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A method to evaluate contact pressure distribution of elastic surfaces using LCP and optimization techniques

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Abstract. Contact occurs in various mechanical components such as gears, bearing, wheel and rail of trains, for which contact fatigue is considered the major cause of failure. For this reason, analyze the stresses in the materials in contact are of great importance at the study of solid mechanics, in particular tribology, because this allows a prediction of fails related to the corresponding surfaces. Yet, the elastic contact of bodies is a problem of particular interest in solid mechanic. The contact between deformable elastic bodies is present in the industry and everyday life, and it is a challenge to identify and estimate the contact area, as well as the pressure and stress distributions at the interface and develop efficient numerical methods for solving the problem. Within this context, this work aims to propose and evaluate a method to calculate the stresses and the contact area between elastic bodies accurately. The adopted method uses a linear complementarity problem (LCP) approach, obtaining its solution through optimization techniques.

Keywords: elastic contact, linear complementarity problem, optimization.

1 Introduction

Engineering contact problems have been studied and developed for decades, especially those related to elastic contact between the surfaces of two bodies. Research about this topic is very important in mechanics, because the contact of bodies occurs in all interfaces that transmit force, movement or both. When two bodies are pressed against each other, there is a contact area between them and one important challenge is obtaining pressure and stress distributions at the interface [1]. The analysis of stresses in the materials in contact is of significant importance for the design of mechanical components. According to [2], the interest to study the contact between two surfaces is based on its high relevance on heat transfer, wear, friction and adhesion, which are phenomena that occur in any type of tribological contact [3].

The elastic behavior of a material is characterized by the absence of permanent deformation after the material is subjected to loading and/or unloading, that is, the elastic contact is a mechanical contact between two bodies and, after tension that make them come into contact, they return to their original state [4]. Johnson [5] explains that when two elastic bodies with positive curvature radius surfaces are in contact, and if their deformations are small enough so that the theory of linear elasticity is applicable, they originate a contact area, which dimensions are very small when compared to the curvature radii of the undeformed surfaces. The zone of interest is located next to the contact area and thus the stresses can be estimated with a good approximation considering each body as a semi-infinite elastic solid, limited by a flat surface and subjected to a concentrated load, that is, an *elastic half-space*.

The Linear Complementarity Problem (LCP) refers to a system of inequalities equations. In recent years, LCP has been studied due to its way of solving optimization problems. Also, over the past 20 years, elastoplastic analysis problems have been converted to LCP to find their solutions, also appearing as a unified solution to quadratic and linear programming problems. The characteristics, complementarity and linearity, provide the basic and fundamental elements for the analysis and understanding of the complex nature of the problems of mathematical programming and of balance, according to [6] and [7].

Therefore, the present research uses the half-space's theory and formulate the problem of frictionless contact between two linearly elastic bodies as an LCP, transforming it into an optimization problem which solution is the discretized values of the surface pressure between the bodies. For this, an algorithm to solve the LCP, based on quadratic programming and named "LCPquad" is developed.

2 Contact problem formulation

Consider two smooth non-conforming isotropic elastic bodies initially in contact a single point P, under the action of some external actions, both bodies deform in the neighborhood of P, as shown the Figure 1, the forces of the bodies are perpendicular to the surface and normal stresses (pressures) are considered.
The mathematical formulation that originated this method is based on the work of [1], which considers three hypotheses: the contact area to be considered is very small in comparison to the dimensions of the two bodies that come into contact, the bodies in contact are assumed as being elastically linear and homogeneous and lastly, ignoring the inertial effects on movement. All of these hypotheses describe a simplified model where the elastic half-space theory can be applied. In addition, two more conditions are required: (i) there is no penetration between the two bodies and thus the distance between the two surfaces is zero in the area in contact, and (ii) the pressure is compressive in the contact are and null outside it. Thus, to develop the problem of the current research, consider $p(x)$ represents the pressure on the surface at point x of the region S studied; $g(x)$ is the distance between the surfaces of the bodies, $u(x)$ is the displacement due to deformation at point x and defining $e(x) = u(x) + g(x)$ as the distance after the deformed contact, also represented in Fig. 1.

