total axial uniform deformation* ε *and its elastic and plastic components* ε_e *and* ε_p *to be found in an elastic-elastoplastic solid with isotropic hardening, initially isotropic, under a given unidimensional monotonic uniform tensile/compressive loading* σ*:*

for $|\sigma| \le \sigma_{E|P}$, $\quad \varepsilon = \varepsilon_e = (|\sigma| / E) \, \mathrm{Sgl}(\sigma)$, $\quad \varepsilon_p = 0$; (5)(a)

for $|\sigma| \ge \sigma_{E|P}$, $\quad \varepsilon = \varepsilon_e + \varepsilon_p = \left(\frac{P_1}{K}\right)^M \mathrm{Sgl}(\sigma) + A_1 \, \mathrm{Sgl}(\sigma)$. (5)(b)

Separately, the elastic and plastic components in eqn. (5)(b) are given by:

$\varepsilon_e = (|\sigma| / E) \, \mathrm{Sgl}(\sigma)$, $\quad \varepsilon_p = \left(\frac{P_1}{K}\right)^M \mathrm{Sgl}(\sigma) + A_1 \, \mathrm{Sgl}(\sigma) - (|\sigma| / E) \, \mathrm{Sgl}(\sigma)$. (5)(c)

Furthermore: for $|\sigma| \ge \sigma_{E|P}$, $\sigma = \sigma_s$,
that is, σ is the plastic limit σ_s itself for $|\sigma| \ge \sigma_{E|P}$ (σ and σ_s are readily explicitly given in Model 2).

Model 2 (Stress Model) *analougously:*

for $|\varepsilon| \le \varepsilon_{E|P}$, $\quad \sigma = (E \, |\varepsilon|) \, \mathrm{Sgl}(\varepsilon)$; (6)(a)
for $|\varepsilon| \ge \varepsilon_{E|P}$, $\quad \sigma = K \, (P_2)^{1/M} \, \mathrm{Sgl}(\varepsilon) + A_2 \, \mathrm{Sgl}(\varepsilon)$. (6)(b)

Furthermore: for $|\varepsilon| \ge \varepsilon_{E|P}$, $\quad \sigma = \sigma_s$, (6)(c)

that is, σ is the plastic limit σ_s itself for $|\varepsilon| \ge \varepsilon_{E|P}$. ($\sigma_s$ is not defined for $|\varepsilon| \le \varepsilon_{E|P}$)

Remarks: InTable 1, ($\sigma_{E|P}$, $\varepsilon_{E|P}$) $\equiv$ (σ_Y, ε_Y), where σ_Y is the common notation for the onset of plastic deformation (i.e., the “yield limit”); $\sigma_{E|P}$ and $\varepsilon_{E|P}$ are always taken as positive (same value for tension and compression). Notes: †: in order to form planes ($\ln(\sigma_c)$, $\ln(\varepsilon_c)$); *: exactly the same equations, just changing nomenclature; **: with ε for ε_p ($\varepsilon_p=\varepsilon-\sigma/E$), see item 5.2.

Table 1: Algebraic expressions for P_i and A_i in eqns. (5)(b) and (6)(b).

