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Simulation-based structural reliability applied to fracture mechanics

Hugo V. F. Azevedo[1], Eduardo T. de Lima Junior[1]

[1]*Laboratory of Scientific Computing and Visualization, Federal University of Alagoas*
Campus A. C. Simões, Tabuleiro dos Martins, 57072-900, Maceió, Alagoas, Brasil
hugo.azevedo@lccv.ufal.br, limajunior@lccv.ufal.br

Abstract. Monte Carlo simulation techniques, applied to structural reliability analysis, have always allowed the solution of complex, large and non-linear problems. However, these techniques demands a high computational cost in problems presenting small probability of failure. In this context, intelligent sampling techniques are used to reduce the number of simulations needed to solve structural problems, reducing the processing time. This paper addresses a study on different combinations of sampling strategies, such as the Latin Hypercube Sampling, Antithetic Variates Sampling, Asymptotic Sampling and Enhanced Sampling, all applied within the Monte Carlo technique. The models are applied to benchmark problems in Linear Elastic Fracture Mechanics, more specifically those ones presenting analytical solution. The advantages and limitations of each method are discussed based on the accuracy of the probabilistic response and the associated computational cost.

Keywords: Linear Elastic Fracture Mechanics, Monte Carlo simulation, Asymptotic Sampling, Enhanced Sampling, Latin Hypercube Sampling.

1 Introduction

Crack propagation analysis is a relevant field of study in structural analysis, being the Linear Elastic Fracture Mechanics (LEFM) a possible approach (Broek [1]). Recent studies address the subject in the light of a probabilistic modeling (Leonel [2]; Huang and Aliabadi [3]; Chocat et al. [4]; Krejsa et al. [5]; Kala [6]), from the theory of structural reliability, to evaluate how the uncertainties inherent to the problem - such as related to crack length and to fracture toughness - affect the integrity of the structure.

Simulation techniques - as the Monte Carlo simulation (MCS) - allow the solution of complex, large and non-linear problems in structural reliability, although the high computational cost associated can become a serious drawback (Santos [7]). MCS is a powerful technique in estimating the probability of failure (P_f) of structural systems, and several intelligent sampling techniques are proposed to reduce the number of simulations and the computational cost (Olsson et al. [8]; Bucher [9]; Naess et al. [10, 11]).

Hence, this article aims to analyze the use of Crude Monte Carlo and two modern sampling techniques - Asymptotic and Enhanced Sampling - over two problems in LEFM. Besides that, the underlying samples required in these techniques are generated using: Simple, Latin Hypercube and Antithetic Variates Sampling. All implementations were made within *Python* programming language.

2 Methodology

According to Barbirato [12], the LEFM is applicable to a series of practical cases to analyze cracking in linear elastic problems. In its formulation, the stress intensity factor, K (parameter that quantifies the magnitude of the stress field at the crack tip), is established for a several loading and geometry configurations. Knowing the intensity factor and the mode of propagation of these cracks is of great interest, since they can lead to the degeneration of the resistance of the structure. The relationship between K and the stress state at the crack tip is shown in eq. (1) (Irwin [13]).

$$\sigma_{ij} = \frac{K}{\sqrt{2 \cdot \pi \cdot r}} \cdot f_{ij}(\theta) \text{ with } K_I = \sigma\sqrt{\pi a} \tag{1}$$

being r the distance between the crack tip and the point considered, θ the mapping angle from the reference point, K_I the stress intensity factor for the mode I (opening) and a the size of the crack. For the particular case of mode I, $K_c = K_I$ and σ is the stress at the remote boundary.

The Theory of Structural Reliability stands out as an important tool to study the durability and safety of products. Within a probabilistic approach, it is possible to verify the probability of failure (P_f) of a structure to a certain limit state, treating the uncertainties of the problem from the statistical description of the design variables. For more details see Melchers and Beck [14].

