XLI CILAMCE
XLI Ibero-Latin American Congress on Computational Methods in Engineering
16-19 November, 2020 FOZ DO IGUAÇU PARANÁ - BRAZIL

Uncertainty quantification of the Collipriest random model by the Fast Crack Bounds and Monte Carlo methodology

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Abstract. Mechanical structures are subjected to cyclical stresses and collapse under fatigue conditions. There are several mathematical models that describe the crack propagation. In general, crack propagation models are classified by the type of loading, which can have constant stress amplitude or variable stress amplitude. In this work, the constant stress amplitude model proposed by Collipriest will be explored. For many engineering applications a reliable estimate of crack behavior is required. Therefore, this work presents theoretical results, which consist of the quantification of uncertainty in the definition parameters in the model used, based on lower and upper bounds, which envelop the estimators of the first and second order statistical moments of the function size of crack, based on the Fast Crack Bounds method. These dimensions are polynomials, defined in the variable number of cycles, which take into account the uncertainties in the parameters that describe the crack propagation models. The Monte Carlo simulation method will be used to obtain the realizations of the crack size function from a set of random samples of the characteristic parameters of the Collipriest model. These achievements will be used to obtain the estimators of statistical moments of crack size. The efficiency of the bounds for the estimators of the statistical moments of the crack size is evaluated through relative deviation functions between the bounds and the approximate numerical solutions of the initial value problem that describes the Collipriest model. In general, the solution of the initial value problem that the crack propagation models describe is obtained through the use of numerical methods, such as the explicit fourth order Runge-Kutta method. In this work, a mathematical software will be used for numerical analysis of the solutions of the crack propagation model of the Collipriest model, therefore an analysis of the computational performance between the Fast Crack Bounds and Runge-Kutta methods and presented by the computational time, graphs and tables.

Keywords: Fracture mechanics, Runge-Kutta, Collipriest, Computational time.

1 Introduction

The objective of this work is to quantify the uncertainty of the crack size function of the Collipriest model, according to the Fast Crack Bounds - Monte Carlo methodology (FCB-MC) and based on the crack propagation regime in constant amplitude. An uncertainty quantification consists of using the FCB method, together with the Monte-Carlo simulation method. From this, numerical bounds are obtained for estimators the statistical moments of the stochastic process crack size of the Collipriest random model, applications for engineering problems. The estimates resulting of the statistical moments will be compared with the approximate solutions using the fourth order Runge Kutta - Monte Carlo (RK4 - MC) methodology. The performance of the FCB - MC methodology, in the quantification of the uncertainty of the Collipriest model and evaluated through computational performance measures. These measures are defined from the computation times, upper and lower dimensions and crack size function obtained by the method RK4 - MC.

2 Collipriest model

Collipriest [1] proposed a crack growth law and describing all the regions in Fig. 1. Based on this law, the propagation of a crack of the Collipriest model is formulated according to eq. (1), using the solution of the following the initial value problem (IPV).

$$\begin{cases} \text{Find } a \in C^1\left(N_0, N_1; \mathbb{R}^+\right) \text{ such that :} \\ \dfrac{da}{dN} = C\left(K_{IC}\Delta K_{I_{th}}\right)^{\frac{m}{2}} exp\left\{ ln\left(\dfrac{K_{IC}}{\Delta K_{I_{th}}}\right)^{\frac{m}{2}} tanh^{-1}\left[\dfrac{ln\dfrac{\Delta K^2}{(1-R)K_{IC}\Delta K_{I_{th}}}}{ln\dfrac{(1-R)K_{IC}}{\Delta K_{I_{th}}}}\right]\right\}, \forall \in \left[N_0, N_1\right] \\ \text{subject to} \\ a\left(N_0\right) = a_0. \end{cases} \tag{1}$$

Where C and m are the material parameters for the Collipriest model. This model performs, with accuracy, the effects of the stress ratio R in the crack propagation regions. The main characteristic of the Collipriest model is that it represents the crack propagation regions I, II and III and considering the average stress, by the stress ratio.

