On the Equivalence of Forchheimer and Inertial Terms in Fluid Dynamics in Porous Media

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Abstract. With the advance of numerical methods and the capacity of computers to solve problems of high computational cost, the dynamics of fluids in porous media started to give up with the simplifying hypotheses and started to not only solve the complete equations, but also allowed to incorporate coupled phenomena such as deformation of porous media, temperature, chemical reaction and electromagnetism. The first steps towards a more rigorous study from the mathematical point of view, necessarily involve the progressive incorporation of more rigorous equations as a state of the art in flow problems. The transition from Darcy's law to Brinkman's equations and even the direct numerical simulations (DNS) of the Navier-Stokes equations on a pore scale, started to take place in the current methods of predicting the hydrogeological behavior of materials. This work seeks, in a succinct and humble way, to highlight and propose an analysis of the possible equivalence between the Forchheimer term to simulate the nonlinear effects on the flow in porous media and the natural convective term of the Eulerian formulation of the conservation of the of momentum in fluids. Despite the equivalence being affirmed in some works, there seems to be a lack in literature concerning rigorous demonstrations of this affirmation. This paper uses tools such as the topology of metric spaces and the weak formulation in finite dimensional spaces in the analysis of the differential equations that govern the problem and seeks to be as clear as possible so that a reader not used to the mathematics involved can understand, at least, the central idea.

Keywords: Porous media, fluid dynamics, nonlinear flows, Brinkman equation.

1 Introduction

The Brinkman equation (1949) was developed for the analytical solution of the problem of a viscous fluid flowing between a pack of identical spheres, seeking to calculate the frictional force between the fluid and the solid. The empirical relationship between pressure gradient and percolation speed given by Darcy did not allow the transition from a flow in a porous medium to a free flow to be simulated nor the use of a non-slip boundary condition. In the free flow condition, it is assumed that the permeability coefficient of the medium is infinite and that by replacing this limit in the equation of motion, the stationary Navier-Stokes equation is obtained without the inertial term. For media with low permeability, it is expected that the dominant term of the equation will then be seepage and that the Darcy equation will be recovered. Brinkman (1949) then proposed that the relationship between pressure gradient and flow velocity was given by:

$$\nabla p = -\frac{\mu}{k}\boldsymbol{u} + \mu' \nabla^2 \boldsymbol{u} \tag{1}$$

Where u(x) is defined in a domain $U \subset \mathbb{R}^n$; $1 \leq n \leq 3$, where $U = \{\boldsymbol{u}: \|\boldsymbol{u}\|_{L^p} < \infty | \, \|D^\alpha \boldsymbol{u}\|_{L^p}, \forall\, \alpha \geq 1\}$, with D^α being the derivatives of order up to α, in the classic sense of the derivative (locally derivable). Equation (1) also allows the adoption of boundary conditions consistent with a physical flow problem.

However, eq.(1) is classified as a second-order linear ordinary differential equation, making the relationship between the pressure gradient and flow velocity strictly linear for all $\boldsymbol{u} \in U$. Laboratory tests carried out at the beginning of the 20th century by Forchheimer (1901) highlighted the non-linear character of the hydraulic gradient (J) - velocity (u) relationship for gradients with higher values. This non-linearity was also observed by Hubbert (1966), Scheidegger (1960) and Lindquist (1933), who proposed that such non-linearity is caused by the appearance of inertial forces that, due to their higher order, were negligible for $u(x) \leq \varepsilon$, ε being as small as desired. Nonlinear formulations for the $J \times u$ relationship have been proposed by researchers such as Kochina (1952), White (1935), Scheidegger (1960), Ergun (1949), Irmay (1964) and Carman (1937).

The non-linearity discussed by the aforementioned researchers were, for the most part, described by empirical relationships, adjusted with parameters without a physical meaning. Other researchers among those cited sought to perform nonlinear regressions to adjust physical parameters such as porosity, drag coefficient, permeability and fluid viscosity. However, none of the above equations were theoretically developed from the fundamental equations of momentum and mass balance. This theoretical gap gave rise to a questioning about the mathematical formulation of the term that would be responsible for the emergence of non-linearities. Considering the inertial term, equation (1) becomes:

$$\nabla p = -\rho_f \boldsymbol{u}(\nabla \cdot \boldsymbol{u}) - \frac{\mu}{k}\boldsymbol{u} + \mu \nabla^2 \boldsymbol{u} \qquad (2)$$

