Basic Fundamental Approach of 2nd Order Geometric Nonlinear Analysis: Concepts and Computational Implementation

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Abstract. One of the main objectives of structural engineering has been to make the structures slenderer and more economical, reducing their weight and the consumption of materials without, however, compromising their stability. The increase in the slenderness of the structural elements makes them more susceptible to large lateral displacements before their rupture occurs. The stability analysis of slender structural systems currently involves the application of the Finite Element Method (FEM). As a consequence, a system of non-linear algebraic equations is generated and its solution is obtained, in general, through incremental-iterative procedures. This article initially presents structural single-degree-of-freedom systems (SDF, scalar variables), subjected to geometric nonlinear behavior, showing their analytical and numerical solutions, using the Principle of Stationary Total Potential Energy. Four SDF systems, with different behaviors, are presented: stable or unstable post-critical behavior and with and without bifurcation. Geometric imperfections are incorporated. The work ends with the presentation of two-degree-of-freedom systems, where the variables become matrix and vector, expanding the concepts of SDF systems for the multi-degree-of-freedom (MDF) ones.

Keywords: Structural Stability, Geometric Nonlinear Analysis, Newton-Raphson method.

1 Introduction

Creating the most economic structures through reduction of material consumption and overall weight without loss of safety and durability has been the main objective of structural engineering.

Weight reduction has been achieved in concrete structures, by increased use of high-strength concrete (generally greater than 20 MPa), especially those with resistances greater than 50 MPa, and steel structures with resistances exceeding 250 MPa. Associated with the use of refined analysis and more accessible and powerful computers, this has led to bolder projects with thinner structural elements. Consequently, compressed structural elements are more susceptible to large lateral deflection, increasing the possibility of loss of stability, demanding nonlinear analysis. This phenomenon is analyzed with adequate depth in the theory of elastic stability, as Ferreira [1] proposed in recent researches. This work aims to present concepts and terminology relevant to the loss of stability of structural systems which are often not approached with adequate depth in the solid mechanics books at undergraduate civil engineering and mechanical engineering courses.

2 Concepts of stability

The stability of equilibrium is a basic concept of rigid body mechanics, which can be easily viewed and intuitively assimilated through the classic problem of spherical mass lying in straight or curved surfaces, as Fig. 1 illustrates.

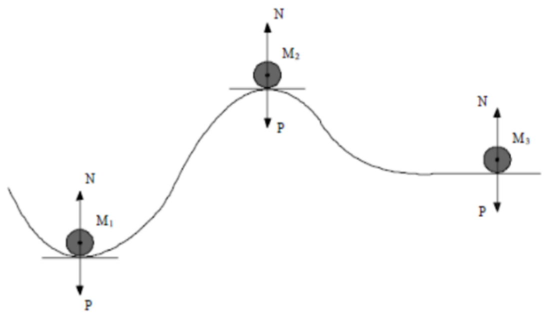


Figure 1. Spherical masses in static equilibrium

The points where the masses M_1, M_2 and M_3 rest have zero slope and represent points of static equilibrium, however, the type of equilibrium achieved at each of these points is essentially different. If the mass M_1 undergoes a small external disturbance, when the cause of trouble is removed it returns to the starting equilibrium position. In this case, the original position is considered a stable equilibrium state. It is observed that in this case, the center of gravity rises, thus increasing the potential energy of the system ($\Delta\Pi > 0$). For mass M_2, unlike what happened with mass M_1, the original position is an unstable equilibrium state, because after a small disturbance, the static forces acting upon the system tend to displace the ball away from the equilibrium position. In this case there was a lowering of the center of gravity and, consequently, a decrease in the potential energy of the system ($\Delta\Pi < 0$). In the third case, when the weight rests on a flat surface, the system is referred as being in a state of neutral stability (or state of indifferent stability), i.e., in any position the ball remains in equilibrium. Here the center of gravity of the ball remains at the same level, and therefore no variation in the potential energy occurs ($\Delta\Pi = 0$, Ferreira [1]). To study the stability of structural systems, three criteria can be used. The static criterion of stability, which examines the equilibrium of forces, the energy criterion of stability, which examines the variation of the total potential energy, and the dynamic criterion of stability, based upon concepts from vibration theory.

The loss of stability is a nonlinear phenomenon. Therefore, to understand accurately, the behavior of the system under this effect, one has to use nonlinear analysis. When analyzing the stability of a structural system, a set of control parameters is used. To understand the overall behavior of the system and identify the possible instability phenomena, the variations of the equilibrium position with respect to changes in each control parameter have to be studied. Thus, the so-called equilibrium paths are obtained. Along these paths, the equilibrium configurations may be qualitative changes with regard to their stability. According to Silveira [2], the border points are called critical points, that which can be of two types: bifurcation points or limit points.

