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# h-adaptative strategy proposal for Finite Element Method applied in structures imposed to multiple load cases

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**Abstract.** This paper proposes an h-adaptive strategy for structures subjected to multiple load cases to evaluate and control intrinsic discretization errors of the Finite Element Method. The proposed strategy consists of an iterative process (i) initially, for each loading case, the ideal size of each finite element is determined by an estimation of its error; (ii) then, an intersection of the size results is conducted, and a new mesh is generated by using the Bidimensional Anisotropic Mesh Generator. These two steps are repeated until the relative global error, based on energy, is less than a previously arbitrated value, considering all load cases separately. The computational routine implementation is carried out in Matlab®. *A posteriori* error estimator based on stress recovery, considering superconverged points, is used for evaluating the discretization errors. Besides that, the definition of the new finite element mesh is achieved by applying two classical h-adaptive techniques (named in this work as ZZ and LB) whose results are compared via an evaluation of local and global quality parameters. Both techniques are underpinned on the equidistribution criterion of the error based on energy norm and depend on a relationship of these discretization errors comparing two subsequent meshes. A Michell's structure is considered and the results obtained show that the proposed h-adaptive strategy results in a mesh with a relative global error below the allowable value for all loads. Finally, comparing both h-adaptive techniques, LB presents a smaller variation of the number of elements between iterations, leading to a more stable process with meshes that have better local and global quality parameters.

**Keywords:** Finite Element Method, h-adaptivity, *a posteriori* error estimators, Multiple Loads.

## 1 Introduction

For the last decades, several numerical approximation methods have been developed to provide acceptable solutions to different boundary value problems, which, often do not have an analytical solution. According to Reddy [1], intrinsic to solutions provided by the Finite Element Method (FEM), there are several sources of numerical errors; such as those related to errors of domain approximation, finite arithmetic and quadrature, and discretization errors. In particular, related to the control and limitation of discretization errors, several adaptive strategies have been proposed which are based on the modification of finite element parameters according to the evaluation of local and global errors provided by estimators or indicators.

Regarding the approximation errors estimative, a possibility is the application of *a posteriori* error estimators based on recovery (Zienkiewicz and Zhu [2]). These estimators are built from the error evaluation according to a given norm that represents the difference between the approximate solution, obtained via FEM, and a recovered solution found by gradient recovery techniques. The recovered solution, in its turn, is obtained from interpolation of the recovered nodal gradients values with shape functions — that are defined for the primary variable interpolation itself. These shape functions, as well known, provide a continuous solution and exhibit a convergence rate above the conventional numerical solution (Zienkiewicz and Zhu [2]; Zienkiewicz and Taylor [3]). For this class of estimators, the quality of the error estimative is related to the quality of the recovered nodal gradient values. In this context, several techniques can be pointed out, e.g. (Zienkiewicz and Zhu [4-5]; Boroomand and Zienkiewicz [6]; Ubertini [7]; Zhang and Naga [8]; Ródenas et al. [9]; Huang and Yi [10]). Being more specific, among the techniques cited, the Superconvergent Patch Recovery (SPR - Zienkiewicz and Zhu [4-5]) is simple and robust, with recovered values that lead to an excellent local error estimative for linear elasticity problems.

Conversely, the limitation and control of discretization errors can be reached by modifying the finite element size and keeping the polynomial degree of the interpolation (Zienkiewicz and Zhu [2]; Zienkiewicz and Taylor [3]). Towards this end, new sizes of elements are calculated by an h-adaptive technique and some of them may be highlighted. Zienkiewicz and Zhu [2] propose a methodology based on *a priori* relation of the error evaluated between two subsequent meshes and considering the optimal mesh error equidistribution criterion. Bugeda [11] and Oñate and Bugeda [12] discuss an h-adaptive technique considering the error density equidistribution criterion, i.e. the ratio between the square of the error per unit area must be equally distributed across the domain. The technique proposed by Li and Bettess [13] and Li et al. [14], on the other hand, consists of determining the dimension of new elements via a mathematical expression taking into account the asymptotic convergence of errors applied at the elementary level together with an estimate of the number of elements of the adapted mesh. Another h-technique recently proposed — which is described in Gonçalves [15] and Pereira et al. [16] — is based on the joint application of the energy error norm concepts, quadratic recovery of the energy error density function and the solution of an optimization problem using the Lagrange Multiplier Method.

