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Strong Stability Preserving Runge-Kutta Methods Applied to Advection-Diffusion Problem

Marlon Mesquita Lopes Cabreira[1] and Renan de S. Teixeira[1]

[1]*Dept. of Mathematics, Federal Rural University of Rio de Janeiro*
BR 465, Km 7, 23897-000, Seropédica/Rio de Janeiro, Brazil
marlonmlc23@gmail.com, rsteixeira@ufrrj.br

Abstract. Heat transfer phenomena are related to several applications in different scientific branches. Advection-diffusion problems with spatial properties variation describe the wave development associated with a physical change in heterogeneous media or multi-layered materials, such as laminated media or conjugated problems. Furthermore, an incompressible thermally developing laminar flow with hydrodynamically developed fluid, known as the Graetz problem, is a typical example of this phenomenon . However, the fluid property variation imposes some mathematical challenges to predict accurate solutions. The Strong Stability Preserving (SSP) methods are capable of numerical schemes to increase their accuracy order while maintaining the original stability properties from their generator Euler method. This methodology is developed basically by rewriting the multistage scheme as a combination of steps in the Euler method and introducing a time step barrier. On the other hand, the Generalized Integral Transformation Technique (GITT) is a hybrid numerical-analytical approach that provides an infinite system of coupled ordinary differential equations (ODE) that needs to be solved numerically by truncating the expansion. Indeed, GITT consists of this solution procedure characterized by a hybrid analytical-numerical nature. Therefore, the present work studies the SSP Runge-Kutta schemes compared to the GITT to solve the Graetz problem in the absence of the axial diffusion.

Keywords: Graetz Problem, Strong Stability Preserving methods, Explicit Runge-Kutta schemes, Generalized Integral Transform Technique.

1 Introduction

The cooling process in heat exchangers is a recurrent topic in scientific and industrial problems. These phenomena have a vast range of applications that encompass the cooling of electronic devices and turbines [1, 2]. The Graetz model is a set of partial differential equations that describe the heat transfer in a laminar flow between parallel plates [2]. This formulation allows evaluating the heat transfer in several engineering devices.

Generally, the non-linear nature of diffusion-advection mathematical models narrows the employment of the classical analytical solution [3–5]. Therefore, discrete numerical schemes as Finite Difference Methods (FDM), Finite Volume Methods (FVM), and Finite Element Methods (FEM) are widely adopted. However, numerical instabilities and low-order temporal schemer require a mesh refinement, but the computational cost increases substantially. Thus, it is highly desirable to control the accuracy while maintaining the numerical stability properties and avoiding a prohibitive computational cost [6]. The Strong Stability Preserving (SSP) methods are an appropriate methodology to raise the numerical order keeping the stability [7].

In Cabreira et al. [2] comparisons were made between solution profiles for the Graetz problem of discrete methods and numerical-analytical techniques, showing that the results were satisfactory despite the considerable computational cost. The first formulation presented for temporal discretization of the SSP method was developed by Shu Osher in 1998. Inspired by the concepts of decreased total variation. In this work written by Shu, second to fifth order Runge-Kutta methods were implemented. In Gottlieb et al. [7], Shu, Gottlieb and Ketcheson made a study of Runge-Kutta methods, where they presented optimal parameters for the second and third order methods of two and three stages respectively and, furthermore, proved that it is not possible to obtain a four-stage, fourth-order method with non-negative coefficients. In the work written by Moraes [6] it shows a comparative study of explicit, implicit and implicit SSP Runge-Kutta methods, in addition to the numerical barrier study for the implemented methods, in addition to the computational cost study of SSP methods. The same result showed that the cost decreased when implementing SSP methods.

The main objective of this work is to study the Strong Stability Preserving Runge-Kutta (SSP Runge-Kutta) method in Graetz's problem. Studies of stability and decay of the approximation error will be carried out. Strong Stability Preserving methods are methods of high temporal order and capable of preserving non-linear numerical stability. As the solution profiles generated by hyperbolic partial differential equations tend to have abrupt variations in properties and discontinuities, the SSP method becomes a great implementation option [7]. Moreover, the numerical results are compared with the solutions provided by numerical-analytical method called Generalized Integral Transformation (GITT) [8, 9].

