XLI CILAMCE
XLI Ibero-Latin American Congress on Computational Methods in Engineering
16-19 November, 2020 FOZ DO IGUAÇU PARANÁ - BRAZIL

Isogeometric Analysis of Functionally Graded Beams using different Micromechanical Models

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Abstract. Functionally Graded Materials are a class of advanced composites with a gradual and continuously varying composition. Its design involves choosing the constituents and a function to describe the volume fraction variation. This work presents an isogeometric formulation for the static analysis of functionally graded beams. The governing equations are based on the Timoshenko Beam Theory for membrane, bending and shear strains. B-spline curves are used as basis functions for the isogeometric formulation. Two micromechanical models are utilized to evaluate the effective material properties.

Keywords: Functionally Graded Materials, Isogeometric Analysis, FSDT, Micromechanics.

1 Introduction

Functionally Graded Materials (FGM) are composite materials with continuously varying composition. This characteristic allows a better distribution of stress and prevents any geometric discontinuity, averting some common problems when dealing with laminated composites, such as delamination and matrix cracking [1]. The effective properties can be evaluated in terms of the volume fraction of the FGM components, using appropriate micromechanical models. There are many models that have already been presented in the literature, with the Rule of Mixtures being the most used one [1, 2].

In order to study the structural responses of functionally graded beams, some mathematical theories were used to simplify the analysis process, asserting hypothesis about the physical behaviour and evaluating the structure displacement field. The Classical Beam Theory (CBT) disregards the effects of shear deformation and is only appropriated to use on thin beams. On the other hand, the Timoshenko Beam Theory, or First-order shear deformation theory (FSDT), considers the transverse shear strains to be constant through the thickness [3], requiring a correction factor.

Although beams can be modeled as one-dimensional problem, some factors can complicate the analysis process to obtain analytical solutions, such as the utilization of FGM. Thereby, computational methods have been employed to evaluate a sufficiently approximated solution. The Isogeometric Analysis (IGA) directly approximates the variables of the displacement field using B-splines or NURBS as basis functions [4, 5], allowing the exact representation of complex geometries.

2 Isogeometric Analysis of Functionally Graded Beams

2.1 Functionally Graded Materials

This study considers a functionally graded beam made of two phases, ceramic and metal, with their volume fractions continuously varying along the thickness according to a power-law function:

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^N, \quad V_m(z) = 1 - V_c(z), \tag{1}$$

where V_c and V_m are the ceramic and metal volume fractions, respectively, z is a coordinate varying along the thickness h, and N is a user-controlled factor to model the variation profile.

According to Shen [1], as the material composition varies, the effective properties associated to it also varies. In this study, in order to estimate the effective properties of the beam along its thickness, two micromechanical models are used. The first one is the Rule of Mixtures (RoM), also known as the Voigt model, which is described by Akbarzadeh and Chen [2]:

$$P(z) = P_m + (P_c - P_m)V_c, \tag{2}$$

where P represents the FGM effective property and the subscripts c and m represent ceramic and metal, respectively.

The second one is the Mori-Tanaka (MT) model, which is also described by Akbarzadeh and Chen [2]. For two-phase materials with a random distribution of spherical particles, the effective bulk (K) and shear (G) moduli are defined as:

$$\frac{K(z)-K_m}{K_c-K_m} = \frac{V_c}{1+V_m\frac{K(z)-K_m}{K_m+\frac{4}{3}G_m}}, \quad \frac{G(z)-G_m}{G_c-G_m} = \frac{V_c}{1+V_m\frac{G(z)-G_m}{G_m+f1}}, \quad f1 = \frac{G_m(9K_m+8G_m)}{6(K_m+2G_m)}. \tag{3}$$

After that, the effective Young's modulus and Poisson's ratio are respectively described as:

$$E\left(z\right) = \frac{9K(z)G(z)}{3K(z)+G(z)}, \quad \nu\left(z\right) = \frac{3K(z)-2G(z)}{2(3K(z)+G(z))}. \tag{4}$$