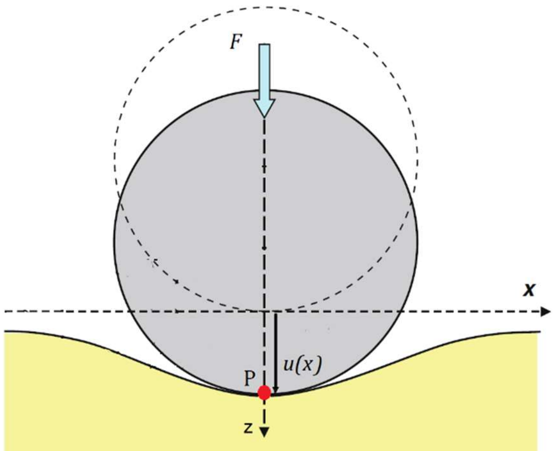

Figure 1: Contact between two elastic bodies and the corresponding displacements (adapted from [8])

The conditions for the contact problem are:

$$e(x) = 0, p(x) \geq 0 \tag{1}$$

for each point in contact, and:

$$e(x) > 0, p(x) = 0 \tag{2}$$

for each point outside contact.
Considering **K** as a matrix that represents the function of influence of normal deformation at points of surfaces, so that **K** depends on the characteristics of the bodies, such as their Poisson's ratios and elasticity moduli. Based on the classic solutions of Boussinesq and Cerruti presented by [5], and considering the problem of normal bidimensional contact, Zhao et al. [1] state that the normal displacement, at z direction, of a point x on the surface S is given by:

$$u(\mathbf{x}) = \int_S K(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) d\mathbf{y}, \quad (3)$$

where $K(\mathbf{x}, \mathbf{y})$ represents the displacement at point $\mathbf{x}$ due to the contact pressure p acting on $\mathbf{y}$ within the region S. For homogeneous linear elastic materials, the influence coefficients are defined by:

$$K(\mathbf{x}, \mathbf{y}) = \frac{1 - v^2}{\pi E} \frac{1}{||\mathbf{x} - \mathbf{y}||}, \quad (4)$$

where ν and E are, respectively, the Poisson's ratio and the combined modulus of elasticity of the two bodies, obtained by:

$$\frac{1}{E} = \frac{1 - {v_1}^2}{E_1} + \frac{1 - {v_2}^2}{E_2}, \quad (5)$$

with E_1 and E_2 being the elasticity's moduli of each body in the half-space, v_1 and v_2 the Possoin's ratios and $\|.\|$ denotes the Euclidean norm. The total force F is calculates as:

$$F = \int_S p(\mathbf{x})\, d\mathbf{x}. \quad (6)$$

Considering what is presented above, to carry out the numerical simulation, it is necessary to perform the discretization of the unidimensional contact S region. Thus, by calculating the pressure value at each point in the domain, it is possible to estimate the value of the total pressure across the region. The greater the number of point where the calculation is made, more accurate is expected to be the value of the total pressure.

Thus, the *N*-dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_N)$ is constructed, which represents the coordinates of the point of the possible contact region S. Likewise the pressure values ($p(x)$), distance between the surfaces ($g(x)$), displacement ($u(x)$) and the distance after the deformed contact ($e(x)$), at each point x_i of the surface, is represented by *N* -dimensional vectors $\mathbf{p}$, $\mathbf{g}$, $\mathbf{u}$, $\mathbf{e}$, respectively.

Taking into account the conditions of eqs. (1) and (2) for the formulation of the linear complementarity problem in the contact, it results in:

$$\mathbf{e} = \mathbf{0}, \quad (7)$$

$$\mathbf{u} + \mathbf{g} = \mathbf{0}. \quad (8)$$

This means that the displacement $\mathbf{u}$ has the same distance of $\mathbf{g}$.

If $\mathbf{u} = \mathbf{K}\,\mathbf{p}$, then the distance between the surfaces after deformation is:

$$\mathbf{e} = \mathbf{K}\,\mathbf{p} + \mathbf{g}. \quad (9)$$

Therefore, to find the value of pressures at the points of the contact area at the region S is equivalent to solve the LCP described below:

$$\mathbf{e} = \mathbf{K}\,\mathbf{p} + \mathbf{g} \quad (10)$$

$$\mathbf{p}^{\mathrm{T}}\mathbf{e} = 0 \quad (11)$$

$$\mathbf{p}, \mathbf{e} \geq \mathbf{0} \quad (12)$$

To calculate the solution, a function f is defined as:

$$f(\mathbf{p}) = \mathbf{p}^{\mathrm{T}}\mathbf{e}, \quad (13)$$

It follows to find a solution to the LCP of eqs. (10), (11) and (12) is associated with solving the quadratic optimization problem with the restrictions described below:

$$\min f(\mathbf{p}) = \mathbf{p}^{\mathrm{T}}(\mathbf{K}\,\mathbf{p} + \mathbf{g}), \quad (14)$$

$$\text{such that } -\mathbf{K}\,\mathbf{p} \leq \mathbf{g} \quad (15)$$

$$\mathbf{p} \geq \mathbf{0} \quad (16)$$

The simulations are performed using the Matlab platform. The results found in the proposed optimization are compared with the ones obtained by the method developed by Almqvist [8, 9].