Where it is considered in this model that the term $\boldsymbol{u}(\nabla \cdot \boldsymbol{u})$ is responsible for the non-linearity of the solution and that ρ_f is the fluid density. Another way of simulating nonlinearity, quite common in recent articles on flow in porous media (Djoko and Razafimandimby (2012), Tsiberkin (2018), Juncu (2015)), is by omitting the inertial term and replacing it with the well-known Forchheimer term. This term arises from the attempt to represent a non-linear flow by Forchheimer (1901), where the hydraulic gradient was expressed by the relation $J = au + bu^2$, for the one-dimensional case. For two-dimensional or three-dimensional cases, it is mathematically incoherent to raise a vector to any power, so to represent a non-linearity instead of u^2, the notation $J = a\boldsymbol{u} + b|\boldsymbol{u}|\boldsymbol{u}$ is adopted. The Forchheimer term is considered in the literature as a term referring to the drag and friction forces between the fluid and the solid matrix. Inserting this relation in eq.(1), we obtain:

$$\nabla p = -C|\boldsymbol{u}|\boldsymbol{u} - \frac{\mu}{k}\boldsymbol{u} + \mu \nabla^2 \boldsymbol{u} \qquad (3)$$

Where *C* is term referring to the drag coefficient. In the oldest literature, in the middle of the 20th century, authors tried to formulate the parameter *C* due to the tortuosity of the medium and the porosity, being an exclusive parameter of the geometry of the solid.

It is notable that both terms reproduce the non-linearity of *J x u*, since in the very mathematical nature these terms are non-linear. However, authors suggest that these are equivalent. The researcher Nield in his work named *"Limitations of the Brinkman-Forchheimer Equation in Modeling Flow in a Saturated Porous Medium and an Interface"* (1991) even proposes that the use of the convective term $\rho_f \boldsymbol{u}(\nabla \cdot \boldsymbol{u})$ is physically incorrect because within in a porous medium the fluid could not transport momentum in a neighborhood $\partial/\partial x_i$ as it could find a solid wall. Although pertinent, the comment can be contested since the momentum is carried only by the fluid and, when encountering an obstacle, the fluid changes its direction, consequently changing the direction to which the momentum will be carried, not necessarily canceling it. Other researchers claim that the term inertial is important for the study of flow instability since the Forchheimer term apparently does not make the differential equation unstable (Tsiberkin, 2018).

The objective of this study is to understand and explain the equivalence of the terms exposed by eq.(2) and eq.(3), if it exists and, in which scenarios it can be considered true. This work will use tools from the theory of topological spaces of finite dimension, the modern theory of partial differential equations and asymptotic analysis of non-linear equations.

2 Functional analysis of approximate solutions in finite-dimensional topological subspaces

In engineering it is no longer common to solve differential equations in an analytical way due to the easy access to numerical analysis softwares. Even if one wants to solve these partial differential equations analytically, depending on the complexity of the domain, solutions may not exist. In this section, the so-called weak solutions of Brinkman's equations will be analyzed from the perspective of functional analysis.

Take $U \subset \mathbb{R}$ and U = [0; 1] as the subspace of the line of the reals with infinite dimension where the PDE's analytical solution lives. Now take $S \subset U$, S = [0; 1] as a subspace of U but with a finite dimension. We assume that a weak solution $u_s(x)$ of eq.(2) and eq.(3) lives in *S*. Let eq.(2) be replaced by the value of the weak solution u_s in the formulation. The two sides of the equation are multiplied by a $v(x)$ belonging to the same subspace of u_s or some similar subspace. The two sides of eq.(2) are integrated across the domain to obtain:

$$\rho_f \int_S u_S \frac{\partial u_S}{\partial x} v \, dS = -\int_S f'(x) v \, dS - \frac{\mu}{k} \int_S u_S v \, dS + \mu \int_S \frac{\partial^2 u_S}{\partial x^2} v \; dS \tag{4}$$

Let the boundary conditions be $u_s(0) = 1$ and $u_s(1) = 0$. Applying the integration by parts in the last term to the right of equality and then applying the divergence theorem, it is rewritten as:

$$\rho_f \int_S u_S \frac{\partial u_S}{\partial x} v \, dS = -\int_S f'(x) v \, dS - \frac{\mu}{k} \int_S u_S v \, dS + \mu \int_{\partial S} \frac{\partial u_S}{\partial x} v \; d\Gamma - \mu \int_S \frac{\partial u_S}{\partial x} \frac{\partial v}{\partial x} \, dS \tag{5.a}$$