2 Mathematical Model

The dimensionless differential model for the problem of forced convection between parallel plates of an incompressible laminar flow developed hydrodynamically and developing thermally has interesting hypoteses, such as the predominance of viscous forms, that is, absence of mixture in the fluid; the specific mass is constant and the coefficient of thermal expansion is zero; as the flow is hydrodynamically developed, the transversal component of the velocity is zero. In addition, there is no power generation or heating by viscous dissipation during the process. Therefore, we have the following formulation:

$$u(y)\frac{\partial T}{\partial x} = \alpha(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}), \qquad \text{where} \quad 0 \leq y \leq \frac{H}{2} \qquad \text{and} \quad 0 \leq x \tag{1}$$

with the following boundary conditions:

$$(\frac{\partial T}{\partial y})_{y=0} = 0, \qquad T(0,y) = T_0, \qquad (\frac{\partial T}{\partial x})_{x\to\infty} = 0 \tag{2}$$

where α is the thermal diffusivity, T_0 and T_s are constants that represent the inlet temperature in the duct and temperature in the duct wall, H is the distance between the parallel plates, but since the flow is laminar, the domain in the transversal direction limited to $0 \leq y \leq \frac{H}{2}$.

The Figure 1 the representation of the temperature profile for a laminar flow between parallel plates with a constant temperature at the beginning of the flow, and with a diffusive characteristic along the duct.

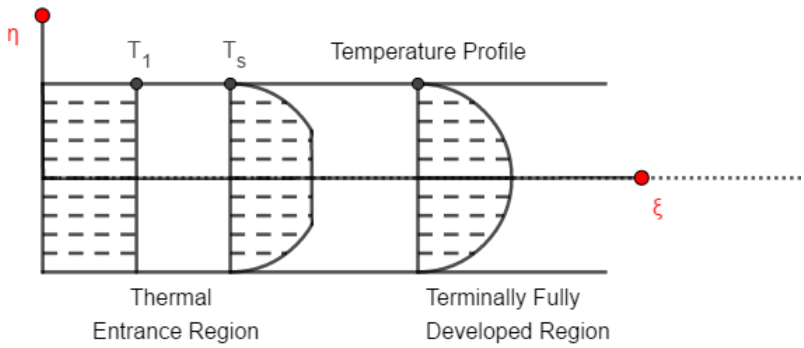

Figure 1. Temperature profile between parallel plates.

The following transformations is performed in the variables:

$$\eta = \frac{y}{\frac{h}{2}}, \quad \xi = \frac{x}{L}, \quad \theta = \frac{T(x,y) - T_s}{T_o - T_s}, \quad u^*(\eta) = \frac{u(y)}{\overline{u}} \tag{3}$$

The velocity profile of the fluid in the duct is given by the Hagen-Poiseuille equation, considering the case of a laminar flow of Newtonian fluids:

$$u^*(\eta) = \frac{u(y)}{\overline{u}} = \frac{3}{2}[1 - (\frac{y}{\frac{H}{2}})^2] = \frac{3}{2}(1 - \eta^2) \tag{4}$$

After due replacement of dimensionless variables and considering cases where axial diffusion can be neglected, for high Péclet numbers, the governing equation has the following formulation.

$$w(\eta)\frac{\partial\theta(\xi,\eta)}{\partial\xi} = \frac{\partial^2\theta(\xi,\eta)}{\partial\eta^2}, \qquad \text{em} \quad 0 \le \eta \le 1. \tag{5}$$

where $w(\eta) = \frac{3}{4}(1-\eta^2)$ and the boundary and entry conditions are given by:

$$\frac{\partial\theta(\xi,0)}{\partial\eta} = 0 \qquad \text{and} \qquad \theta(\xi,1) = 0. \tag{6}$$

$$\theta(\xi,\eta) = 1, \qquad \text{in} \quad \xi = 0. \tag{7}$$

2.1 SSP Explicit Runge-Kutta Methods

The methods used in this work are explicit. However, the Shu-Osher formulation was implemented more generally. This representation is known as ”modified Shu-Osher formulation” [6].

$$u^{(i)} = v_i u^n + \sum_{j=0}^{m} \alpha_{ij}(u^{(j)} + \Delta t\frac{\beta_{ij}}{\alpha_{ij}}F(u^{(j)})) \qquad \text{where} \quad 1 \le i \le m+1. \tag{8}$$

Evidently, the equation must respect.

$$v_i u^n + \sum_{j=0}^{m} \alpha_{ij} = 1 \qquad \text{where} \quad 1 \le i \le m+1. \tag{9}$$

It is possible to obtain a convex combination of advanced Euler steps whenever α_{ij}, β_{ij} and v_i are positive [7]. Now, the constants of the methods used in this paper for the discretization of the Graetz problem will be displayed.