ALGEBRAIC EXPRESSIONS FOR P_i AND A_i IN THE STRAIN AND STRESS BEHAVIOR MODELS						
set #	σ_c, ε_c QUANTITIES USED†	STRAIN MODEL $\varepsilon=\varepsilon(\sigma)$, eqn. (5)(b)		STRESS MODEL $\sigma=\sigma(\varepsilon)$, eqn. (6)(b)		some similar formulations (developed basically as a relation of the type:)
		P_1 ($\lvert\sigma\rvert$-$\lvert\sigma_{taken}\rvert$)	A_1 ($\lvert\varepsilon_{taken}\rvert$)	P_2 ($\lvert\varepsilon\rvert$–$\lvert\varepsilon_{taken}\rvert$)	A_2 ($\lvert\sigma_{taken}\rvert$)	
i	$\sigma-\sigma_{E\|P}$, $\varepsilon-\varepsilon_{E\|P}$	$\lvert\sigma\rvert - \sigma_{E\|P}$	$\varepsilon_{E\|P}$	$\lvert\varepsilon\rvert - \varepsilon_{E\|P}$	$\sigma_{E\|P}$	
ii	$\sigma-\sigma_{E\|P}$, $\varepsilon-\sigma/E$	$\lvert\sigma\rvert - \sigma_{E\|P}$	$\lvert\sigma\rvert / E$	$\lvert\varepsilon\rvert - \lvert\sigma\rvert/E$	$\sigma_{E\|P}$	Ramberg-Osgood* ($\varepsilon=\varepsilon(\sigma)$); Ludwick* ($\sigma=\sigma(\varepsilon)$)
iii	σ, $\varepsilon-\varepsilon_{E\|P}$	$\lvert\sigma\rvert$	$\varepsilon_{E\|P}$	$\lvert\varepsilon\rvert - \varepsilon_{E\|P}$	0	
iv	σ, $\varepsilon-\sigma/E$	$\lvert\sigma\rvert$	$\lvert\sigma\rvert / E$	$\lvert\varepsilon\rvert - \lvert\sigma\rvert/E$	0	Hollomon or simple power curve** ($\sigma=\sigma(\varepsilon)$)
v	$\sigma-\sigma_{E\|P}$, ε	$\lvert\sigma\rvert - \sigma_{E\|P}$	0	$\lvert\varepsilon\rvert$	$\sigma_{E\|P}$	

4 Formula compactation

First of all, as a point of reference, eqns. (5)(a) and (5)(b) using the quantities of set (ii) are exactly a Ramberg-Osgood equation (as given in Lemaitre and Chaboche (1985)). Adjusting nomenclatures: $K = K_y$, $M = M_y$, $\sigma_{E|P} = \sigma_y$, and rewriting, it's obtained:

$$\varepsilon = \varepsilon_e + \varepsilon_p , \qquad \forall\sigma , \qquad 11 \qquad (7)$$

$$\text{where:}\quad \varepsilon_e = \sigma / E , \quad \varepsilon_p = \left\langle \frac{\sigma - \sigma_{E|P}}{K} \right\rangle^M \text{Sgl}(\sigma), \quad \text{and} \quad \langle f(x)\rangle = \begin{cases} 0, & f(x)\le 0 \\ f(x), & f(x)>0 . \end{cases}$$

Equation (7) is what could be called a "monolithic-compound" (or false monolithic) formulation, in opposition to equations (5)(a) to (6)(b) which make up formulations split into cases. For such monolithic formulation, models 1 and 2 have to be rewritten as a "single" formula, in a similar way to eq. (7) (but not exactly alike):

***Compact writing** of eqns. (5)(a), (5)(b), (6)(a), (6)(b) from models 1 and 2 as a single formula, analogous to eq. (7):*

$$\upsilon = \upsilon_{elast} + \upsilon_{elastopl} , \qquad 12 \qquad (8)$$

where: $\upsilon_{elast} = \langle\, e\,|\chi|\, \rangle_{\chi_{E|P}}\ \text{Sgl}(\chi)$,

$\upsilon_{elastopl} = {}_{\chi_{E|P}}\langle\, k\, p^m\, \text{Sgl}(\chi) + a\, \text{Sgl}(\chi)\, \rangle$,

a, e, k, m, p, υ, χ are given in Figure 3,

$$\langle f(x)\rangle_{x_A} = \begin{cases} f(x), & f(x)\le x_A \\ 0, & f(x)>x_A \end{cases} \qquad \text{and} \qquad {}_{x_A}\langle f(x)\rangle = \begin{cases} 0, & f(x)\le x_A \\ f(x), & f(x)>x_A . \end{cases}$$

Equation (8) can be inserted in a fluxogram for the stress-strain diagram characterization and simulation, as exemplified in Fig. 3. Figure 3 describes the use of eq. (8). Some points in this figure are of interest to notice.