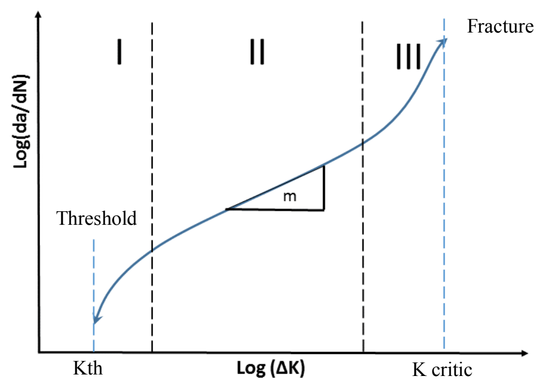

Figure 1. Sigmoid diagram da/dN.

3 Fast Crack Bounds – Monte Carlo methodology

The Fast Crack Bounds (FCB-MC) methodology consists of efficiently finding two functions that define upper and lower bounds for the crack size function of any crack propagation law. The computational cost of the FCB method is often less than that required by numerical integration with the fourth order Runge Kutta method, according to Ávila [2] and Machado [3]. In this work, the numerical results show that the bounds represent, satisfactorily, the crack evolution. The lower and upper bounds of the crack size function must satisfy the following inequalities, according to eq. 2.

$$\bar{a}\left(N\right) \le a\left(N\right) \le \underline{a}\left(N\right), \forall N \in \left[N_0, N_1\right] \tag{2}$$

The FCB-MC methodology, starting from hypotheses and adequate analyzes of the representation by the Taylor series and retaining the second order term, with Lagrange's rest, obtains functions which results in lower and upper bounds for the crack size function for some law of crack propagation, according to eq. 3.

$$\begin{cases} \underline{a}\left(N\right)=a_0+\dfrac{da}{dN}\left(N_0\right)\left(N-N_0\right)+\dfrac{1}{2}\dfrac{d^2a}{dN^2}\left(a_0\right)\left(N-N_0\right)^2;\ \eta\in\left[N_0,N_1\right]; \\ \overline{a}\left(N\right)=a_0+\dfrac{da}{dN}\left(N_0\right)\left(N-N_0\right)+\dfrac{1}{2}\dfrac{d^2a}{dN^2}\left(a^*\right)\left(N-N_0\right)^2. \end{cases} \tag{3}$$

Where $a^*=\beta a_0$, and β equal to coefficient and "N" the quantity of cycles.

3.1 Fast Crack Bounds method applied to the Collipriest model

The mathematical formulation consists of determining the equations that represent the upper and lower bounds of the Collipriest crack propagation model using the FCB-MC methodology. Starting from the Collipriest eq. 1 and by the integration of the Taylor series with the Lagrange rest in eq. 3, considering hypotheses i, ii, iii and iv, consequently the eq. 4.

$$\begin{aligned} &\left(i\right)\frac{m}{2}\geq 0 \therefore m\geq 2; \\ &\left(ii\right)0<f\left(a\right)\leq f\left(x\right)\leq f\left(y\right); \\ &\left(iii\right)f'\left(a_0\leq f'\left(x\right)\leq f'\left(y\right)\right); \\ &\left(iv\right)f\in c'\left(\mathbb{R}^+\right). \end{aligned}$$

From eq.4 it is possible to find the upper and lower bounds. For the upper bound, mathematical inequality $\geq$ and $\Delta K\left(a_0\right)$ is changed. However, for the lower bound, inequality $\leq$ and $\Delta K\left(a^*\right)$ is changed. Therefore, the equations of the lower and upper bounds for the FCB-MC methodology of the Collipriest model are found.