The MCS is one of the simulation techniques that involves the use of random numbers to generate random events to be evaluated based on an experiment. In the context of structural reliability, a N size random sample is generated for each variable, and the limit state equation is evaluated for the several sets of values. For this, an indicator function $I[\mathbf{x}]$ is introduced, so that $I[\mathbf{x}] = 1$ if $\mathbf{x}$ belongs to the failure domain and $I[\mathbf{x}] = 0$ if the point lies on the safe domain. In this sense, the P_f estimate $(\bar{P}_f)$ can be calculated using an estimator for the expected value of $I[\mathbf{x}]$, defined such that:

$$\bar{P}_f = \frac{1}{N}\sum_{i=1}^{N} I[\mathbf{x}] = \frac{N_f}{N} \tag{2}$$

where x is the vector of random variables (r.v.) of the problem and N_f is the number of times that the random sample generated corresponds to a point on the failure domain Ω_f. Besides that, a parameter called reliability index (β), geometrically defined as the distance between the origin of the reduced normal space and the design point (point of higher probability of occurrence in Ω_f), is also defined. The standard normal cumulative distribution function Φ allows to relate P_f and β, as can be seen in eq. (3).

$$\beta = \Phi^{-1}(1 - P_f) \tag{3}$$

The MCS requires the generation of samples from r.v. vector $\mathbf{x}$. The first method used in this generation is the Simple Sampling (SS). It consists of the direct application of the MCS procedure, with the most basic type of random number generation for each r.v. of the vector $\mathbf{x}$. Usual SS algorithms generate random numbers between 0 and 1, then obtain the samples then obtain the samples by inverting the cumulative distribution function $F_X(x)$ of the defined random variables.

The second method is using the Latin Hypercube Sampling (LHS). The domain of each r.v. is divided in bands (Olsson et al. [8]). In the simulation each band must be sampled only once so that a homogeneous coverage of the domain of the random variables is guaranteed. This type of sampling results in a more sparse dispersion of the points, reducing considerably the number of points required to cover the entire domain of the r.v.

At last, the third method is called Antithetic Variates Sampling (AVS). This method is based on generating the simulation of two sets of random numbers, the first being $U = \{u_1, u_2, \ldots, u_n\}$; the second being a complementary set $\bar{U} = \{1 - u_1, 1 - u_2, \ldots, 1 - u_n\}$. For structural reliability, the method is based on generating a non-biased estimator $P_{f;c}$, formed by the arithmetic mean of two other non-biased estimators $P_{f;a}$ and $P_{f;b}$, by making $P_{f;a} = f(u_i)$ and $P_{f;b} = f(1 - u_i)$, with $i = 1, \ldots, n$. It is worth mentioning that the variance of $P_{f;c}$ is smaller than the combined variance of $P_{f;a}$ and $P_{f;b}$, because the correlation between U and $\bar{U}$ is negative.

For the intelligent sampling techniques, Asymptotic Sampling (AS) and Enhanced Sampling (ES) strategies are addressed in this paper. The AS was introduced by Bucher [9] to estimate low probabilities of failure. It is based on the asymptotic behavior of the P_f, as the standard deviations of random variables and, consequently, the P_f, tends to zero (Sichani et al. [15, 16]).

The idea of the technique is to cause an excitement in the failure function analysis, forcing the MCS to overcome the safety domain barrier. In its formulation, a f factor is initially introduced that is inversely proportional to the standard deviation σ of the r.v. of interest, $f = \frac{1}{\sigma}$. Next, it is known that the β becomes linearly proportional to factor f, as the dispersion of vector $\mathbf{X}$ and β itself increases. Following Bucher [9], the functional relationship between β and f is approximated from:

$$\beta = Af + \frac{B}{f} \tag{4}$$

where A and B are constants computed by using a usual fitting technique (least squares, for example), using (f, β) support points. Therefore, as $f \to \infty$, the asymptotic behavior is guaranteed. Finally, after finding the values of the regression coefficients A and B, the reliability index for the original problem is estimated by taking $f = 1$, so that $\beta = A + B$.

Despite the benefits proposed by AS, it should be observed the possibility of it presenting inconsistencies, since the modification of the standard deviation of random variables creates the possibility of simulating values not consistent with the physical properties of the variables and often this leads to errors in estimating the probability of failure of each support point (Santos [7]).

