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Study of the enriched mixed finite element method using comparisons of computational cost and errors with formulations in continuous and discontinuous functions and mixed scheme on quadrilateral finite elements.

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Abstract. Mixed finite element formulations are used to approximate stress and displacement variables simultaneously for Poisson problems. The purpose of this article is to analyze new discrete mixed approximation based on the application of enriched version of classic Poisson-compatible spaces. With that purpose we decided to measure the computational cost of applying four formulations for two Poisson problems with known exact solution. The first model considers a smooth sinusoidal solution and the second model has a high gradient solution. The objective is not to compare which formulation is better, but rather to highlight characteristics of computational cost and the errors obtained for both the primal and dual variables. Weak formulations correspond to the use of the FEM using continuous and discontinuous functions, using the mixed method and the enrichment mixed method. In the algorithms developed, we computed the error in L^2 and H^1 norms and we measured the computational cost of the assembly and solving processes. When analyzing these costs together with the errors obtained, we visualized that the cost of enriched version is less expensive computationally than non-enriched version, however they getting the same approximation errors.

Keywords: Enriched mixed finite element, Mixed finite element, Galerkin discontinuous, Computational cost

1 Introduction

In applications, many problems are formulated as boundary value problems for second order elliptic differential equations, one simplest form is the Poisson equation. Different finite element formulations have been developed to solving numerically these problems. For the approximation of primal variable a H^1-conforming weak formulation has proven to be efficient. The H^1-conforming weak formulation is based on continuous finite element approximation spaces. Another good option is to use methods which are mainly based on discontinuous approximation spaces, non conforming, called Galerkin discontinuous spaces (GD), Forti [6].

When the accuracy of the flux (dual variable) is the quantity of interest, an approximation by taking the gradient of H^1 or GD approximate solutions leads to lower-order accuracy. An alternative formulation is the dual mixed H(div)-conforming method, Brezzi [1], it is based on simultaneous approximations of the primal and the dual variables. In its formulation the approximation spaces for primal and dual variable are required to be compatible (satisfying the equilibrium condition) to obtain a numerical scheme which is both consistent and stable. Using the H(div)-conforming formulation, Devloo [4] has proposed to enrich the space of the dual variable by increasing the order of the polynomials that form the vector bubble polynomials inside the finite elements. This enrichment means to increase by one the degree of the polynomials used to construct the vector bubble functions inside the elements. In this work, we intend to analyze Devloo's recent proposal numerically. In our analysis we have verified the theoretical convergence rate for smooth solutions.

However, comparing the errors obtained between the mixed method and the enriched mixed method, we observe that the enriched method of order k corresponds to the mixed method of order $k + 1$. Thus we carry out a simple study to determine the orders (levels) of the methods that must be compared. With this, the enriched version has a lower number of dofs and therefore lower computational cost, obtaining practically the same errors.

Motivated by these characteristics, we decided to compare the processing time, assembly process and resolution of the linear system together, with three other formulations. The computational cost of the assembly and solver processes was calculated in clocks ticks. The clock ticks depend on the configuration of the system and the application, therefore, the numerical experiments presented have been carried out on the same computer.

The implemented formulations are: (a) the classical H^1 weak formulation (continuous); (b) the non-symmetric discontinuous Galerkin formulation by Baumann, Oden and Babuska, Baumann [2]; (c) a mixed H(div)-conforming formulation, de Siqueira [3]; (d) a enriched mixed H(div)-conforming formulation, Devloo [4] Farias [5]. Static condensation is always used for continuous, mixed H(div)-conforming and enriched mixed H(div)-conforming formulations. For more details over how the static condensation is applied in the continuous and mixed H(div)-conforming formulations see Forti [6], and in enriched mixed H(div)-conforming formulation see Farias [5].