$$\begin{cases} a=a_0+C\left(K_{IC}\Delta K_{I_{th}}\right)^{\frac{m}{2}}\exp\left[\ln\left(\dfrac{K_{IC}}{\Delta K_{I_{th}}}\right)^{\frac{m}{2}}\tanh^{-1}\left(\dfrac{\ln\dfrac{\Delta K^2}{\left(1-R\right)K_{IC}\Delta K_{I_{th}}}}{\ln\dfrac{\left(1-R\right)K_{IC}}{\Delta K_{I_{th}}}}\right)\right]\left(N-N_0\right)+ \\ \dfrac{1}{2}\left\{C\left(K_{IC}\Delta K_{I_{th}}\right)^{\frac{m}{2}}\exp\left[\ln\left(\dfrac{K_{IC}}{\Delta K_{I_{th}}}\right)^{\frac{m}{2}}\tanh^{-1}\left(\dfrac{\ln\dfrac{\Delta K^2}{\left(1-R\right)K_{IC}\Delta K_{I_{th}}}}{\ln\dfrac{\left(1-R\right)K_{IC}}{\Delta K_{I_{th}}}}\right)\right]^2\right\} \\ C\left(K_{IC}\Delta K_{I_{th}}\right)\dfrac{\ln\left(\dfrac{K_{IC}}{\Delta K_{I_{th}}}\right)^{\frac{m}{2}}}{\ln\dfrac{\left(1-R\right)K_{IC}}{\Delta K_{I_{th}}}}\left[\dfrac{1}{1-\left(\dfrac{\ln\dfrac{\Delta K^2}{\left(1-R\right)K_{IC}\Delta K_{I_{th}}}}{\ln\dfrac{\left(1-R\right)K_{IC}}{\Delta K_{I_{th}}}}\right)}\right] \\ \left[\dfrac{1}{a}+2\dfrac{f'}{f}\left(a\right)\right]\left(N-N_0\right)^2. \end{cases} \tag{4}$$

4 Runge Kutta – Monte Carlo methodology

The Runge Kutta - Monte Carlo methodology (RK4-MC) is used to find an estimate of the statistical moments of the "crack size" stochastic process, using the fourth order Runge Kutta method. To obtain the estimators of the statistical moments, random samples are generated for each parameter with uncertainty. To evaluate the performance of the methodology, relative deviation functions are defined for the first and second statistical moments of the upper and lower bounds, considering to the statistical moments of the numerical solution, obtained through the RK4-MC methodology. In Avila and Santos [2] crack propagation is evaluated by the stochastic method associated with the RK4-MC methodology. Thus for the i-th sample of the parameters, the approximate numerical solution of the process is given by eq. 5 implemented for the Collipriest model.

$$\begin{cases} \mathit{Find}\, a_{k+1}(\omega_i) \in \omega_i = (\omega_i, \omega_i', \omega_i'', \omega_i''', \omega_i'''') \in \Omega : \\ a_{k+1}(\omega_i) = a_k(\omega_i) + \left(\frac{\Delta N}{6}\right)\left(K_1(\omega_i) + 2K_2(\omega_i) + 2K_3(\omega_i) + K_4(\omega_i)\right), \\ \text{for } k = 0, 1, \ldots, m-1, \textit{ with}, \\ K_1(\omega_i) = C\left(KI_c \Delta KI_{th}\right)^{\frac{m}{2}} \exp\left\{ \ln\left(\frac{KI_c}{\Delta KI_{th}}\right)^{\frac{m}{2}} \tanh^{-1}\left[\frac{\ln\left(\frac{\Delta K^2}{(1-R)KI_c \Delta KI_{th}}\right)}{\ln\left(\frac{(1-R)KI_c}{\Delta KI_{th}}\right)}\right]\right\}; \\ K_2(\omega_i) = a_j(\omega_i) + \left(\frac{\Delta N}{2}\right) K_1(\omega_i); \\ K_3(\omega_i) = a_j(\omega_i) + \left(\frac{\Delta N}{2}\right) K_2(\omega_i); \\ K_4(\omega_i) = a_j(\omega_i) + (\Delta N) K_3(\omega_i); \\ a_o(\omega_i) = a(N_o, \omega_i), \quad \omega_i \in \{\omega_1, \omega_2, \ldots, \omega_N\} \end{cases} \tag{5}$$