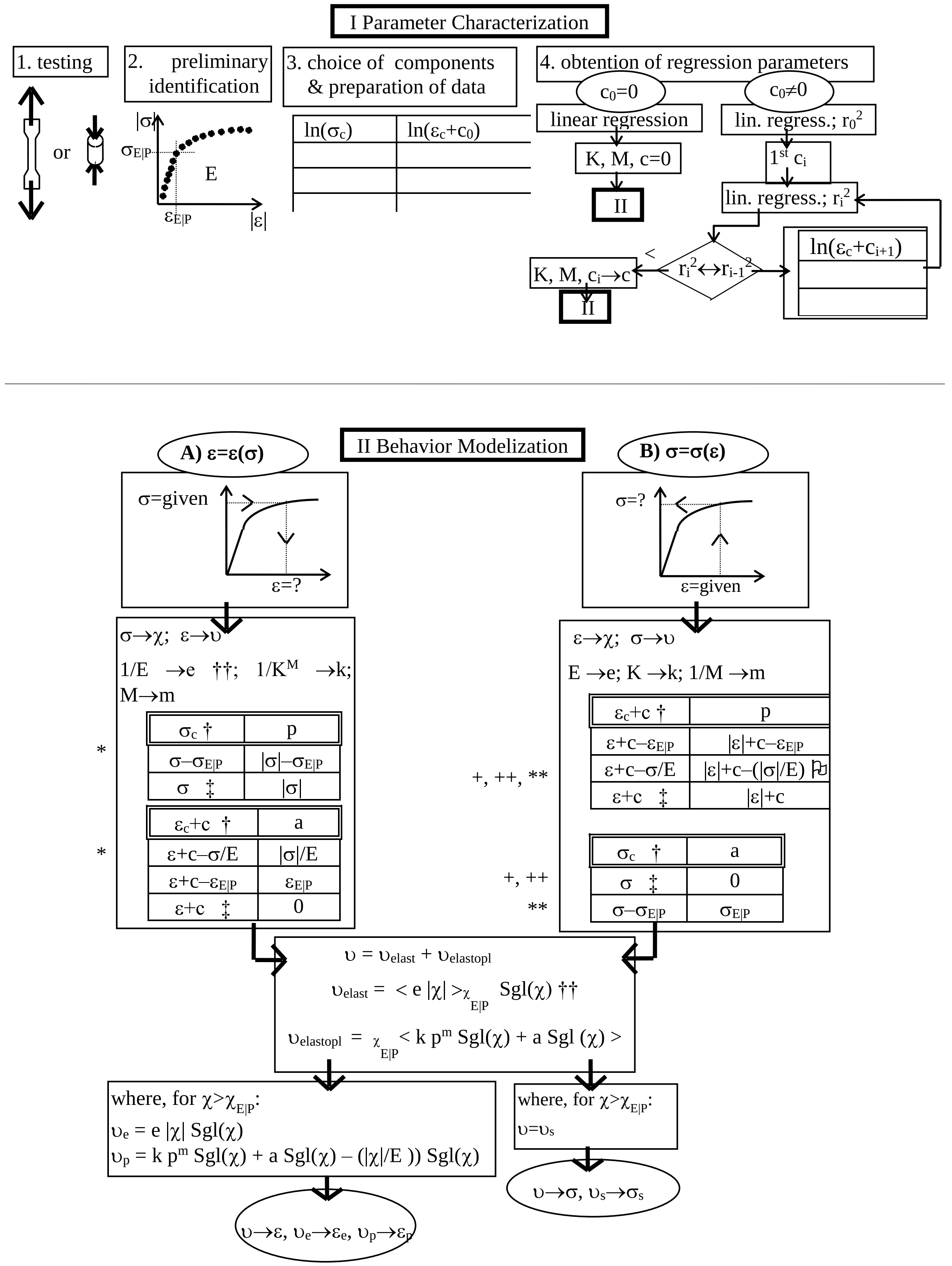

Fig. 3: Schematic activities for parameter characterization and behavior modelization of a simple monotonic stress-strain curve using component quantities and the compact notation of eq. (8).