This work aims to propose an adaptive strategy in linear elasticity problems considering structures subjected to multiple independent loads. It is worth emphasizing that, most studies in literature address error evaluation for a single loading, unlike the current paper which handles as many error fields and parameter maps as loads applied. In general lines, the proposed strategy consists of two main stages. First, an evaluation of approximation errors for all loading cases is performed separately using *a posteriori* error estimator based on SPR of the stress field. Second, a unique mesh of parameters is obtained to satisfy the error convergence criteria considering the results of all loading cases simultaneously. In addition to the second stage, a comparison of two h-adaptive techniques pointed out in the literature — ZZ technique (Zienkiewicz and Zhu [2]) and the LB technique (Li and Bettess [13]; Li et al. [14]) — is carried out by evaluating global and local parameters of mesh quality.

# 2 h-adaptative strategy applied to structures subjected to multiple load cases

## 2.1 *A posteriori* error estimation based on stress recovery

Considering a two-dimensional linear elasticity problem, defined in a Cartesian coordinate system $\boldsymbol{x} = (x, y)$, the vector error function, $\boldsymbol{e}(\boldsymbol{x})$, can be defined as the difference between the approximate displacement solution obtained via FEM, $\boldsymbol{u}^F = \left(u_x^F, u_y^F\right)^T$, and the analytical displacement solution, i.e. $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^F$, (Zienkiewicz and Zhu [2]; Zienkiewicz and Taylor [3]). The measurement of the error function can be carried out by different norms. In the current work, the energy error norm is used, $\|\boldsymbol{e}\|$, which is written as (Zienkiewicz and Zhu [2])

$$\|\boldsymbol{e}\| = \left[\int_{\Omega} \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^F\right)^T \boldsymbol{D}^{-1} \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^F\right) d\Omega\right]^{1/2}, \tag{1}$$

where $\boldsymbol{\sigma}$ is the analytical stress field, $\boldsymbol{\sigma}^F$ is the approximate stress field via FEM, $\boldsymbol{D}$ is the material constitutive stiffness tensor, and $\Omega$ characterizes the problem domain. The representation of the absolute error in energy via a relative measure, called global relative error, $\eta^A$, is given by (Zienkiewicz and Zhu [2]) as

$$\eta^A = \|\boldsymbol{e}\| / \|\boldsymbol{E}\|, \quad with \quad \|\boldsymbol{E}\| = \left(\int_{\Omega} \boldsymbol{\sigma}^T \boldsymbol{D}^{-1} \boldsymbol{\sigma} \, d\Omega\right)^{1/2}. \tag{2}$$

Here, $\|\boldsymbol{E}\|$ is the norm that expresses the analytical measure of the total energy of the system.

Since the analytical solution, $\boldsymbol{u}$, is generally not known, hence the analytical evaluation of the energy error norm, eq. (1), cannot be applied. Nevertheless, there are some alternatives in the literature to estimate discretization errors. In this work, a classical model of *a posteriori* error estimator based on recovery, proposed by Zienkiewicz and Zhu [2]), is considered — whose formulation is presented below

$$\|\boldsymbol{e}^{est}\| = \left[\int_{\Omega} \left(\boldsymbol{\sigma}^R - \boldsymbol{\sigma}^F\right)^T \boldsymbol{D}^{-1} \left(\boldsymbol{\sigma}^R - \boldsymbol{\sigma}^F\right) d\Omega\right]^{1/2}, \tag{3}$$

where $\boldsymbol{\sigma}^R$ is the recovered stress field obtained by interpolating recovered nodal stress with the same shape functions used to interpolate the displacement field. This recovered field converges to the analytical solution more quickly than $\boldsymbol{\sigma}^F$. Thereby, a smooth and continuous tension field between the elements of the mesh can be obtained. Here, the SPR technique (Zienkiewicz and Zhu [4-5]) is used to obtain the nodal values of recovered stresses.