4.1 Modeling the uncertainty of the Collipriest random variables

The modeling of the Collipriest uncertainty is performed based on the random variables of the parameters C, m, a_0, ΔK_{th} and K_c from model, based on the FCB-MC methodology and RK4-MC methodology, according to eq. 6. The random variables are uniform distribution and the generated values have the same probability of occurrence in the events, according to Machado [4]. Random variables are characterized as uniform and with statistical independence.

$$C(\omega) = \mu_c + \sqrt{3}.\delta_C.\xi_1(\omega), \forall \omega \in \Omega. \tag{6a}$$

$$C(\omega) = \mu_c + \sqrt{3}.\delta_C.\xi_1(\omega), \forall \omega \in \Omega. \tag{6b}$$

$$C(\omega) = \mu_c + \sqrt{3}.\delta_C.\xi_1(\omega), \forall \omega \in \Omega. \tag{6c}$$

$$C(\omega) = \mu_c + \sqrt{3}.\delta_C.\xi_1(\omega), \forall \omega \in \Omega. \tag{6d}$$

$$C(\omega) = \mu_c + \sqrt{3}.\delta_C.\xi_1(\omega), \forall \omega \in \Omega. \tag{6e}$$

Where $\{\mu c, \mu m, \mu a_0, \mu \Delta K_{th}, \mu K_c\}$ and $\{\delta c, \delta m, \delta a_0, \delta \Delta K_{th}, \delta K_c\}$ are the expected values. The random variables are represented by ξ_n, uniform in U [-1,1] and statistically independent. The dispersion coefficient of $\delta_c = 1/10\mu_c$ and $\delta_c = 3/10\mu_c$ used to represent the value of the uncertainty variable.

4.2 Numerical and computational implementation

The RK4-MC and FCB-MC methodologies are applied to the Collipriest model. From the respective mathematical formulations in eq. 4 and eq. 6 algorithms were developed and implemented in matlab. To evaluate the implemented methods and algorithms, tests were performed with problems whose solution is known and described in the chapter results. The algorithm implemented to obtain the fourth order Runge Kutta numerical solution has a greater computational effort in solving the main function of ordinary differential equation, according to Burden and Faires [5], this influences the computational performance and consequently the results of this work. As part of the results there is an analysis of the computational times of the methodologies using an intel core i5 computer hardware and 8 Gb RAM.

5 Numerical results

The results are based on the numerical values obtained of the RK4-MC and FCB-MC methodology. The upper and lower bounds are calculated according to the estimates of the statistical moments of the crack size function of the Collipriest model. The numerical solutions, the relative deviations between the bounds of the statistical moments, are associated to the functions of the stress intensity factor (SIF) of the metal plates requested by tension, and presenting surface crack. As part of the numerical result, there is an analysis of the computational times of the methodologies, according to the software and hardware mentioned above.

Classical deterministic examples are used to evaluate the performance of the dimensions, such as: infinite plate with central crack, finite plate with a central crack and finite plate with edge crack. The parameters of the plate material are the same as those used in Rubaie's works [6]. These parameters are described in Tab. 1 and belong to the Inconel 600 alloy.