2.2 B-Splines

The B-Spline basis functions are defined using a set of nondecreasing parametric values known as knot vector $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$, where $\xi_i \in \mathbb{R}$, p is the polynomial order of the B-spline and n is the number of basis functions, or control points, used to describe the curve. Given the knot vector, the B-splines basis functions are evaluated using the Cox-de Boor recursion formula, according to Piegl and Tiller [5]:

$$N_{i,0}(\xi) = \begin{cases} 1, & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0, & \text{otherwise} \end{cases}, \tag{5}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1}^{p-1}(\xi). \tag{6}$$

Thus, a B-spline curve is defined by the summation of each basis function $N_{i,p}(\xi)$ associated to the control points $\mathbf{p}_i$:

$$C(\xi) = \sum_{i=1}^{n} N_i(\xi)\,\mathbf{p}_i. \tag{7}$$

2.3 Governing Equations

In this study, a First-order shear deformation theory (FSDT) is utilized. According to Wang and Lee [3], the displacement field at any point of the beam can be described as:

$$\begin{aligned} u(x,z) &= u_0 - z\theta(x) \\ w(x,z) &= w_0, \end{aligned} \tag{8}$$

where u_0 and w_0 denotes, respectively, the axial and transverse membrane displacements and θ is the cross section rotation about the y-axis. The strain-displacement relations can, then, be described as follows:

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon_0 + z\kappa \\ \gamma_{xz} &= w_{,x} - \theta, \end{aligned} \tag{9}$$

where

$$\varepsilon_0 = u_{0,x}, \qquad \kappa = -\theta_{,x}. \tag{10}$$

The strain energy of the beam is expressed as:

$$U = \frac{1}{2}\int_L \int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})\, dA\, dx. \tag{11}$$

Writing it in terms of the generalized strains:

$$U = \frac{1}{2}\int_L (\boldsymbol{\varepsilon}^T\, \mathbf{C}\, \boldsymbol{\varepsilon} + \gamma_{xz}^T\, GA_s\, \gamma_{xz})\, dx, \tag{12}$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix}\varepsilon_0 & \kappa\end{bmatrix}^T, \qquad \mathbf{C} = \begin{bmatrix} A & B \\ B & D\end{bmatrix}, \qquad A_s = \frac{5}{6}\, A, \tag{13}$$

and

$$[A, B, D] = \int_A E\, [1, z, z^2]\, dA. \tag{14}$$

2.4 Isogeometric Formulation

According to Hughes and Bazilevs [4], the generalized displacement **u** of an isogeometric model can be approximated as:

$$\mathbf{u}(\xi) = \sum_{i=1}^{n} N_i(\xi)\, \mathbf{u}_e, \tag{15}$$

where $\mathbf{u}_e = \{\, u_0 \;\, w \;\, \theta\}^T$ is the vector of degrees of freedom at any control point.

Applying Eq. (15) in Eq. (10), the element strains can be written in terms of the generalized strains as follows:

$$\varepsilon_0 = \sum_{i=1}^{n} \mathbf{B}_m\, \mathbf{u}_e, \qquad \kappa = \sum_{i=1}^{n} \mathbf{B}_b\, \mathbf{u}_e, \qquad \gamma_{xz} = \sum_{i=1}^{n} \mathbf{B}_s\, \mathbf{u}_e, \tag{16}$$

where

$$\mathbf{B}_m = [N_{i,x} \;\; 0 \;\; 0], \qquad \mathbf{B}_b = [0 \;\; 0 \;\; -N_{i,x}], \qquad \mathbf{B}_s = [0 \;\; N_{i,x} \;\; -N_i]. \tag{17}$$

Thereby, the global stiffness matrix is given by:

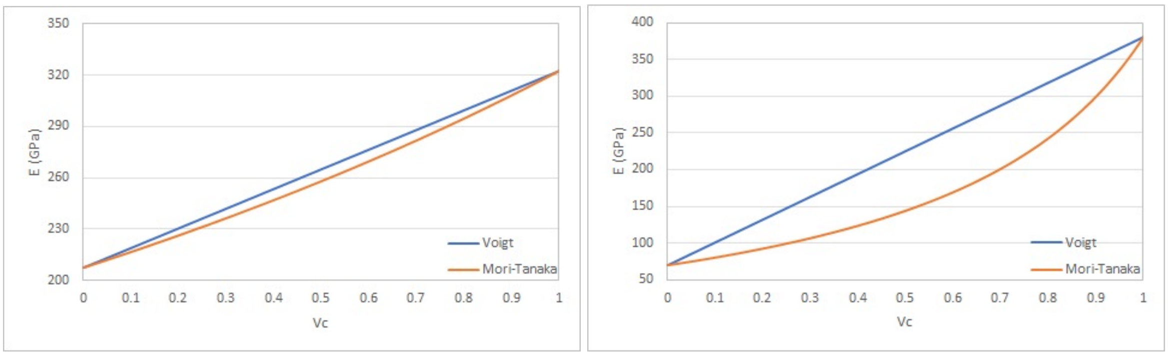
$$\mathbf{K} = \int_L \left(\begin{bmatrix}\mathbf{B}_m & \mathbf{B}_b\end{bmatrix}^T \mathbf{C} \begin{bmatrix}\mathbf{B}_m & \mathbf{B}_b\end{bmatrix} + \mathbf{B}_s^T\, GA_s\, \mathbf{B}_s \right) dx. \tag{18}$$

3 Numerical examples

Two different micromechanical models are used to evaluate the mechanical properties of FGM beams. The effective material properties are compared in Fig. 1, whilst the properties of both phases are given in Table 1. In order to make a thorough study, two different FGMs will be considered.

Table 1. Material properties of FGM

FGM I			FGM II		
Material	E (GPa)	ν	Material	E (GPa)	ν
Si_3N_4 [c]	322.76	0.28	Al_2O_3 [c]	380	0.3
SUS304 [m]	207.89	0.28	Al [m]	70	0.3



(a) FGM I

(b) FGM II

Figure 1. Effective Young's modulus according to the ceramic volume fraction for both micromechanical models

As seen in Fig. 1, given the same ceramic volume fraction, the Voigt technique estimates a higher Young's Modulus, resulting in a stiffer structure. The values obtained by Mori-Tanaka are up to 2.62% less when compared to RoM for FGM I and 36.38% less when compared for FGM II.

A static analysis process of FG beams is performed based on a FSDT. A clamped-free (CF) beam is subjected to a uniform load F = -1 kN/m, comprised of two distinct compositions of FGM stated in Table 1. The length-thickness ratio is L/h = 15 and the length-width ratio is L/b = 16.

In order to support the analysis, IGA based on cubic B-splines basis functions is adopted. The mesh uses 10 control points and the stiffness matrix is evaluated using the Gauss Quadrature with full integration. The non-dimensional displacements are given by the relations $\overline{u}_0 = u_0/w_{CBT}$, $\overline{w} = w/w_{CBT}$ and $\overline{\theta} = \theta/\theta_{CBT}$, where the CBT subscript relates to the Classical Beam Theory for purely ceramic material. The results for FGM I and FGM II are, respectively, shown in Table 2, Table 3 and Fig. 2.

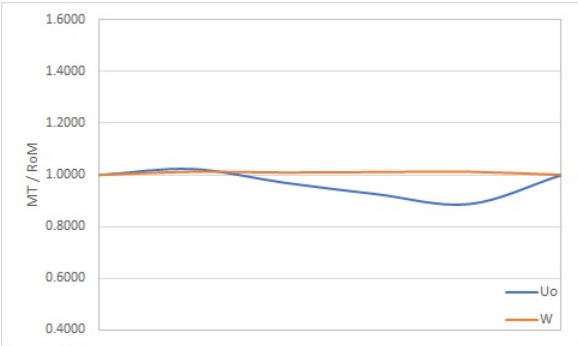
Table 2. Displacements of a cantilever beam - FGM I