Almqvist [8] developed the "LCP_CM" algorithm, where the function "LCPSolve" [9] solves problems of elastic contact mechanics using the LCP as a mathematical model. The "LCPSolve" consists in solving the linear complementarity problem described in eqs. (10), (11) and (12) so that the function takes the matrix $\mathbf{K}$ and the vector $\mathbf{g}$ as arguments. The function has return variables as the vectors $\mathbf{e}$ and $\mathbf{p}$, found by complementary rotation. The third return is a vector of dimension 1×2, where the first component is 1 if the algorithm runs successfully and the second component is the number of iterations performed in the outer loop.

The proposed method created the function "LCPmqc" based on "LCP_CM" to insert the input data and "LCPquad" for post-processing and graphs of the elastic contact mechanic problem. Then, the mathematical model of LCP described in eqs. (10), (11) and (12) is transformed into a quadratic optimization model to then be solved.

3 Numerical results

In this section, four test cases are performed. The corresponding data are the same as [9]. For the Test 2, Test 3 and Test 4, variations are made regarding the radii of curvatures of the bodies (R_1 and R_2), number of nodes, elasticity moduli (E_1 and E_2) and Poisson's ratios (ν_1 and ν_2). The initial data for the test cases are shown in Table 1.

Table 1. Data for the test cases

Propriety	Test 1	Test 2	Test 3	Test 4
R_1(m)	0.05	0.01	0.5	0.05
R_2(m)	0.03	0.01	0.0005	0.03
E_1(Pa)	210×10^9	82.7×10^9	115.7×10^9	78×10^9
E_2(Pa)	210×10^9	82.7×10^9	115.7×10^9	78×10^9
ν_1	0.3	0.345	0.321	0.293
ν_2	0.3	0.345	0.321	0.293
Number of nodes	129	500	500	7

The graphs of Figs. 2 to 6 show the pressure variation along the contact region S obtained with the proposed method (LCPquad) and that obtained with the method developed by [9, 10] (LCPSolve). In the horizontal axis is x, in the range of [-4, 4] $\times 10^{-4}$ meters, which represents the domain of the S region and in the vertical axis is the pressure distribution, normalized in relation to the Hertzian pressure, therefore this quantity is dimensionless.
Figure 2 depicts the pressure distribution of both methods for Test 1. Figure 2 (a) represents the pressure distribution by "LCPSolve" method in red and Fig. 2 (b) represents the pressure distribution by "LCPquad" in magenta. It is possible to observe a great similarity in the pressure values in both methods, built on a scale of 10^{-4} meters. In Fig.3, the pressure curves of both methods are plotted on the same graph and zoomed to better observe the difference between them, which is very small, in the order of 3×10^{-4}.

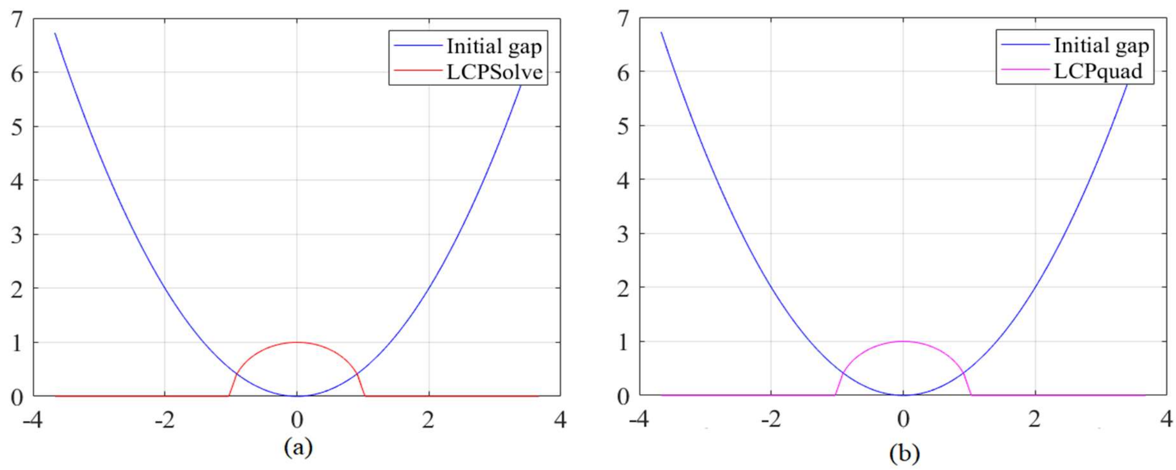

Figure 2. Dimensionless pressure distribution for Test 1 obtained via LCPSolve (a) and LCPqua (b)