Below, we have the explicit SSPRK(2,2) method.

$$\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} v = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

Now, we have the constants of the explicit SSPRK(3,3) method.

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \end{pmatrix} \beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ 1 \\ \frac{3}{4} \\ \frac{1}{3} \end{pmatrix}$$

Finally, the constants of the explicit SSPRK(5,4) method.

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{3,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{4,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{5,4} & 0 & 0 \\ 0 & 0 & \alpha_{6,3} & \alpha_{6,4} & \alpha_{6,5} & 0 \end{pmatrix} \beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{2,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{3,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{4,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_{5,4} & 0 & 0 \\ 0 & 0 & 0 & \beta_{6,4} & \beta_{6,5} & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ 1 \\ v_{3,1} \\ v_{4,1} \\ v_{5,1} \\ 0 \end{pmatrix}$$

Table 1. Coefficients α and β of the SSPRK(5,4) method [7].

Coefficient	Value	Coefficient	Value
$\alpha_{3,2}$	0.555629	$\beta_{2,1}$	0.391752
$\alpha_{4,3}$	0.379898	$\beta_{3,2}$	0.368410
$\alpha_{5,4}$	0.821920	$\beta_{4,3}$	0.251891
$\alpha_{6,3}$	0.517231	$\beta_{5,4}$	0.544974
$\alpha_{6,4}$	0.096059	$\beta_{6,4}$	0.063692
$\alpha_{6,5}$	0.386708	$\beta_{6,5}$	0.226007

Table 2. Coefficients v of the SSPRK(5,4) method [7].

Coefficient	Value
$v_{3,1}$	0.444370
$v_{4,1}$	0.620101
$v_{5,1}$	0.178079

3 Results and discussions

The Order Graph is used to verify the variation of the error in view of the mesh refinement, these types of representations are widely used for numerical proof. Therefore, we know in the literature that the SSPRK(2,2) method has a second order in its formulation, however the SSPRK(3,3) method has a third order and finally the SSPRK(5,4) method has a fourth order , even if it has 5 stages. Thus, we see in the graphs that the order line has a decay equal to the order of implementation of the methods, a result that confers the correct implementation of the formulations in the computational mode.

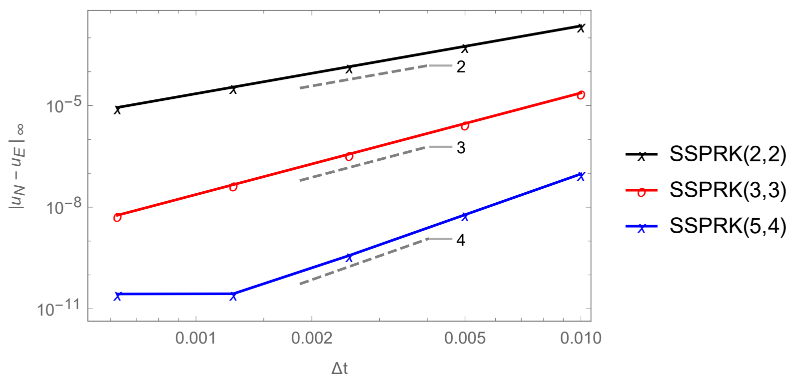

Figure 2. SSPRK Method Order Graphic

To certify the validity of the implementation of the explicit SSP Runge Kutta method, we compare it with a numerical analytical method known as GITT. In the graphs we see four different end times. In the figure 3a and 3b we have the first showing the final time of $T_f = 0.001$ and the second showing the final simulation time of $T_f = 0.004$. Figure 3c and 3d shows the final time of $T_f = 0.01$ and $T_f = 0.04$ respectively.

We see that the results are satisfactory, because in the three SSP Runge Kutta methods covered in this paper, we have profiles almost the same as the analytical numerical solution (GITT). Looking at the graphs, it appears that for end times very close to the entrance of the duct, we have a sharper curve, in contrast, profiles of solutions away from the beginning of the duct have a smoother curve. This fact is mainly due to physical reasons, the fluid exchanges heat with the duct wall and, over time, they enter into thermal equilibrium.