Furthermore, the overall analytical relative error $\eta^A$, presented in eq. (2)a, can be approximated using eq. (3) together with the orthogonality property of the error concerning the numerical solution (Oden and Reddy [17]), i.e.

$$\eta = \|\boldsymbol{e}^{est}\| / \|\boldsymbol{E}\|, \quad with \quad \|\boldsymbol{E}\| \approx \left(\|\boldsymbol{E}^F\|^2 + \|\boldsymbol{e}^{est}\|^2\right)^{1/2}, \tag{4}$$

where $\|\boldsymbol{E}^F\|$ is the total energy evaluated according to an approximate field of stress or strains (Zienkiewicz and Zhu [2]).

### 2.2 h-adaptative techniques and mesh quality parameters

This section briefly presents two h-adaptive techniques exposed in the literature, here called ZZ (Zienkiewicz and Zhu [2]) and LB (Li and Bettess [13]; Li et al. [14]). For both, a mesh is admitted as convergent if the relative error in energy, $\eta$, is less than or equal to the allowable error, $\eta_a$. Furthermore, the techniques aim to find out a mesh with discretization errors equally distributed among the domain. Some details of them are given below.

ZZ technique estimates the new size of elements, $h_{new}$, for problems with no singularities, as follows

$$h_{new} = \left( h_{old} / \xi_K^{(1/p)} \right), \quad with \quad \xi_K = \|\boldsymbol{e}\|_K / e_a \,, \tag{5}$$

where $h_{old}$ is the size of the current element, *p* is the polynomial degree of the finite element approximation, $\|\boldsymbol{e}\|_K$ is the energy error for element *K*; and $e_a$ is the permissible limit error per element, whose formulation is written as

$$e_a = \eta_a \|\boldsymbol{E}\| / \sqrt{N} \,, \tag{6}$$

with *N* equal to the number of elements of the mesh under analysis. Therefore, $\xi_K$ is a refinement parameter that indicates whether the element will undergo refinement $(\xi_K > 1)$ or derefinement $(\xi_K < 1)$ and whose formulation adopted is based on the classic proposal of Zienkiewicz and Zhu [2].

The LB technique, in its turn, determines the new size of elements according to the following

$$h_{new} = h_{old} \left( \frac{\eta_a \|\boldsymbol{E}\|}{\sqrt{N_{new}} \|\boldsymbol{e}\|_K} \right)^{1/(p+(d/2))} , \tag{7}$$

where *d* is the physical dimension of the problem (in this case, *d* = 2); and $N_{new}$ is the number of elements in the new mesh that is estimated as

$$N_{new} = \left( \eta_a \|\boldsymbol{E}\| \right)^{-d/p} \left( \sum_{K=1}^{N} \left( \|\boldsymbol{e}\|_K \right)^{d/(p+d/2)} \right)^{(p+d/2)/p} . \tag{8}$$

The notation of eq. (7) and (8) is presented in Díez and Huerta [18]. According to Li et al. [14], there are two main differences between these two techniques. First, ZZ depends on the number of elements in the current mesh, while LB seeks to distribute errors based on the number of elements in the future mesh. Second, the exponent that determines the rate of increase or decrease of the elements is established *a priori* for both of them; however, they are based on different relationships.