Table 1. Inputs parameters of metal plate – alloy 600 steel

Parameters	Numerical value
Random variable $\Delta\sigma$	65 Mpa
Plate thickness b	100 mm
Stress ratio R	0,1 mm
Initial cycle N_0	0 cycles
Final cycle N_1	900000 cycles
Initial crack length a_0	2,5 mm
Random variable c	$5{,}61.10^{-12}$
Random variable m	2,62
Random variable K_c	40,08 $MPa.m^{1/2}$
Random variable ΔK_{th}	6,38 $MPa.m^{1/2}$

The numerical results of the FCB-MC and RK4-MC methodologies are show in graphs for the three types of deterministic examples. The first example is an infinite width plate with a central crack in Fig.2, the second example is a finite plate with a central crack in Fig. 3 and the third example is a finite plate with an edge crack in Fig.4. These examples have a correction factor, according to Peterson [7], whose functions are indicated in the figures. The material that constitutes these plates is an alloy of Inconel 600, whose parameters are shown in Tab. 1. Through these input data it was possible to measure the performance of both methodologies, looking for parameters of the first statistical moment of the stochastic crack size process. This performance can be evaluated according to the numerical values presented through graphs in Fig. 2, Fig. 3 and Fig. 4, and by the computational time for the random parameters of the Collipriest model, according to Tab. 2. 20000 simulations are carried out for each sample from 0 to 900000 cycles.

In the classic and deterministic example infinite plate with central crack and random parameter crack length a_0, according to Fig. 2, with dispersion coefficient in 1/10 and $a^* = 1/10a_0$, it is observed that the randomized parameters enveloped the numerical solution and complying with the inequality shown in eq. 2. That is, the bounds $\hat{\mu}_{\bar{a}}$ and $\hat{\mu}_{\underline{a}}$ envelop the numerical solution $\hat{\mu}_a$. This situation shows the behavior of the FCB-MC methodology in quantifying the uncertainty of the random variable crack size a_0 of the Collipriest model.

The computational times are measured for each methodology and compared to each other, according to Tab. 2. Note that the execution time of the FCB-MC methodology algorithm in relation to the RK4-MC is in the order of 3700% faster.

Table 2. Computational time performance – FCB-MC versus RK4-MC methodologies

Methodology	Infinite plate central crack	Finite plate central crack	Finite plate side crack
RK4-MC	751,5 s	829,9 s	870,8 s
FCB-MC	20,4 s	46,4 s	26,7 s
rate (%)	3700 %	1800 %	3300 %

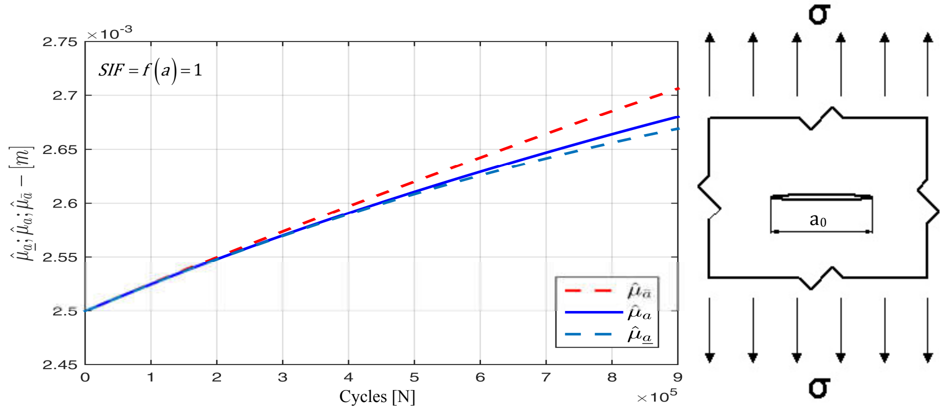

Figure 2. Results of RK4-MC vs. FCB-MC bounds - random parameter a_0.

In the classic and deterministic example finite plate with central crack and random parameter crack length a_0, shown in Fig. 3, using dispersion coefficient of 1/10, $a^* = 1/10 a_0$, all randomized parameters enveloped the numerical solution complying with the inequality shown in eq. 2. That is, the bounds $\hat{\mu}_{\bar{a}}$ e $\hat{\mu}_{\underline{a}}$ envelop the numerical solution $\hat{\mu}_a$. The computational execution time is approximately 1800% faster, according to Tab.2.