5 Conclusion

The essence of this work is the systematic gathering and study of stress-strain diagram variables and their components inside an environment of worked out composed, component quantities and related characteristic subdomains. The resulting models are made of juxtaposed or adjoined expressions, in the form of algebraic or numeric, independent equations. The key point is the environment, which is itself quite simple and straightforward, defined in a couple of lines (relations (1) to (3)). Almost all the formulas that follow are the application of this environment to use logarithmic linearizations. Those formulas show that the environment is able of consistently setting forth the classical logarithmic formulations, and further ones not so famous nor tried out, all of that from a single source (models 1 and 2 as schematized in Fig. 3). Notice also that the planes so defined form a sequence of finite domains. In a very specific sense, those finite domains are a means of "discretization" of the problem domain, into its distinctive type of curves (from two curves in Fig. 1 to five in Fig. 4), analogous to using a plate element linked to truss elements. Nevertheless, such analogy applies only in a general sense. Remark that the formulation of each plane ("domain") can also be regarded as "inclusive", that is, it includes the previous domain, or better saying its domain, not in formlution In another view, notice that the characteristic subdomains so defined form a sequence of inter-imbeded subdomains, which are a means for the "discretization" of the problem domain into its distinct types of curves, from two curves in Fig. 1 to five curves in Fig. 4. The formulation of each domain in this case is independent but starts at the final point of the previous formulation, which is the link between them.

As an application, the logarithm linearization-based equationing developed explores comprehensively the use of the component quantities. In this way, variables and relations of common use (power curve, Hamberg-Osgood) are unified and expanded into a broader context. It is specifically used one algebraic relation, eqn. (4), a logarithmic linearization. The general form of eqn. (4), the component quantities defined in eqns. (1), (2) and (3), and Hooke's law give forth eqns. (5)(a) to (6)(c). The latter equations by their turn comprise in fact five sets of elastic plus plastic stress-strain relations, using the parameters in Table 1.

The environment itself has been accomplished starting with the use of the parameter $\sigma_{E|P}$ to define a new variable, σ_{ep}, in terms of σ and $\sigma_{E|P}$ ($\sigma_{ep}=\sigma-\sigma_{E|P}$). The next step is the identification and definition of the deformation variable ε_{ep}, similar to σ_{ep}. The elastoplastic components σ_{ep} and ε_{ep} together are to be identified and visualized as a subdomain in the stress-strain diagram comprising solely the nonlinear portion of the stress-strain curve. All sets of constitutive quantities basically follow this scheme: they form planes where just an exclusively type of curve appears, in this case exclusively non-linear curves. Since it is not possible to write down an equation which is at th same time linear and non-linear (which is the problem in stabllishing one single equation for stress-strain diagrams), those planes are a better approach, since for each separate domain more accurate formulations can be developed.

A matter of concern in using some of the formulations from the bibliography is the use of curve (3) of Fig. 2 as accounting for a previous plastic deformation, what is a conceptual mistake but in the very specific case of a material without an elastic domain. This conceptual mmistake may have unforseen effects when using this basic equationing as part of a more comprehensive formulation.

Those expedients are not new taken separately. They can be found one of a type at each one of several places in the literature. Also the nomenclature and symbology are for the most borrowings from here and there. Nevertheless, the use of these expedients all at once, that is, the explicit reference and use of separate variables for the component quantities systematically and the comprehensive identification of the roles of those variables and of their respective subdomains, that is not easily found in the literature.

REFERENCES

Adams, C., Beese, J.G. (1974), Empirical Equations for Describing the Strain-Hardening Characteristics of Metals Subjected to Moderate Strains. Trans. ASME: Journ. of Eng. Materials and Technology 96, 123-126.

American Society for Testing and Materials (ASTM) (1993), Method number A370-93: Mechanical Testing of Steel Products, ASTM.

Associação Brasileira de Normas Técnicas (ABNT) (1992), NBR 6152-out/1992: Materiais Metálicos: Determinação das Propriedades Mecânicas à Tração, ABNT.

Datsko, J. (1966), Material Properties and Manufacturing Processes, John Wiley, New York.

Dieter, G.E. (1981), Metalurgia Mecânica, 2nd ed., Guanabara-Dois, Rio de Janeiro.

Dieter, G.E. (1988), Mechanical Behavior of Materials under Tension, in Metals Handbook: Volume 8: Mechanical Testing, 9th ed., American Society for Metals (ASM).

Hayden, H.W., Moffatt, W.G., Wulff, J. (1965), The Structure and Properties of Materials: Volume III: Mechanical Behavior, John Wiley, New York.