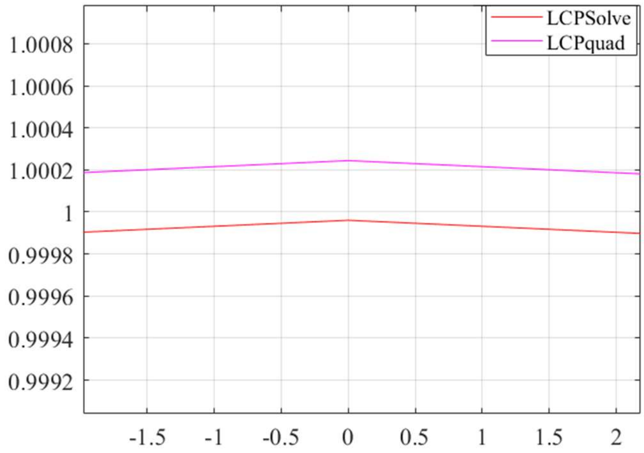

Figure 3. Zoom in the pressure distribution graph of Test 1

In Test 2, compared to Test1, the radii of curvature, the moduli of elasticity of the bodies and the number of nodes are changed, aiming to analyze the behavior of both methods front of these variations. The pressure distributions obtained from both methods are also very similar, as shown in Fig.4.

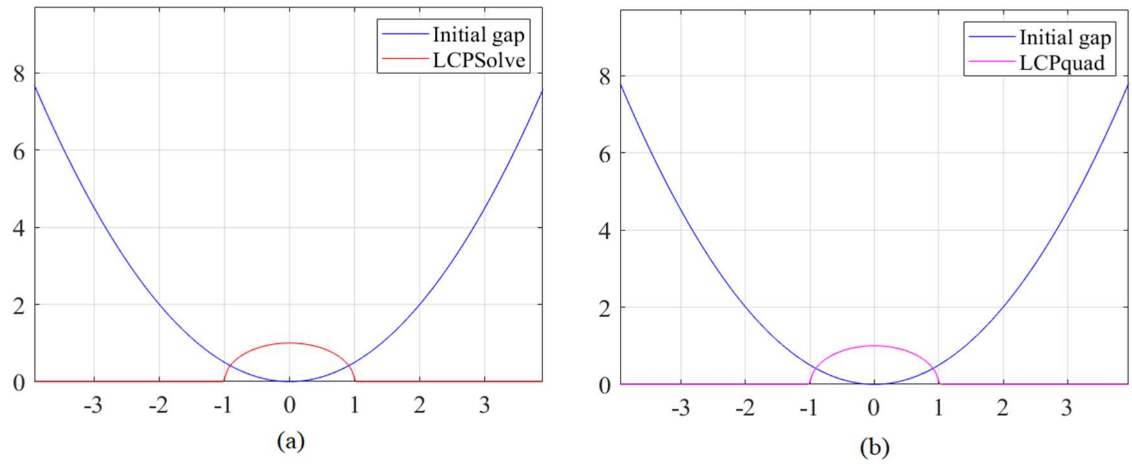

Figure 4. Dimensionless pressure distribution for Test 2 obtained via LCPSolve (a) and LCPqua (b)

In order to carry on the method evaluation, in Test 3 two bodies with very different curvature radii are considered. Body 2 has a radius 10^{-4} times smaller than the radius of body 1. At this situation, the pressure calculation by the proposed method (LCPquad) is constant over the interval [-1, 1]. Despite the difference, both graphs are increasing close to the point x = -1, decrease close to the point x = 1 and he maximum pressure value occurs at the point x = 0, as illustrated in Fig. 5.