The third term on the right of equality disappears identically because the two boundary conditions are of the Dirichlet type, remaining:

$$\rho_f \int_S u_S \frac{\partial u_S}{\partial x} v \, dS = -\int_S f'(x) v \, dS - \frac{\mu}{k} \int_S u_S v \, dS - \mu \int_S \frac{\partial u_S}{\partial x} \frac{\partial v}{\partial x} \, dS \tag{5.b}$$

It should be noted that the subspace where u_s lives should only have the first derivative in relation to space and that this is Lebesgue-integrable. Suppose now that *S* is a closed subspace of *U*, complete and provided with a norm defined by an inner product. Let this norm also be defined such that as a norm in $L^p \; \forall \; 1 \leq p < \infty$. From these assumptions, it is admitted that *S* is a Banach space because it is complete, more specifically a Hilbert space because it has a norm in an inner product, and mainly, it is a Sobolev space because the norm applied to its derivatives is also Lebesgue-integrable . *S* is defined such that:

$$S = \left\{ u_S : \|u_S\|_{L^p} < \infty \, , \|D^\alpha u_S\|_{W_p^1} < \infty \, , \qquad \forall \alpha \leq 1, \;\; u_S(0) = 1 \, , u_S(1) = 0 \right\} \tag{6}$$

The derivative $D^\alpha u_S$ should not be interpreted in the classic sense of local differentiability, but rather as a weak derivative. The weak derivative is admitted existing in almost the entire domain, and may not exist in sets of measure zero (points). The definition of the norm in (19) is given by:

$$\|u_S\|_{L^p} = \left(\int_S |u_S|^p \, dS \right)^{\frac{1}{p}} \tag{7}$$

And the Sobolev norm:

$$\|D^\alpha u_S\|_{W_p^1} = \left(\int_S \left| \frac{\partial^\alpha u_S}{\partial x^\alpha} \right|^p \, dS \right)^{\frac{1}{p}}, \qquad \alpha = 0,1 \ldots, n \tag{8}$$

Here the solution norm and its derivatives have been separated, but the existence of a norm in Sobolev eq.(8) necessarily induces eq.(7).

Let the inner product (scalar product) in L^p and W_p^1 be defined by $(\cdot, \cdot)$. Again, $\rho_f = \mu = k = 1$. Equation (5.b) becomes:

$$(u_s u_s', v) = (f', v) + (u_S \, , v) + (u_S' \, , v') \tag{9.a}$$

The L^1 norm is applied to both sides of the equation. By the definition of the norm L^1 in Sobolev spaces, $\left(\int_X |(u_s, v) + (u_s', v)|^1 dx\right)^{\frac{1}{1}} = \|u, v\|_{W_1^1}$. Then one has:

$$\|u_s u_s', v\|_{L^1} = \|f', v\|_{L^1} + \|u, v\|_{W_1^1} \tag{9.b}$$

The term (f', v) can be discarded from the analysis without loss of generality since *f* is not a function of $u_s(x)$. Applying Schwarz's inequality:

$$\|u_s u_S'\|_{L^2} \, \|v\|_{L^2} \leq \|u\|_{W_2^1} \, \|v\|_{W_2^1} \tag{9.c}$$

Considering that all the norms in a space of finite dimension are equivalent to each other, it is possible to propose that $\|v\|_{W_2^1} \leq \alpha \|v\|_{L^2}$.. Simply put, you get:

$$\|u_s u_S'\|_{L^2} \leq \alpha \|u_S\|_{W_2^1} \xrightarrow{Hölder} \|u_S\|_{L^p} \, \|u_S'\|_{L^q} \leq \alpha \|u_S\|_{W_2^1} \tag{9.d}$$

For $1 = \frac{1}{p} + \frac{1}{q}$. The result in eq.(9.d) implies obvious and less trivial conclusions. The first conclusion that can be drawn from (9.d) is that the norm of the nonlinear term is smaller than the norm of the solution itself. It is also noted from eq.(9.d) that the non-linear term is coercive and limited, indicating continuity in the given space. This conclusion is obvious because it is customary to discard second order nonlinear terms as they are smaller than first order terms. However, it is noted that the norms in comparison are not defined in the same topological subspace. The norm for the non-linear term is taken in 2-Lebesgue and the norm for the solution and its derivatives is taken in Sobolev. This comparison in different norms is known in the literature as Sobolev's inequality theorem (Brenner, 2008), which supports the well-known Sobolev's embeddings (Ramanujan, 1997). This theorem allows us to reach the corollaries:

i) u_s' and u_s exist and are continuous, then $u_s' u_s$ exists and is continuous in $S \in W_2^1(U)$.

ii) There is a function equivalent to $u_s u_s'$ and it is of class C^2.