In this paper, an evaluation of the h-adaptive techniques is performed by comparing five quality mesh parameters shown in eq. (9), whose formulations are defined in Pereira et al. [16] and are analogous to those presented in Oñate and Bugeda [12]: global relative error $(\eta)$, number of degrees of freedom $(DOF)$, deviation of refinement parameters $(\xi_d)$, an average of refinement parameters $(\xi_m)$ and maximum refinement parameter $(\xi_{max})$.

$$\xi_d = \sqrt{\frac{1}{N} \sum_{K=1}^{N} \left( \xi_K - 1 \right)^2} \,, \; \xi_m = \frac{1}{N} \sum_{K=1}^{N} \xi_K \; \text{ and } \; \xi_{max} = max\left( \xi_1, \xi_2 \ldots \xi_N \right) . \tag{9}$$

One should note that for an ideal mesh, and considering the optimal mesh criterion for error equidistribution, is expected that $\eta = \eta_a$, $\xi_d = 0$ e $\xi_m = \xi_{max} = 1$. Therefore, from these reference values, the quality of the meshes generated by each h-adaptive technique can be evaluated.

### 2.3 h-adaptative strategy

In this work, an h-adaptive strategy is proposed and discussed in view of its application in structures subjected to multiple load cases. Such a procedure aims to limit and equidistribute discretization errors by defining a unique finite element mesh that satisfies a previously arbitrated convergence criterion considering all loads separately. Given an initial finite element model, Algorithm A describes the main steps of this strategy:

**Algorithm A**

Step 1) For each loading case, a displacement solution $\boldsymbol{u}^F$ — of a linear elasticity problem suitably restricted and subjected to a plane stress state — is obtained via FEM.

Step 2) For each loading case, the energy error is estimated, both globally and at the elementary level, according to the *a posteriori* error estimator based on stress recovery considering superconvergent points (SPR - Zienkiewicz and Zhu [4-5]).

Step 3) For each loading case, the convergence criterion is checked, i.e. the overall relative error must be below the allowable error. If the convergence criterion is met, the results are saved and the algorithm is finalized. Otherwise, proceed to Step 4.

Step 4) An h-adaptive technique is selected (ZZ or LB).

Step 5) For each loading case, a mesh of parameters with the new sizes of elements, that may guarantee the convergence criterion, is calculated based on the h-adaptive technique selected.

Step 6) By applying a minimum operator on the new sizes of elements values of all meshes of parameters, obtained in step 5; a unique mesh of parameters is achieved with the smallest element size values.

Step 7) A new finite element mesh is generated via a mesh generator taking into account the intersection mesh of parameters. Return to Step 1.

The algorithm described is carried out twice, once for each h-adaptative technique. Furthermore, the results presented consider an acceptable percentage error arbitrated as 5%; and the only limitation imposed on the calculation of the new size of elements is that they do not exceed 20% of the diagonal of the problem. This limitation aims to avoid very large element sizes in regions with low errors.

### 2.4 The numerical model for finite element analysis

Figure 1 presents the mechanical model studied, which is based on a classic problem presented by Michell [19] and consists of a beam with a circular hole - where is applied homogeneous essential boundary conditions. For natural boundary conditions, three independent multiple load cases are applied considering a constant pressure distributed over a central edge of 8 mm at the beam's right end (represented as a black region in Fig. 1). It is noteworthy that $P_1$ and $P_2$ impose bending on the beam; and $P_3$, in its turn, imposes a traction condition.

The finite element analysis was performed in Matlab® considering a two-dimensional linear elasticity problem under plane stress state. The initial mesh was obtained using the BAMG mesh generator (Bidimensional Anisotropic Mesh Generator [20-21]), considering the domain discretization with 443 constant strain triangle elements. Table 1 summarizes the mechanical properties, geometric data, and the loads' magnitude.

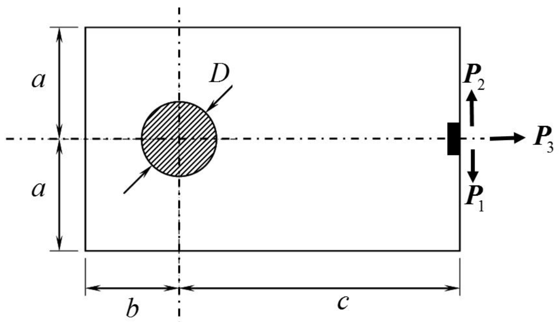



Figure 1. Mechanical Model scheme (Adapted from Pereira et al. [22]).