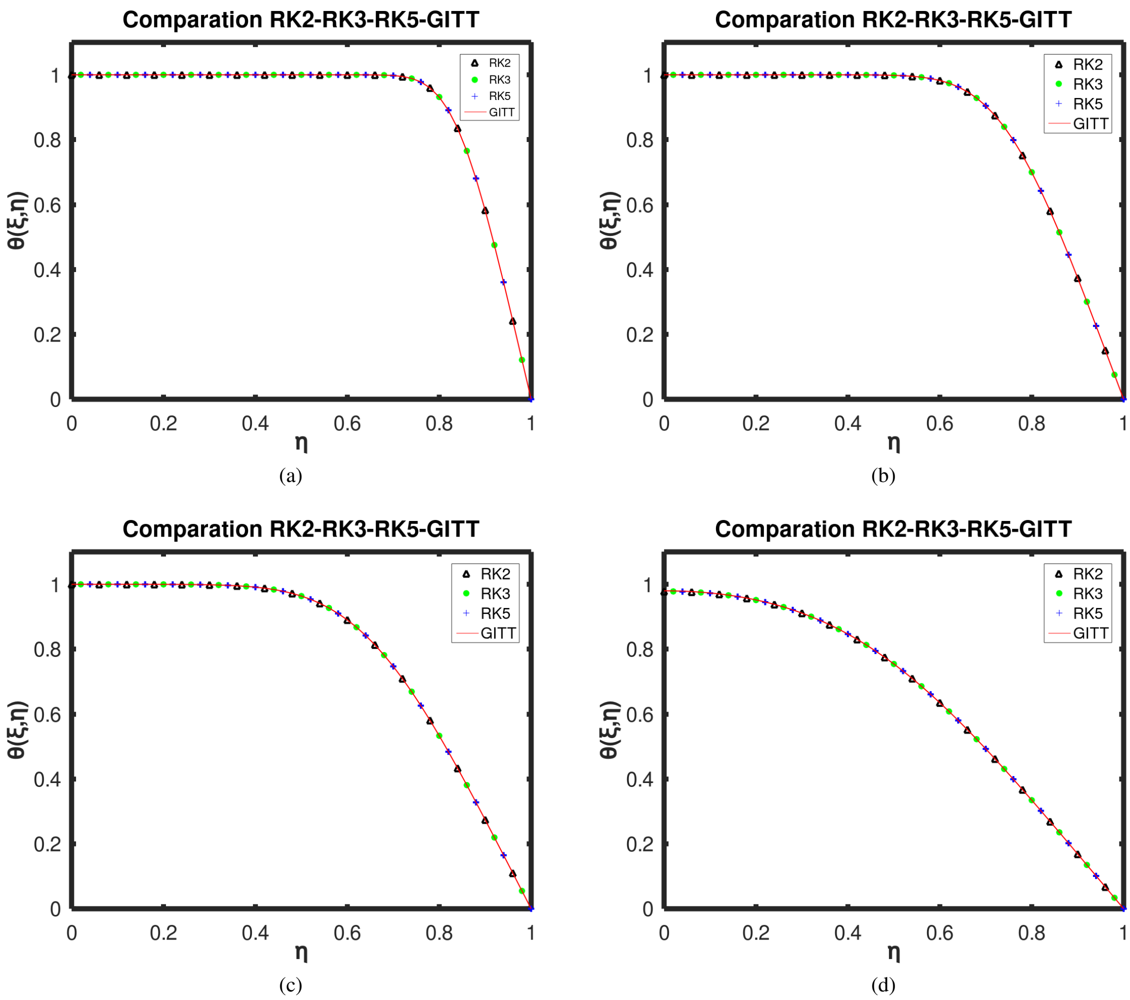

Figure 3. Comparison between SSPRK and GITT methods [7].

We know that for the SSP method to be efficient, the non-linear restriction of the step in time must be respected. That said, analyzes were carried out in order to find a numerical SSP barrier for each method implemented. First, the Graetz problem was solved with the Explicit Euler where we used the respective value of its step in time as a reference ($d\xi_{Ref} = 0.000001$) to find a numerical barrier. In the figure 4a4b4c we have the first graph showing stability in the SSP method for a step in a time equal to the reference time and also for a 20% increase in the reference time. However, for an increase of 25% we see an instability of the method. Therefore, we have a numerical barrier for the SSPRK(2,2) method with a value of $d\xi = 1.2 * d\xi_{Ref}$. In the second figure, we see that the SSPRK method (3.3) has stability for an equal time step and for a 56% increase in the reference time step, however if we increase 57 % we see numerical instability, so we have a barrier of $d\xi = 1.56 * d\xi_{Ref}$. Finally, for the SSPRK(5,4) method we see stability for an equal value and for a value 132 % higher than the reference step, but for a value 133% higher, the method has instability, with that, we see that the method has a numerical barrier of $d\xi = 2.33 * d\xi_{Ref}$.