Applying the same reasoning in eq.(3), that is, the Brinkman equation with the Forchheimer term, we obtain the weak formulation:

$$\mathrm{C}\int_S (u_S)^2 v\, dS = -\int_S f'(x) v\, dS - \frac{\mu}{k}\int_S u_S v\, dS - \mu \int_S \frac{\partial u_S}{\partial x}\frac{\partial v}{\partial x}\, dS \tag{10}$$

Assuming $C = \mu = k = 1$ and using the definition of internal products again, one obtains:

$$(u_S^2, v) = (f', v) + (u_S\, , v) + (u_S'\, , v') \tag{11.a}$$

The term (f', v) is ignored by the same arguments used previously, applying the L^1 norm on both sides and using the Schwarz inequality:

$$\|u_S^2\|_{L^2}\, \|v\|_{L^2} \leq\, \|u_S\|_{W_2^1}\, \|v\|_{W_2^1} \tag{11.b}$$

Where, after simplifications:

$$\|u_S^2\|_{L^2} \leq\, \alpha \|u_S\|_{W_2^1} \tag{11.c}$$

By Sobolev's inequality theorem, the corollaries naturally appear:

i) u_s exists and is continuous, then $u_s \cdot u_s = u_S^2$ exists and is continuous in $W_2^1(U)$.

ii) There is a function equivalent to u_S^2 of class C^2.

The two terms, inertial and Forchheimer, seem to be totally equivalent in relation to the topological space in which they live. Note, however, that in eq.(9.d) there is a restriction on the derivative of u and that this restriction is not imposed in eq.(11.c). The restriction in the derivative requires that this space to be a subspace of the less restricted space and therefore, the functionals applied to them would only be equal through a Hahn-Banach extension. The fundamental difference between the terms is then in their behavior when submitted to a linear operator belonging to the dual space of S, called S^*. Let be $\mathcal{L} \in S^*$ an operator such that it is coercive and limited. Let $S_{\mathcal{L}}$ be the space of the images of the terms of S submitted to $\mathcal{L}$. This space is also complete and closed. Let S be composed of Forchheimer's terms and of inertial terms, since it has been shown that they belong to the same bigger space, that is, $S = S_{Forchheimer} \cap S_{inertial}$.. The space of the images, resulting from the application of $\mathcal{L}$ in S, can then be separated into a space where the images of the Forchheimer terms are and another where the inertial terms are. Let $S_{\mathcal{L}_{Forchheimer}}$ be the space of the images corresponding to the Forchheimer term and $S_{\mathcal{L}_{inertial}}$ be the space of the images corresponding to the inertial term. Then one has:

$$S_{\mathcal{L}_{Forchheimer}} \cap S_{\mathcal{L}_{inertial}} = \{\emptyset\} \tag{12}$$

This relationship indicates that the image subspaces resulting from the application of the linear operator in Forchheimer and inertial terms are separable. Two subspaces are separable when their terms are disjoint, that is, they do not share neighborhood. This abstract analysis can be better represented in Figure 4.