2 Finite element formulations

Consider the model Poisson problem written in the form

$$-\nabla \cdot (\mathbf{K}\nabla u) = f, \text{ in } \Omega \tag{1}$$

$$u = 0, \text{ on } \quad \partial\Omega, \tag{2}$$

defined in a region $\Omega \subset \mathbf{R}^2$, $\mathbf{K}$ is a symmetric, bounded, and uniformly positive definite matrix, and $f \in L^2(\Omega)$. Let be M_h a mesh of the quadrilateral elements on Ω where N_e is the number of elements. Each quadrilateral element is represented by Ω_e and $Q_p(\Omega_e)$ is a set of hierarquical polynomial functions of degree p on Ω_e. Details over the construction of shape functions see in Devloo [7]

2.1 H^1-conforming formulation

The weak formulation for problem (1)-(2) reads: find $u_h \in H_0^1(\Omega)$ such that

$$(\mathbf{K}\nabla u_h, \nabla v_h)_\Omega = (f, v_h), \ \forall v_h \in H_0^1(\Omega).$$

2.2 Galerkin discontinuous formulation

The discrete version for the GD weak formulation is constructed over the broken polynomial spaces

$$V_p(M_h) = \{v \in L^2(\Omega); v|_{\Omega_e} \in P_p(\Omega_e), \forall \Omega_e \in M_h\}.$$

The discrete weak formulation presented in Baumann [2] for the model problem is: Find $u_h \in V_p(M_h)$ such that

$$\sum_{e=1}^{N_e} \int_{\Omega_e} \nabla u_h \cdot \nabla v_h d\Omega_e + \int_{\partial\Omega} (u_h \nabla v_h \cdot \mathbf{n} - v_h \nabla u_h \cdot \mathbf{n}) ds +$$

$$+ \int_{\Gamma_i} (< \nabla v_h \cdot \mathbf{n} > [u_h] - < \nabla u_h \cdot \mathbf{n} > [v_h]) ds = \sum_{e=1}^{N_e} \int_{\Omega_e} f v_h d\Omega_e$$

for any $v_h \in V_p(M_h)$. Γ_i is the union of all interelement boundaries called interfaces. Each interface Γ_{rl} is a common boundary (codimension 1) between two quadrilateral elements Ω_{e_r} (right element) and Ω_{e_l} (left element). The operators are: $< \nabla v_h \cdot \mathbf{n} >= \frac{1}{2} (\nabla v_{e_r} \cdot \mathbf{n}_{e_r} + \nabla v_{e_l} \cdot \mathbf{n}_{e_l})$ and $[v_h] = \Big(v_h|_{\partial\Omega_{e_r} \cap \Gamma_{rl}} - v_h|_{\partial\Omega_{e_l} \cap \Gamma_{rl}} \Big)$.
$\mathbf{n_e}$ refers to the outward unit normal of the boundary $\partial\Omega_e$.

2.3 H(div)-conforming mixed finite element formulation

Given finite-dimensional approximation spaces $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, $U_h \subset L^2(\Omega)$, consider the discrete mixed formulation for the model problem: Find $\sigma_h \in \mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$, $u_h \in U_h \subset L^2(\Omega)$ satisfying

$$(\sigma_h, \mathbf{q})_\Omega - (u_h, \nabla \cdot \mathbf{q})_\Omega = 0, \ \forall \mathbf{q} \in \mathbf{V}_h,$$

$$-(\nabla \cdot \sigma_h, v)_\Omega + (f, v)_\Omega = 0, \ \forall v \in U_h.$$

For $k \in \mathbf{N}$, vector and scalar polynomial approximations spaces $\hat{\mathbf{V}}_k$ and $\hat{U}_k$ are defined in master element $\hat{K}$, which are assumed to be divergence compatible: $\nabla \cdot \hat{\mathbf{V}}_k \equiv \hat{U}_k$. To compose $\mathbf{V}_h$ and U_h, the order of the polynomials in $\hat{\mathbf{V}}_k$ is $p = k + 1$ when in $\hat{\mathbf{U}}_k$ is $p = k$.