Helman, H., Cetlin, P.R. (1983), Fundamentos da Conformação Mecânica dos Metais, Guanabara-Dois, Rio de Janeiro.

Hertzberg, R.W. (1983), Deformation and Fracture Mechanics of Engineering Materials, John Wiley, New York.

Hollomon, J. H. (1945), Tensile Deformation. Transactions of the AIME 162, 268-290.

Kleemola, H.J., Nieminen, M.A. (1974). On the Strain-Hardening Parameters of Metals. Metallurgical Transactions 5, 1863-1866.

Lemaitre, J., Chaboche, J.-L. (1985), Mécanique des Matériaux Solides, Bordas, Paris.

Lopes, L. V., Al-Qureshi, H.A. (1990), Método para Caracterização do Comportamento Plástico de Material Metálico Segundo a Expressão de Swift. IV Simpósio de Conformação Plástica de Metais.

Massonet, C., Save, M. (1965), Plastic Analysis and Design: Vol. I: Beams and Frames, Blaisdell Publishing, New York.

Neal, B. G. (1977), The Plastic Methods of Structural Analysis, 3rd ed., Chapman and Hall, London.

Oliveira, L. C., Oliveira, S. A. G. (2001), Using Subdomains for the Formulation of Stress-Strain Dia grams, XVI COBEM.

Ramberg, W., Osgood, W.R. (1943), Description of Stress-strain Curves by Three Parameters. National Advisory Commitee for Aeronautics (NACA) Technical Note no. 902.

APPENDIX

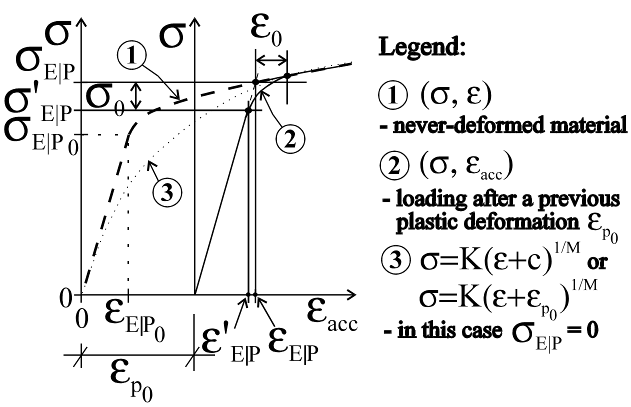

Fig. 2: Previous Plastic Deformation or Reloading.

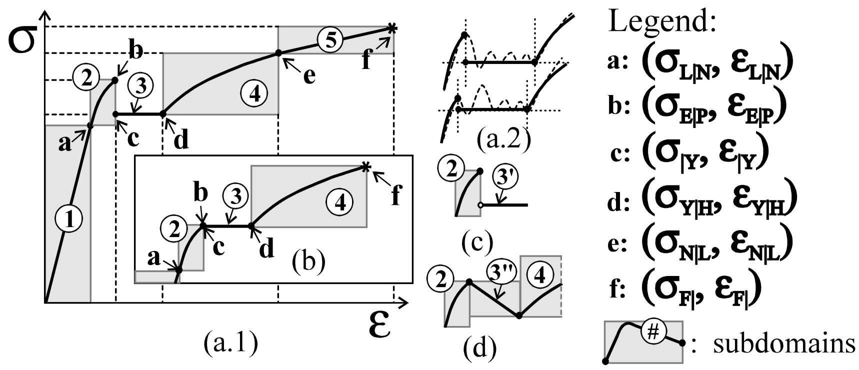

Fig. 4: $\sigma\times\varepsilon$ curve split into a series of pre-defined characteristic subdomains: (a.1) with a drop in stress; (a.2) $\sigma_{|Y}$ approximated as the first value just before the yielding plateau (and not the greater value, ABNT (1990)); (b) no drop in stress, exclusively non-linear hardening; (c) univoque relation $\varepsilon = \varepsilon(\sigma)$; (d) an illustrative possibility of diversity in subdomain modeling, in this case the passage from elasticity to strain-hardening plasticity as a negative slope straight line.