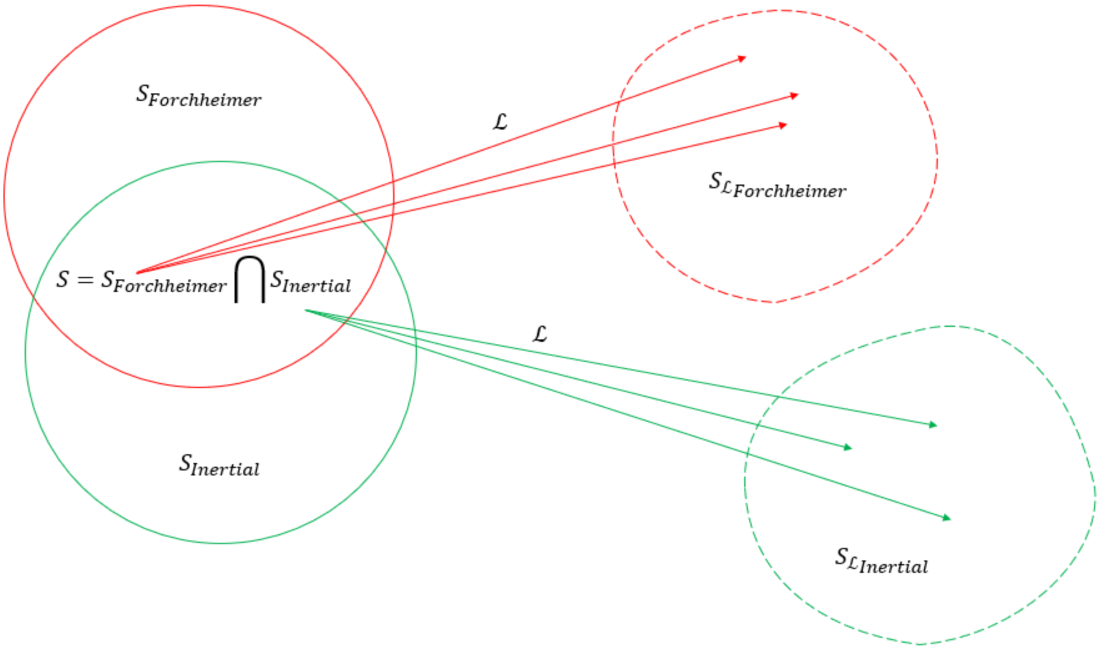

Figure 4. Representation of separable subspaces using a linear operator (author, 2020)

Being $S \subset \mathbb{R}$, it is known that the space of the images generated by any linear operator is represented by the axis of the ordinates in a Cartesian graph. The notion of abstract vector space becomes more understandable when the axis of the abscissa corresponds to space S and the axis of ordinates space $S_{\mathcal{L}}$. As demonstrated, the images of the Forchheimer and inertial terms are separable, that is, they are not the same in almost the entire domain, and can be the same in different points that, for the purpose of topological analysis, do not indicate equivalence (for example, two distinct functions can intercept at more than one point and this does not indicate that these functions are identical).

The corollaries induced by the results eq.(9.d) and eq.(11.c), state that for both formulations, there is a C^2 function that is equivalent to $u_s(x)$. A continuous, limited and C^2 function within a Sobolev space, indicates that this function can be expanded in a Taylor series, depending on its derivatives. However, the derivatives used in the expansion do not have the classic sense, but the weak derivative sense. Let $u_s(x)$ be such that its expansion around any point y is:

$$u_S(x) = u_s(y) + u_s'(y)|x-y| + \frac{u_S''(y)|x-y|^2}{2} + \cdots \frac{u_S^n|x-y|^n}{n!} \qquad (13.a)$$

It is guaranteed that, because there is an equivalent function of class C^2, the first and second derivatives are existing and continuous in S. Therefore, $u_s(x)$ can be represented by a second order polynomial within $W_2^1(U)$. These Sobolev polynomials indicate that, when building a vector space to solve the Brinkman equation, assuming that S is complete and finite-dimensional, $u_s(x)$ admits the form:

$$u_0\phi_0 + \sum_{i=1}^{n} \phi_i(x)u_i(x) = u_S(x) \qquad (13.b)$$

Where $\phi_i(x)$ spans the space. This result introduces that to solve the Brinkman equation with the Forchheimer term or the inertial term, the finite element formulation under Bubnov-Galerkin's theory must use interpolation functions that are polynomials of order of at least two and, for a isoparametric formulation, the same order is maintained for the shape functions. Therefore, linear elements with three nodes and Lagrange polynomials are adopted. The development of the finite element formulation is not within the scope of this work; however, it was carried out in view of all the mathematical foundations described so far.

Both formulations were solved with the finite element tool, programmed in MatLab 2019. Nonlinearity was treated with the Newton's method. The values of k and μ were kept constant, varying only ρ_f to strengthen the nonlinearity of the inertial term and C to strengthen the nonlinearity of the Forchheimer term. The result was represented in Figure 5.