2.4 Enriched mixed finite element formulation

Consider two divergence compatible spaces, a vector polynomial space $\hat{\mathbf{V}}_k$ and a scalar polynomial space $\hat{U}_k$, defined in $\hat{K}$ as in 2.3, $k \in \mathbf{N}$. Suppose that a direct decomposition $\hat{\mathbf{V}}_k = \hat{\mathbf{V}}_k^{\partial} \oplus \mathring{\hat{\mathbf{V}}}_k$ holds, where $\mathring{\hat{\mathbf{V}}}_k$ indicates the flux functions with vanishing normal components over $\partial\hat{K}$. Otherwise, the functions in $\hat{\mathbf{V}}_k^{\partial}$ are assumed to have normal components over $\partial\hat{K}$ of degree k.
Under these conditions, a enriched version $\hat{\mathbf{V}}_k^{+}$, is defined by adding to $\hat{\mathbf{V}}_k$ higher degree internal shape functions of the original space at level $k+1$, while keeping the original border fluxes at level k, $\hat{\mathbf{V}}_k^{+} = \hat{\mathbf{V}}_k^{\partial} \oplus \mathring{\hat{\mathbf{V}}}_{k+1}$. The corresponding enriched potential spaces are now $\hat{U}_k^{+} = \nabla \cdot \hat{\mathbf{V}}_k^{+} = \hat{U}_{k+1}$. Then, the approximation spaces $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, and $U_h \subset L^2(\Omega)$ are defined in terms of local space configurations $\{\mathbf{V}_k^{+}(K), U_k^{+}(K)\}$, backtracked from $\{\hat{\mathbf{V}}_k^{+}, \hat{U}_k^{+}\}$ by adequated transformations. For more details, see Castro [8], Farias [5].

3 Numerical results: Poisson problem models

3.1 Poisson problem with sinusoidal exact solution

Consider the Poisson problem (1)-(2) with exact solution:

$$u_{exact} = Sin\,(\pi x)\,Sin\,(\pi y)\,, \tag{3}$$

and choose $f = -\Delta u_{exact}$, $\Omega = [-1, 1]^2$, and Dirichlet boundary conditions accordingly.

The exact solution is showed in “Figure 1a)”.

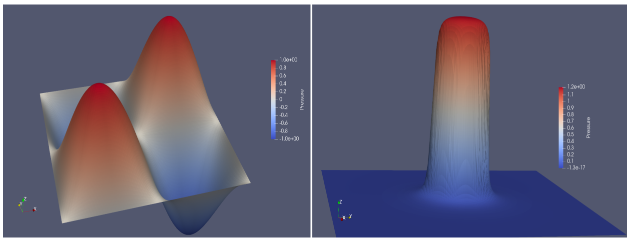

Figure 1. a) The exact solution, Equation (3) (left). b) The exact solution with high gradient, Equation (4) (right).

The model 3.1 is used to verify the characteristics of the enriched mixed method. The first verification carried out is on the behavior of the convergence of the method when we increase the order, from $p = 2$ to $p = 7$. According to the theory, the convergence is presented in the figure ”Figure 2”.

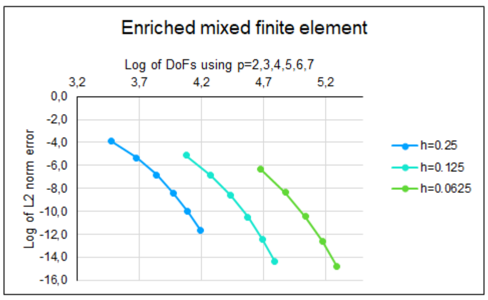

Figure 2. Enriched mixed method: Log(Error) vs Log(CDoFs) - three uniform h-refinements, varying $p = 2, ..., 7$.

It was verified that the mixed method of order k and the enriched mixed method of order k have the same number of condensed degrees of freedom. Also, it was verified that the rate of convergence of the enriched mixed is equal to $k+1$ while the mixed is equal to k, see 1.
From this result, we suspect that the comparison of the enriched mixed method of order k and the mixed method of order k are not equivalent. We observe that the enriched mixed method uses order $k+1$ to construct $\hat{U}_k^{+}$ and $\hat{\mathbf{V}}_k^{\partial}$, and it uses order $k+2$ to construct $\mathring{\hat{\mathbf{V}}}_{k+1}$. This means that the equivalent version of the enriched mixed method of order k is the mixed method of order $k+1$. Based on this conclusion, we decided to also compare the mixed

Table 1. Validation of the rate of convergence of the error in L^2 norm, $p = 2$.

h	Enriched Mixed		Mixed		Galerkin discontinuous		H1	
	Error $\|\|.\|\|_{L^2}$	Rate of converg.	Error $\|\|.\|\|_{L^2}$	Rate of converg.	Error $\|\|.\|\|_{L^2}$	Rate of converg.	Error $\|\|.\|\|_{L^2}$	Rate of converg.
0.25	1.24E-04		2.14E-03		7.54E-02		3.86E-03	
0.125	7.89E-06	3.981	2.69E-04	2.992	2.35E-02	1.680	4.90E-04	2.979
0.0625	4.95E-07	3.995	3.37E-05	2.998	6.28E-03	1.905	6.15E-05	2.995
0.03125	3.10E-08	3.999	4.21E-06	2.999	1.60E-08	1.975	7.69E-06	2.999

method of order $k + 1$ with the classic FEM of order $k + 2$ for a not very smooth model, but without reaching discontinuity. We consider a model with a high gradient in the next section.