Table 1. Overall numerical model parameters.

| Parameter | Value |
|---|---|
| Young's Modulus [Pa] | 100 |
| Poisson's Ratio [-] | 0.3 |
| $a$ [mm] | 20 |
| $b$ [mm] | 15 |
| $c$ [mm] | 40 |
| $D$ [mm] | 20 |
| Thickness [mm] | 1 |
| $P_1$ [N] | 1 |
| $P_2$ [N] | 1 |
| $P_3$ [N] | 1 |

# 3 Numerical results and discussion

Figure 2 presents the initial finite element mesh and its respective elementary error field for $P_1$ and $P_3$ loads — together with the result of two adaptive iterations (number of iterations required to satisfy the convergence criterion pointed out in Step 3 of Algorithm A) for both h-adaptive techniques studied in this work. Adding up, the mesh quality parameters are shown in Fig. 3. Attention should be drawn that the initial mesh is quasi-uniform; therefore, $P_1$ and $P_2$ error fields are not exactly symmetrical, although very close. Even so, it was decided to present only results for $P_1$ and $P_3$. As expected for both loads, a greater error magnitude is observed near the regions where the boundary conditions are applied. Indeed, for the bending case, this perturbation was more effective near the natural contour (likewise to the traction case) as well as the essential contour. Nevertheless, in absolute terms and for the initial coarse mesh, the errors were 18.47% and 29.58%, respectively for $P_1$ and $P_3$.