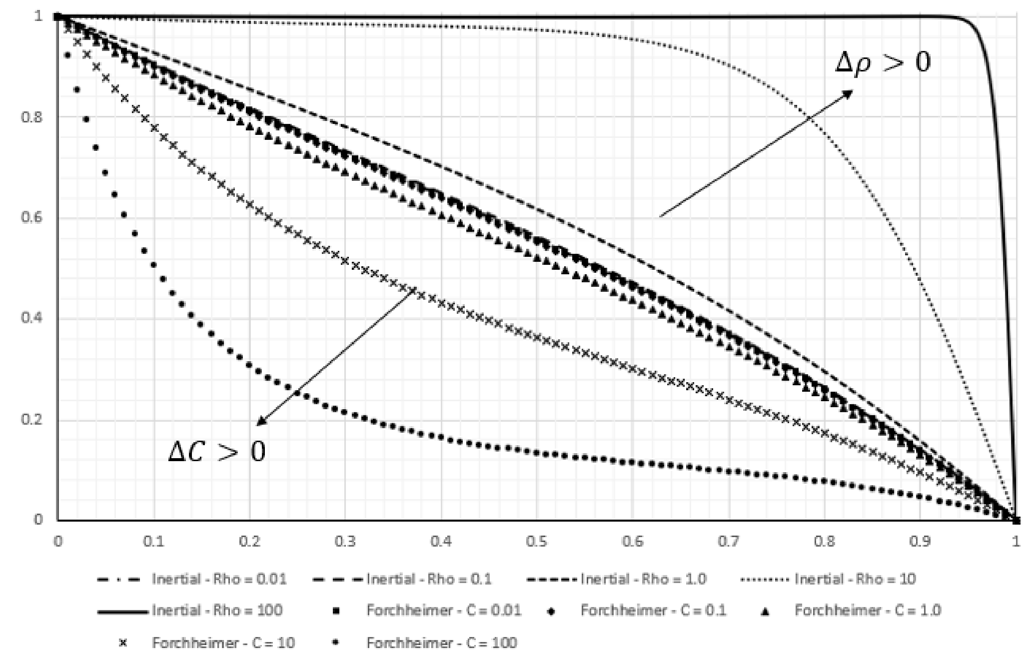

Figure 5. The inertial and Forchheimer formulations with increasing nonlinear term (author, 2020).

It can be seen in Figure 5 that for moderate values of ρ_f and C, in relation to the other coefficients in the PDE, the solutions are not identical, but they are quite similar. Both solutions respect the boundary conditions and have a smooth distribution within the domain, monotonically decreasing. For values in which ρ_f and C have an order of magnitude at least three times greater than the other coefficients, the solutions, despite respecting the boundary conditions, present different behaviors, including in relation to the concavity of the function. Note that the solution for the inertial term has a concavity facing downwards and the solution for the Forchheimer term has a concavity facing upwards. For extreme values of ρ_f and C the behavior of the two solutions no longer seems to represent the same physical phenomenon. Note that for the concavity facing downwards, represented by the solution with the inertial term, it has a second positive derivative in almost the entire domain, indicating that the viscous term is positive in almost the entire domain, and therefore dissipative, as predicted by thermodynamics . The same cannot be said for the formulation provided with the Forchheimer term, since the upwards-facing concavity indicates that the system is not dissipating energy by viscous friction. However, Forchheimer's formulation is always stable due to purely mathematical reasons.

3 Conclusion

The analysis of the nature of Forchheimer's and inertial terms, showed that they have different meanings and importance within the momentum balance equation. For this reason, the two terms would not be equivalent. However, the topological analysis of the terms, which is deeper and more rigorous in the mathematical nature of each term, closes this work, emphasizing that the solutions are different for the formulation provided with the Forchheimer term or with the inertial term but that, for engineering purposes, with well-behaved and coherent values of fluid density and drag coefficient, both can accurately approximate the flow behavior. It can be noted, in Figure 2, that in the absence of nonlinear terms, or for limit values of nonlinear coefficients tending to zero, both solutions approximate Darcy's linear solution. The formulation with the Forchheimer term is then indicated for situations where it is necessary to calculate the drag force that the fluid exerts on solid grains, in linear or slightly non-linear regimes. For the study of strongly non-linear solutions and hydrodynamic instability, it is recommended the formulation with the inertial term. A formulation with both terms must be studied later. Analyzes in two and three-dimensional spaces are recommended to understand the differences between the solutions when structures such as vortexes or more complex boundary conditions are admitted in the problem.

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