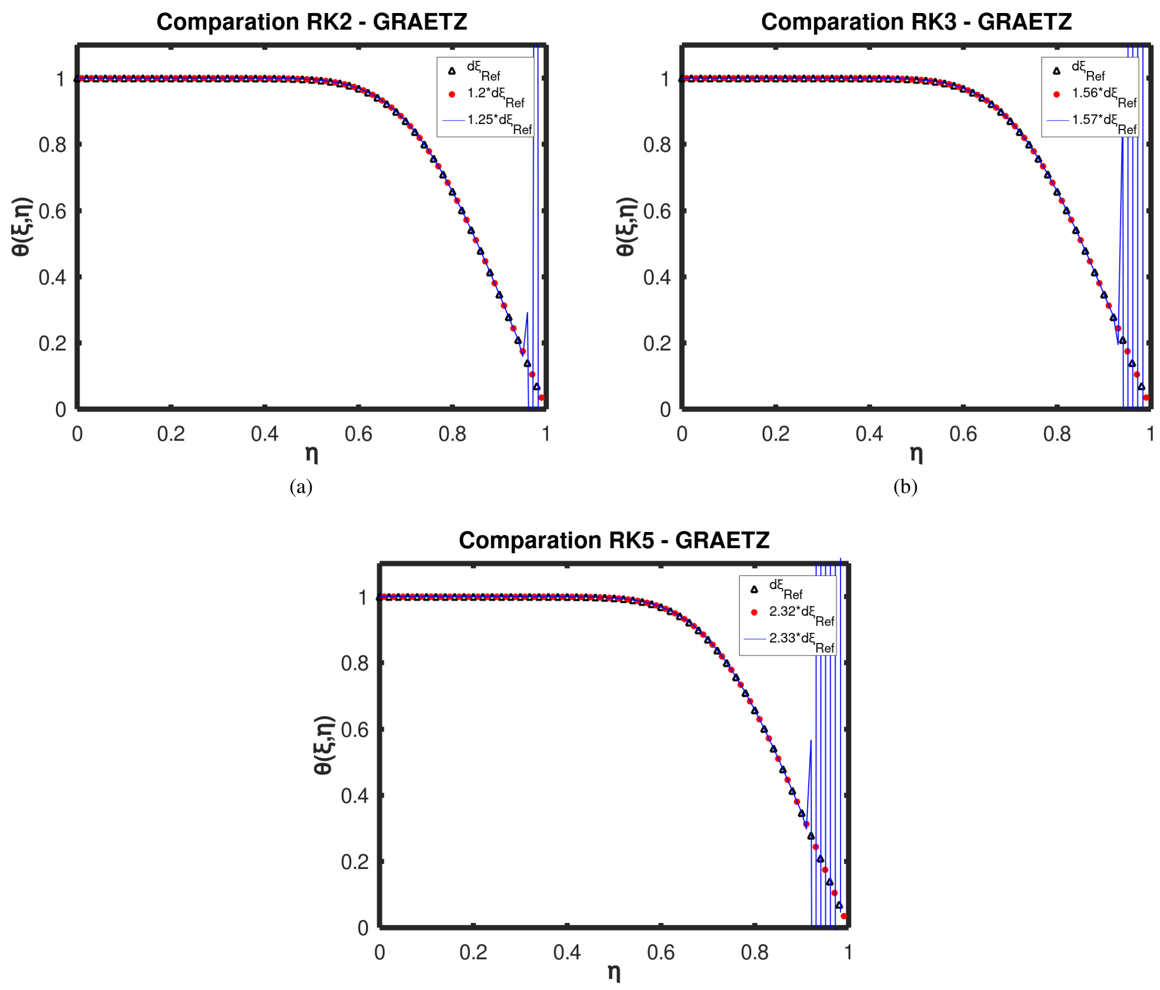

Figure 4. Numerical barrier study of the SSPRK methods [7].

4 Conclusions

The objective of this work was to propose solutions to the Graetz equation with SSP Runge Kutta methods two stages, three stages and 5 stages. First, we analyze the order chart of the implemented methods. We see a satisfactory result, because the slope of the line is consistent with the order of the method used. Then, solution profiles were proposed at four different times and compared with a numerical analytical method called GITT. From the analysis of the graphs, we have an excellent result, because the error between the comparison of the methods is very small. Showing that the SSP Runge Kutta method is great for solving diffusion problems. Finally, the numerical barrier of the SSP method was verified. Using a reference time step from the explicit Euler method, the implemented SSP methods have a loss of stability with a certain percentage increase in the reference time step.

Thus, we show that the problem was successfully implemented and the results obtained are in accordance with the conditions of the problem.

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