3.2 Poisson problem with high gradient in the exact solution

Consider the Poisson problem (1)-(2) with exact solution:

$$u_{exact} = 0.4 \left[\frac{\pi}{2} - \arctan \left[10 \left(1 - 20(x^2 + y^2) \right) \right] \right], \tag{4}$$

and choose $f = -\Delta u_{exact}$, $\Omega = [-1, 1]^2$, and Dirichlet boundary conditions accordingly. Strong gradients in the proximity to the circumference centered at the origin, with radius $\frac{1}{4}$. The exact solution is showed in "Figure 1b)"

Table 2. Error in L^2 norm for model with high gradient. $h = 0.0625$

k	Enriched Mixed $p = k$	Mixed $p = k + 1$	GD $p = k + 2$	$H1$ $p = k + 2$
2	0.888524	0.888524	0.896943	0.888532
3	0.375360	0.375360	0.378554	0.375351
4	0.263106	0.263106	0.261382	0.263105
5	0.114205	0.114205	0.113495	0.114202
6	0.053405	0.053405	0.053905	0.053403
7	0.021689	0.021689	0.021839	0.021687

Table 2 shows the errors in the L^2 norm obtained for the four formulations for the high gradient model. Observe how close the values are when comparing with the corresponding order for the classical FEM method, the mixed method and the enriched mixed method. The errors of mixed method and enriched mixed method are practically the same (they differ in millionths). Only the discontinuous Galerkin formulation has errors that are slightly different from the other methods and with less approximation.

Table 3. Error in $H1$ norm for model with high gradient. $h = 0.0625$

k	Enriched Mixed $p = k$	Mixed $p = k + 1$	GD $p = k + 2$	$H1$ $p = k + 2$
2	2.66244	2.65971	2.98315	2.90992
3	1.35876	1.35681	1.47133	1.44733
4	0.881271	0.880457	0.981287	0.97537
5	0.470638	0.469961	0.523247	0.514667
6	0.272517	0.272356	0.318146	0.310241
7	0.164937	0.164794	0.195961	0.191546

Table 3 shows the errors in the H^1 norm for the high gradient model. Note that the approximation of the mixed methods in terms of the H^1 norm are very close. The mixed method approach is the best. The classical FEM method is no longer as approximate as the mixed ones and the discontinuous Galerkin is the least accurate.

Table 4. Computational cost comparing with the time processing of the mixed method

k	Mixed ($p = k + 1$) clock ticks	Enriched Mixed ($p = k$) percent over Mixed	GD ($p = k + 2$) percent over Mixed	$H1$ ($p = k + 2$) percent over Mixed
2	36716408	70%	191%	15%
3	61778717	76%	298%	18%
4	103661465	82%	456%	21%
5	184144380	84%	530%	23%
6	317469280	87%	568%	24%
7	536448242	90%	592%	25%

Table 4 comparatively presents the computational cost (clock tickts) of the enriched mixed methods, discontinuous Galerkin and classical FEM relative to the time used by the mixed method. We observed that the enriched mixed method was up to 70% smaller than the mixed method, and yet the errors obtained were very similar (Table 3). The discontinuous Galerkin method has the worst performance (static condensation is not useful).

4 Conclusions

The enriched mixed method was verified to satisfy the theoretical properties of convergence. The enriched mixed method of order k has been shown to achieve an approximation equivalent to that of the mixed method of order $k + 1$. However, its computational cost is up to 30% lower, since it considers fewer degrees of freedom because it has fewer vector functions associated with the one-dimensional sides of the master element. Certainly, the classical finite element method has a much lower computational cost, even considering order $k + 2$ or greater and a better approximation in norm L^2 of the primal variable.

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