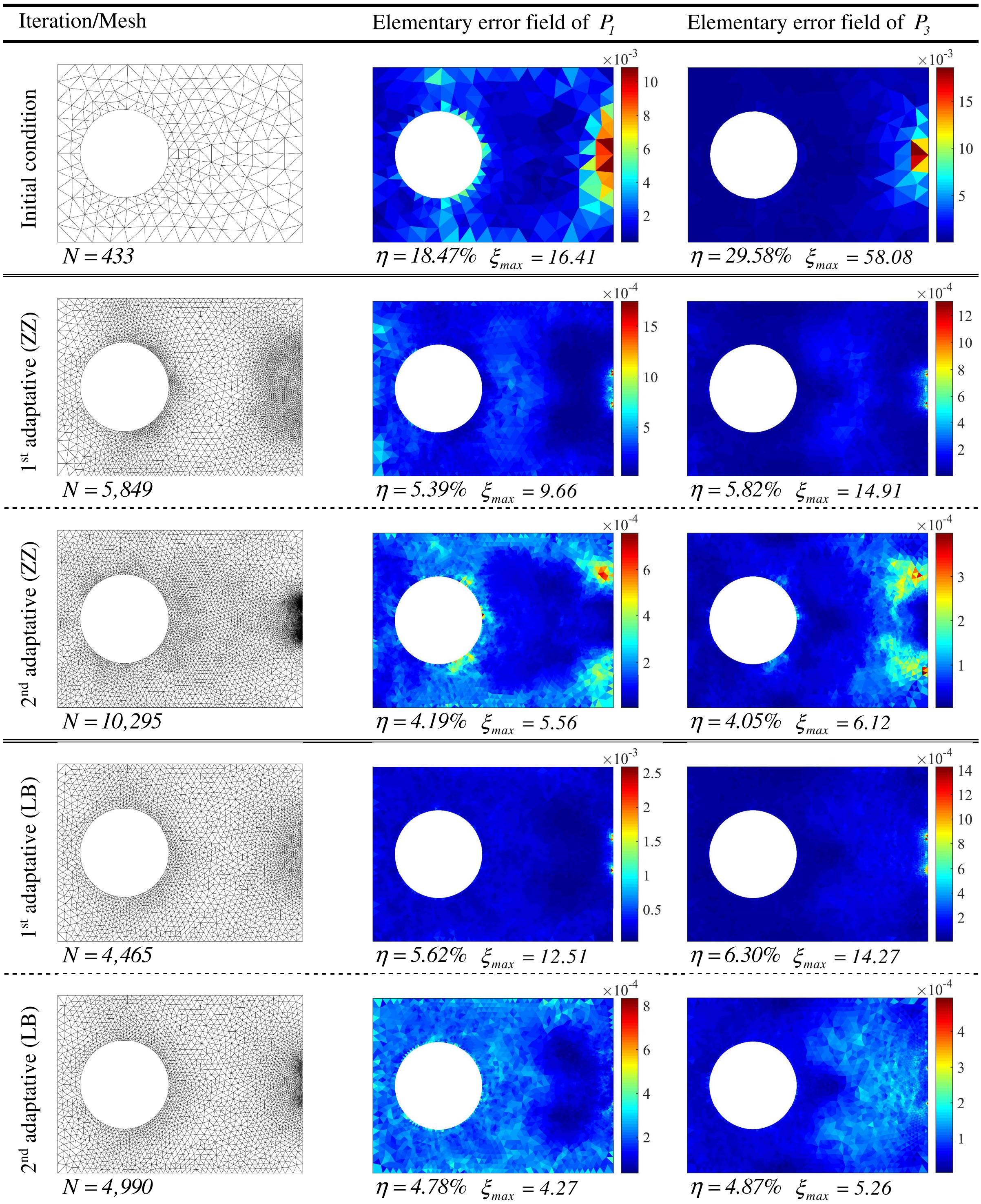



Figure 2. Finite elements meshes, and elementary error fields of $P_1$ and $P_3$ loads, for initial condition and the adaptative iterations of ZZ and LB approaches.

Evaluating the h-adaptive techniques, both presented adaptative meshes with a refinement concentration (i.e. minor elements) in regions of the domain close to where the boundary conditions were applied since these are regions with greater variations of the stress field. However, when compared to ZZ, LB leads to a final mesh with fewer degrees of freedom (approx. 50% less) required to guarantee the prescribed percentage error.

Still in this context, although both techniques achieved, for all loads, a percentage error below 5% already in the second adaptive iteration, as shown in Fig. 2 and Fig. 3, the superiority of the LB methodology is evident when evaluating the quality parameters — i.e. values of $\xi_d$ and $\xi_m$ respectively closer to zero and the unit, coupled with values of $\xi_{max}$ being lower for LB. This result indicates a finite element mesh closer to the optimal mesh, in which all refinement parameters are expected to be equal to the unit. To put it another way, in such conditions, errors are equally distributed over the domain and equal to the prescribed limit error. In line with this fact, evaluating qualitatively $P_1$ and $P_3$ elementary error field of the convergent iteration in Fig. 2, LB provided better equidistribution of errors.

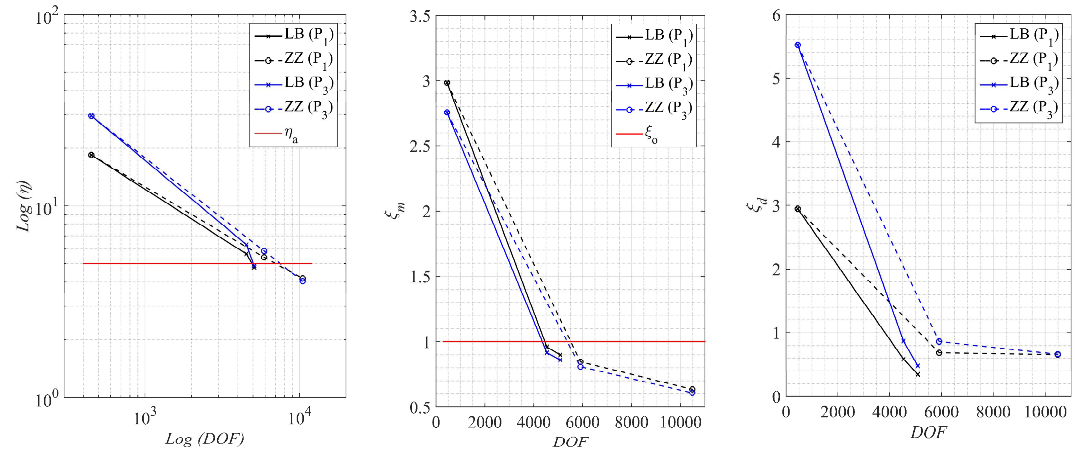
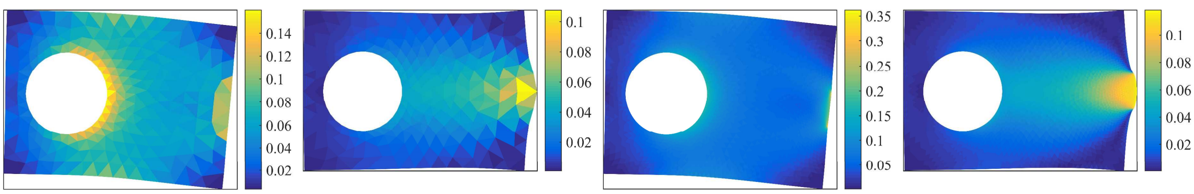