Proposed by Naess et al. [10, 11], the ES technique aims to reduce the computational cost of MCS while maintaining its main advantages, from the exploration of the regularity of the probabilities of failure in the tails of probability distributions, so that it is possible to use an approximation procedure to estimate small probabilities of failure. The formulation of the technique takes the original limit state function $M = g(X)$ to create a parameterized function class, defined as $M(\lambda) = M - \mu_M(1 - \lambda)$, where the parameter λ satisfies $0 \leq \lambda \leq 1$ and μ_M is the mean value of the safety margin function M. Therefore, the relationship between the probability of failure and λ is assumed as:

$$P_f(\lambda) \approx q \cdot exp\left[-a\left(\lambda - b\right)^c\right] \text{ with } \lambda \to 1 \tag{5}$$

where q, a, b and c are constants to be determined, with some non-linear regression method using the pair $[\lambda, p_f(\lambda)]$. Hence, the value of P_f of the problem is estimated by taking $\lambda = 1$, so that $P_f = P_f(1)$.

It should be noted that the choice of the interval for f and λ depends on each problem. One should test all possible values of these parameters and see which interval ensures convergence. The choice of how many support points also depends on the nature of each problem.

3 Application Analysis

To apply the techniques presented herein, two benchmark LEFM problems (Fig. 1) are addressed. The first example (a) deals with a plate under tension with central crack and the second (b) deals with a three-point bending test of a notched beam. The limit state equations for both problems are given by eq. (6) and eq. (7), respectively. All simulations were made on a computer Intel®Core^{TM}i7-7500U CPU 2.90 GHz 64 bits with 8 GB of RAM.

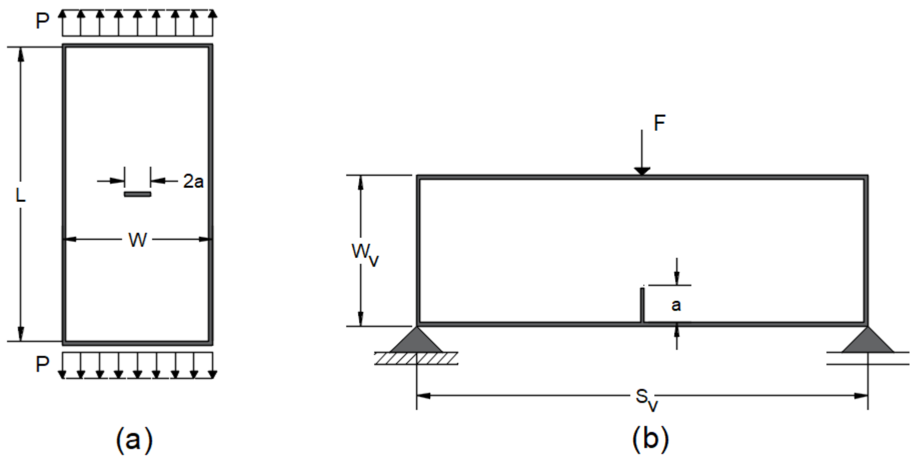

Figure 1. (a) Center-cracked plate under tension; (b) notched beam under three-point bending test

Both limit state equations are written in terms of the applied external load with stress intensity factor K_c. The relation between K_c and applied load for each case considered is defined by Broek [1].

$$G_{(a)} = \frac{K_c}{\sqrt{\pi \cdot a} \cdot \left[1 + 0.256 \cdot \left(\frac{a}{W}\right) - 1.152 \cdot \left(\frac{a}{W}\right)^2 + 12.2 \cdot \left(\frac{a}{W}\right)^3\right]} - P \tag{6}$$

$$G_{(b)} = \frac{K_c \cdot 2 \cdot \left(1 + 2 \cdot \frac{a}{W_v}\right) \cdot \left(1 - \frac{a}{W_v}\right)^{3/2} \cdot B_v \cdot W_v^{3/2}}{S_v \cdot 3 \cdot \left(\frac{a}{W_v}\right)^{1/2} \cdot \left[1.99 - \frac{a}{W_v} \cdot \left(1 - \frac{a}{W_v}\right) \cdot \left(2.15 - 3.93 \cdot \left(\frac{a}{W_v}\right) + 2.7 \cdot \left(\frac{a}{W_v}\right)^2\right)\right]} - F \tag{7}$$