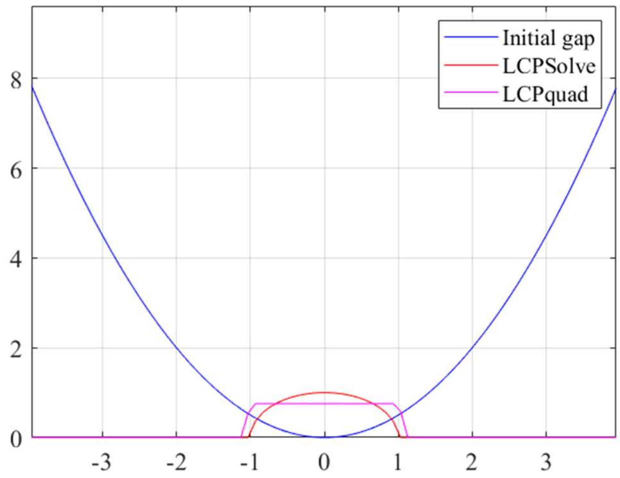

Figure 5. Dimensionless pressure distribution for Test 3

In Test 4, the number of nodes is reduced to just 7. The results of both methods are very similar, and the graphs are very close to each other. For this reason they are also shown separately to analyze the behavior of both. It is noted in the two methods that the maximum pressure value (at x=0) a peak occurs, which value is close to 1 (see Fig. 6).

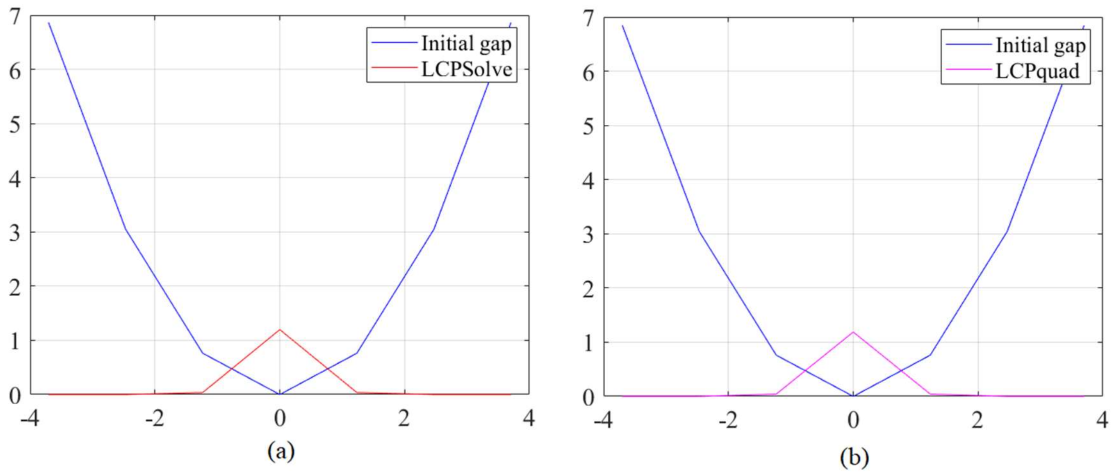

Figure 6. Dimensionless pressure distribution for Test 4 obtained via LCPSolve (a) and LCPqua (b)

Figures 2, 4, 5 and 6 also show, in blue, the initial gap between the bodies.

4 Conclusions

The objective of the proposed method (LCPquad) has been achieved, since it is another alternative to obtain the pressure distribution in contact problems as shown by the various tests, which results are summarized below.

In Fig. 2 of Test 1, it is possible to observe a great similarity in the pressure distribution in both methods built on a scale of a 10^{-4} meters. This fact is confirmed in Fig. 3 where it is zoomed 1000 times.

There is at Test 2 a variation in the elasticity moduli and number of nodes and it equalized the radii of curvature of both bodies, the pressure value along the surface is also very similar in both methods, as shown in Fig. 4.

In Test 3, two bodies with very different radii of curvature are used and this difference more clearly, even so the pressure behavior along the surface is equivalent: increasing close to x=-1, decrease close to x=1 and maximum pressure value at x=0, as illustrated in Fig. 5 which ensures the similarity between the methods

In Test 4, the number of nodes is reduced to just 7. The results of both methods that at the maximum pressure value ($x = 0$) a peak occurs, which value is close to 1 (see Fig. 6).

After performing all the tests, it can be concluded that the numerical simulation for the calculation of contact pressure of elastic problems modeled through the LCP, using the quadratic optimization method, is very close to the LCP solution method developed by [9]. Thus, the usage of quadratic optimization can be considered as an efficient alternative for calculation pressures in contact problems.

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