	N	0	1	2	5	10	∞
$\overline{u}_0$	RoM	0	0.0040	0.0045	0.0036	0.0025	0
	MT	0	0.0041	0.0043	0.0034	0.0022	0
$\overline{w}$	RoM	1.0046	1.2413	1.3003	1.3612	1.4177	1.5595
	MT	1.0046	1.2549	1.3112	1.3746	1.4332	1.5595
$\overline{\theta}$	RoM	1	1.2357	1.2943	1.3547	1.4109	1.5525
	MT	1	1.2492	1.3051	1.3680	1.4264	1.5525

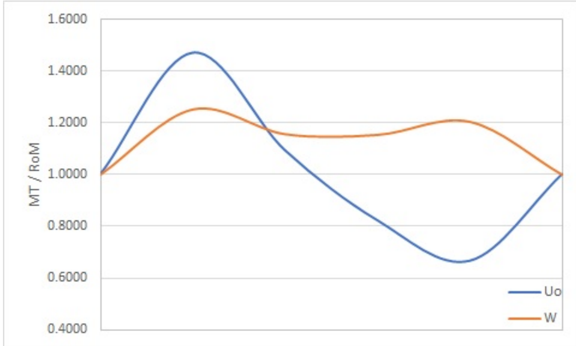
Table 3. Displacements of a cantilever beam - FGM II

	N	0	1	2	5	10	∞
$\overline{u}_0$	RoM	0	0.0205	0.0341	0.0410	0.0355	0
	MT	0	0.0301	0.0372	0.0337	0.0236	0
$\overline{w}$	RoM	1.0046	2.0146	2.5820	3.0555	3.3952	5.4537
	MT	1.0046	2.5212	2.9857	3.5241	4.0872	5.4537
$\overline{\theta}$	RoM	1	2.0063	2.5711	3.0403	3.3766	5.4286
	MT	1	2.5101	2.9716	3.5056	4.0657	5.4286

As seen in Table 2 and Table 3, the displacements using RoM are always smaller for non-isotropic material when compared to Mori-Tanaka, as expected by the behaviour of the Young's Modulus variation through the



(a) FGM I



(b) FGM II

Figure 2. Ratio between Mori Tanaka and Rule of Mixtures displacements

thickness. Also, the maximum deflections from CBT and FSDT for $N = 0$ are different because the CBT disregards the shear strains, only accounting for the bending effects. Nevertheless, the membrane displacement portion shows that the coupling process is very distinguished for both materials, presenting the highest percentage difference. For isotropic materials, this displacement is nonexistent as the center of mass is congruent to the center of gravity, which is not true for FGM.

In addition, even though the results corroborate with the expected behaviour, it is important to remark that, due to the components properties, the difference between the micromechanical models in FGM II can be 20 times greater, for the same N, when compared to FGM I, ratifying the Young's modulus values dissemblance discussed previously.

4 Conclusions

This study utilizes a B-spline isogeometric formulation based on the FDST for analysis of functionally graded beams. Two micromechanical models were employed to evaluate the effective properties and their influence on the FGM behavior. The results show that the appropriate choice of the micromechanical model can be determinant for the analysis outcome. The discrepancy between the micromechanical models results are greatly influenced by the difference between the properties of the FGM components and by the FGM volume fraction profile. Finally, although widely used in the analysis of FG structures, the Rule of Mixtures may overestimate the structural stiffness and underestimate the beam displacements.

Acknowledgements. The financial support from CNPq is gratefully acknowledged.

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References

[1] Shen, H. S., 2009. *Functionally Graded Materials: Nonlinear analysis of Plates and Shells*. Taylor Francis Group, LLC.
[2] Akbarzadeh, A. H., A. A. & Chen, Z. T., 2015. Effect of micromechanical models on structural responses of functionally graded plates. *Composite Structures*, vol. 119, pp. 598–609.
[3] Wang, C. M., R. J. N. & Lee, K. H., 2000. *Shear Deformable Beams and Plates: Relationships with Classical Solutions*. Elsevier Science Ltd.
[4] Hughes, T. J. R., C. J. A. & Bazilevs, Y., 2005. Isogeometric analysis: Cad, finite elements, nurbs, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, vol. 194, pp. 4135–4195.
[5] Piegl, L. & Tiller, W., 1995. *The NURBS Book*. Monographs in Visual Communication. Springer.