For the first problem (a), $W = 4,0\ m$ is the width of the plate; $2a \sim N(0.1; 0.03)\ m$ the initial crack extension; $P = 50.0\ kN/m$ the applied surface force and $K_c \sim N(50.0; 15.0)\ kN/m^{3/2}$ the critical stress intensity factor. For the second problem (b), $S_v = 5\ m$ is the beam span; $W_v = 1.25\ m$ the height of the beam; $a \sim LN(0.1; 0.02)\ m$ the initial crack extension; $B_v = 1$ the domain thickness; $F = 115\ kN$ the value of the concentrated load and $K_c \sim LN(500.0; 100.0)\ kN/m^{3/2}$ the critical stress intensity factor. It is noteworthy that the $N(\mu, \sigma^2)$ and $LN(\mu, \sigma^2)$ notations represent, respectively, r.v. that have the Normal and Log-Normal distributions, with mean μ and variance σ^2.

The results are described in terms of the reliability index, obtained by the different techniques presented. The reference value $\beta_{ref;1} = 1.966$ for the first problem can be seen in Leonel [2]. For the second problem, the reference value $\beta_{ref;2} = 2.070$ is taken from a MCS analysis, with $N = 5 \cdot 10^6$. For the associated computational cost, the total processing time (PT) is computed.

The values obtained are shown in Table 1. For the Simple MCS, the values of β are obtained for $N = 1 \cdot 10^6$ for both problems. For the ES, the parameter λ varies from 0.7 to 1, with 10 support points for both problems. The parameter f for the AS varies from 0.6 to 0.9, with 8 support points for the first problem, and from 0.8 to 0.9, with 12 support points for the second. For both AS and ES, $N = 3000$.

Table 1. Reliability analysis results for both problems

Estimation Technique		β	Error	Time (s)	β	Error	Time (s)
			Problem 1			Problem 2	
Simple Sampling	Crude	1.984	0.915%	608.424	2.068	0.096%	1290.329
	Asymptotic	1.991	1.271%	11.385	2.079	0.434%	18.318
	Enhanced	1.985	0.966%	5.852	2.061	0.434%	79.057
Latin Hypercube Sampling	Crude	1.983	0.864%	1022.292	2.068	0.096%	1133.089
	Asymptotic	1.968	0.101%	27.977	2.691	30%	37.295
	Enhanced	1.969	0.152%	13.180	2.061	0.434%	54.381
Antithetic Variates Sampling	Crude	1.987	1.068%	1950.818	2.072	0.096%	2088.017
	Asymptotic	1.998	1.627%	22.395	2.791	34.830%	43.962
	Enhanced	1.977	0.559%	60.800	2.054	0.772%	59.557

From the analysis of the first problem, it can be seen that all intelligent sampling techniques performed properly. In general, the ES showed the lowest percentage errors, such that none of them was equal to or greater than 1%, and the lowest PT using both SS and LHS. Moreover, the largest error observed refers to the AS using AVS to generate the samples, which is nevertheless a small error and demonstrates the accuracy of the method for a small number of simulations made.

For the second problem, it is observed that the use of Crude MCS provides a good convergence and does not present a difference for the three sampling methods, since the relative error is the same for the three analyses. The AS technique associated to SS leads to good results, with a percentage error of 0.434%, but generating the samples by the LHS and AVS, the accuracy is significantly reduced, with errors of 30% and 34.83%. For the ES, all sample generation methods show good convergence, and the result using SS was the same as the result using LHS, both providing a small percentage error around 0.434%. In terms of PT, AS performed better using SS, but since LSH and AVS had unsatisfactory results in the combination, in general ES worked properly with lower error and time.

In summary, the AS did not provide consistent results for the analysis of the second problem, using both LHS and AVS to generate the underlying samples. This fact can be explained by the limitation exposed by Santos [7], since the modification of the standard deviation of random variables allows the generation of samples not consistent with the analyzed problem.

4 Conclusions

This paper presents a combination of new methodologies for the structural reliability analysis of two problems of LEFM. It is observed that the generation of samples by the LHS technique had a better convergence in most cases. Sampling by AVS had a good performance, but SS performed better than AVS in almost all cases. Therefore, it can be said that all three techniques, combined with intelligent sampling techniques, are good methods for analyzing structural reliability problems with fewer simulations and processing time required, being the difference between ES, AS and Crude MCS very notorious.