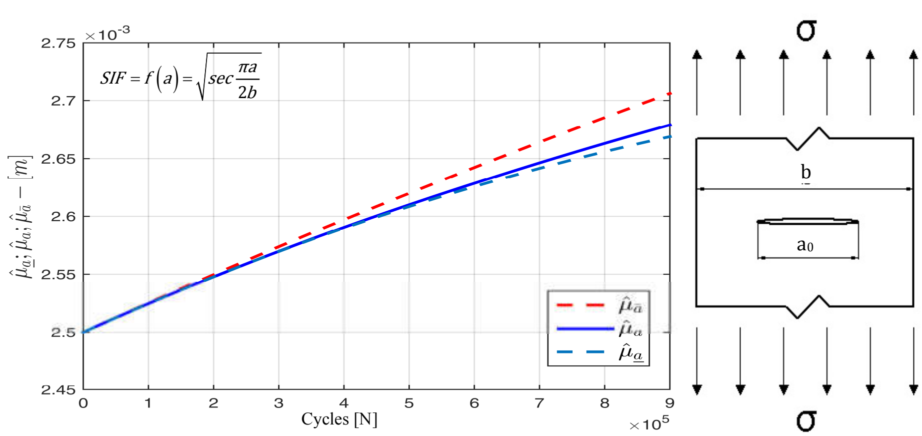

Figure 3. Results of RK4-MC vs. FCB-MC bounds - random parameter a_0.

In the classic and deterministic example finite plate with side crack with random parameter crack length a_0, show in Fig. 3, with dispersion coefficient in 1/10, $a^* = 1/10a_0$, and all randomized parameters enveloped the numerical solution complying with the inequality shown in eq. 2. The computational execution time is approximately 3300 % faster, according to Tab.2.

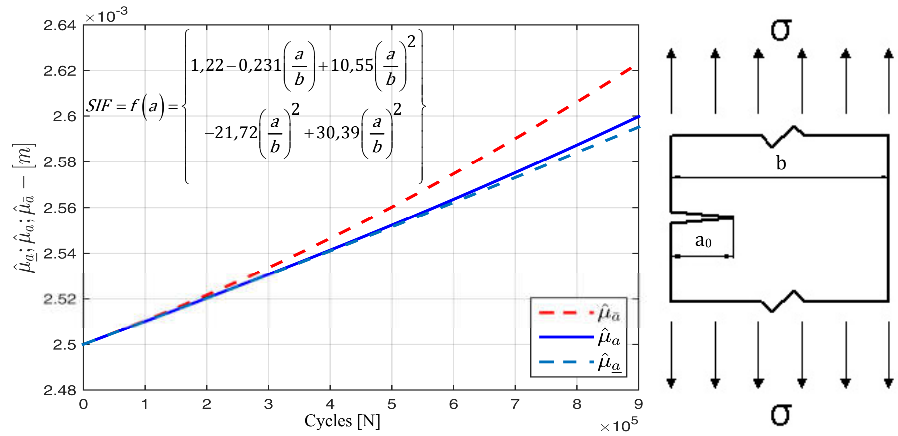

Figure 4. Results of RK4-MC vs. FCB-MC bounds - random parameter a_0.

6 Conclusions

The methodology used to quantify uncertainty uses, together, the Fast Crack Bounds and Monte Carlo simulation methods. Checking the obtained results, according to Tab. 2, it was possible to verify how efficient the FCB-MC methodology is good for the evaluation of a given crack size function. This efficiency was measured through the computational times and the numerical results of the bounds in the statistics first moment, shown as a graph in the chapter results. Efficiency was observed for the three deterministic examples analyzed. This means the FCB-MC methodology has a good performance, faster and requires less computational effort in mathematical solution compared to the RK4-MC methodology.

Acknowledgements. Thank UTFPR, the PPGEM and Numat for cooperation in the development of this work.

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