Figure 3. Comparison between quality parameters of the h-adaptive techniques.

Concerning the application of ZZ in linear elasticity problems, it is worth mentioning that the works of Onãte and Bugeda [12] and Díez and Huerta [18] point out that this technique tends to present an oscillatory characteristic in the topology of its generated meshes, which means that refinement-derefinement-refinement might occur across different regions of the domain. Indeed, for Michell's problem [19] solved in the context of multiple load cases, the analysis of the first and second adaptive iterations presented, as expected, a remarkable oscillation between the topologies of the meshes. When observing the elementary error fields of the second adaptive iteration for $P_1$ and $P_3$, if a new mesh was generated it would also undergo a significant change since ZZ aims to the equidistribution of errors. On the other hand, LB presents a greater smoothness in the topological modification between the generated meshes, which indicates a trend of stability of the solution with better equidistributed errors.

Finally, Fig. 4 compares the von Mises stress field obtained for the initial mesh with the convergent mesh result of LB for the $P_1$ e $P_3$ loads. The refinement obtained with this technique exhibit higher stress values; and in a much more relevant way for the bending loading. In fact, in cases where there is no singularity in the domain, it is expected that h-refinement results in a solution with less approximation error. Within this problematic, the use of both techniques is successful for a more accurate result that guarantees a prescribed global percentage error for all applied loading cases separately.



(a) (b) (c) (d)

Figure 4. von Mises stress field: (a) Initial mesh ( $P_1$ ). (b) Initial mesh ( $P_3$ ).
(c) Convergent mesh for LB ( $P_1$ ). (d) Convergent mesh for LB ( $P_3$ ).

# 4 Conclusions

The current study proposes an h-adaptive strategy for linear elasticity problems subjected to multiple load cases to determine a finite element mesh that simultaneously satisfies several convergence criteria on the global error. The number of criteria to be met is equal to the number of independent loads being applied to the structure. The h-adaptive strategy is evaluated by applying two different h-adaptive techniques, named as ZZ and LB. Furthermore, a quantitative comparison between both techniques was carried out with global and local quality parameters.

In general, regardless of the h-adaptive technique used, the proposed strategy leads to a resulting mesh with errors below the arbitrated allowable value for each loading case. Moreover, one should note an advantage of the proposed strategy concerning the definition of a unique mesh that guarantees the convergence criteria for all the applied loads. Hence, the computational cost is reduced since the assembly of the stiffness matrix is carried out only once.

In particular, a comparison between the LB and ZZ techniques points out that LB provides a resulting mesh with better quality parameters and more uniformly distributed error fields. Nonetheless, when the ZZ technique is applied, there is an oscillatory characteristic of refinement-derefinement-refinement between subsequent meshes.

Finally, for future work, the h-adaptive strategy proposed can be extended to control errors of discretization in problems of a more complex nature involving several independent error fields, e.g. topological optimization problems involving local restrictions of tension and multiple cases of loading or even fluid dynamics problems.

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