Regarding the intelligent sampling techniques, the ES showed the best results for both problems analyzed, with the lowest percentage errors obtained and being faster (in terms of PT) in the combination ES+SS, ES+LHS for problem 1 and ES+LHS and ES+AVS in problem 2. The AS proved to be a good technique for the first example with the lowest PT in AS+AVS, but presented pronounced errors in the second problem analyzed. It has to be cited that the ES has an advantage over the AS, since it uses the same set of underlying samples in the analysis of the supporting points, because the parametrization is done in the limit state function and not in the r.v. of the problem. This allows ES to use larger samples in the analysis of each support point without computational penalty.

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References

[1] Broek, D., 1982. *Elementary engineering fracture mechanics*. Springer Netherlands, Dordrecht.

[2] Leonel, E. D., 2009. *Modelos Não Lineares do Método dos Elementos de Contorno para Análise de Problemas de Fratura e Aplicação de Modelos de Confiabilidade e Otimização em Estruturas Submetidas à Fadiga*. PhD thesis, Universidade de São Paulo.

[3] Huang, X. & Aliabadi, M. H., 2011. Probabilistic fracture mechanics by the boundary element method. *Int. J. Fract.*, vol. 171, n. 1, pp. 51–64.

[4] Chocat, R., Beaucaire, P., Debeugny, L., Lefebvre, J.-p., Sainvitu, C., Breitkopf, P., & Wyart, E., 2016. Reliability Analysis in Fracture Mechanics According. In *VII Eur. Congr. Comput. Methods Appl. Sci. Eng.*, number June, Crete Island.

[5] Krejsa, M., Koubova, L., Flodr, J., Protivinsky, J., & Nguyen, Q. T., 2017. Probabilistic prediction of fatigue damage based on linear fracture mechanics. *Frat. ed Integrita Strutt.*, vol. 11, n. 39, pp. 143–159.

[6] Kala, Z., 2019. Global sensitivity analysis of reliability of structural bridge system. *Eng. Struct.*, vol. 194, n. November 2018, pp. 36–45.

[7] Santos, K. R. M., 2014. *Técnicas de amostragem inteligente em simulação de Monte Carlo*. Dissertação de Mestrado - Universidade de São Paulo, São Carlos.

[8] Olsson, A., Sandberg, G., & Dahlblom, O., 2003. On Latin hypercube sampling for structural reliability analysis. *Struct. Saf.*, vol. 25, n. 1, pp. 47–68.

[9] Bucher, C., 2009. Asymptotic sampling for high-dimensional reliability analysis. *Probabilistic Eng. Mech.*, vol. 24, n. 4, pp. 504–510.

[10] Naess, A., Leira, B. J., & Batsevych, O., 2009. System reliability analysis by enhanced Monte Carlo simulation. *Struct. Saf.*, vol. 31, n. 5, pp. 349–355.

[11] Naess, A., Maes, M., & Dann, M. R., 2013. Enhanced Monte Carlo for Reliability-Based Design and Calibration. *Comput. Methods Appl. Sci.*, vol. 26, n. May, pp. 149–159.

[12] Barbirato, J. C. C., 1999. Método dos elementos de contorno com a reciprocidade dual para a análise transiente tridimensional da mecânica do fraturamento. *São Carlos. Tese (Doutorado)-Escola de Engenharia de São Carlos-Universidade de São Paulo.*

[13] Irwin, G. R., 1957. Analysis of stresses and strains near the end of a crack transversing a plate. *Trans. ASME, Ser. E, J. Appl. Mech.*, vol. 24, pp. 361–364.

[14] Melchers, R. E. & Beck, A. T., 2018. *Structural Reliability Analysis and Prediction*. John Wiley & Sons Ltd, third edit edition.

[15] Sichani, M. T., Nielsen, S. R., & Bucher, C., 2011a. Applications of asymptotic sampling on high dimensional structural dynamic problems. *Struct. Saf.*, vol. 33, n. 4-5, pp. 305–316.

[16] Sichani, M. T., Nielsen, S. R., & Bucher, C., 2011b. Efficient estimation of first passage probability of high-dimensional nonlinear systems. *Probabilistic Eng. Mech.*, vol. 26, n. 4, pp. 539–549.