3 Stable-symmetric bifurcation system

3.1 Analytical nonlinear solution

In real cases, the structures often have imperfections: there are no perfectly straight bars. Thus, such imperfection is simulated by means of an initial angular deflection φ_0 in the bar, with respect to its vertical position, as Fig. 2 indicates.

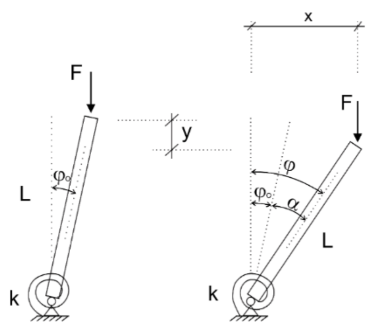


Figure 2. System showing a no weight rigid bar with imperfections and circular spring

The total potential energy (Π) depends on the angular displacement (φ). For $\varphi = 0$ and $\varphi_0 = 0$, which corresponds to the vertical bar (original equilibrium position), the solution generates a graph at the coordinates $FL\ /\ k\ \times\ \varphi$ called the fundamental path. For $F = k\varphi\ /\ Lsin\varphi$, the solution is called post-critical, and its graph at coordinates $FL\ /\ k\ \times\ \varphi$ called secondary equilibrium path or post-critical path, as shown on Tab. 1.

Table 1. Stability study

Solution: $\varphi = 0$		Solution: $F = \frac{k\varphi}{Lsin\varphi}$
$\frac{d^2\Pi}{d\varphi^2} = k - FL$		$\frac{d^2\Pi}{d\varphi^2} = k(1 - \varphi\text{cotg}\varphi)$
$k - FL > 0$ or $F < k/L$	$k - FL < 0$ or $F > k/L$	For $\lvert\varphi\rvert < \pi, \frac{d^2\Pi}{d\varphi^2} \geq 0$
Stable equilibrium Fundamental path	Unstable equilibrium Fundamental path	Stable equilibrium Post-critical path

Table 2 shows the equations for the potential energy of elastic spring, the potential of F and the total potential energy ($\Pi = U + V$).

Table 2. Total potential energy on system with imperfections

Potential energy of elastic spring	Potential of load F	Total potential energy
$U = \frac{(\varphi - \varphi_0)^2}{2}$	$V = FL(cos\varphi - \cos\varphi_0)$	$\Pi = \frac{(\varphi - \varphi_0)^2}{2} + FL(cos\varphi - \cos\varphi_0)$

For the energy criterion of stability, the system is in equilibrium (stable, unstable or indifferent) when the equation $d\Pi/d\varphi = 0$ is satisfied, or when:

$$k(\varphi - \varphi_0) - FLsin\varphi = 0 \tag{1}$$

This equation shows that $\varphi\ =\ 0$ is no longer the problem solution. Isolating F in eq. (1),

$$F = \frac{k(\varphi - \varphi_0)}{Lsin\varphi} \tag{2}$$

Figure 3 shows the critical paths of the imperfect system illustrated in Fig. 2, along with the those of the perfect system.

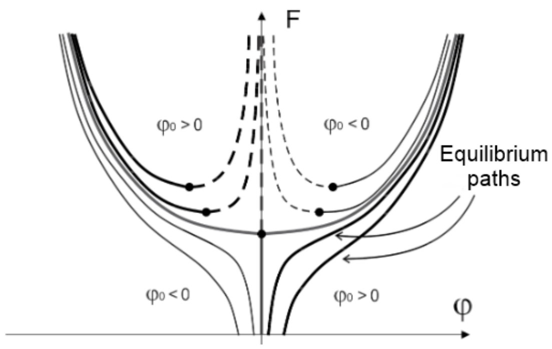


Figure 3. Nonlinear solution of system with imperfections and circular spring

It is observed that there is no bifurcation, the solutions of the nonlinear imperfect system bordering the solutions of the perfect system (there is an asymptotic approximation). The paths above the perfect system curve are called complementary equilibrium paths, as said by Croll and Walker [3], and can only be achieved in dynamic problems. This example presented the equations for the total potential energy and the equations resulting from applying the energy criterion of stability.

The formulation of ABNT NBR 14762:2010 [4] for cold formed profile considers the use of the post-critical behavior of plates, which make up the steel profile (Fig. 4), for the calculation of the column's resistant capacity.

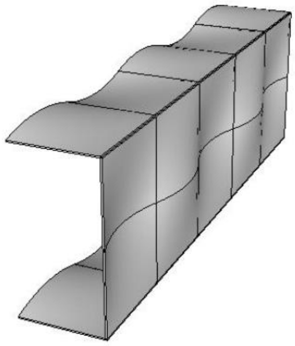


Figure 4. Post-critical behavior of a cold formed profile

3.2 Numerical nonlinear solution using Newton-Raphson procedure

The numerical solution is performed by an iterative process involving two stages. The first step involves the incremental offsets from a given load increase, where unbalanced forces appear. The second stage is a correction process, by searching the equilibrium of forces, from a convergence criterion, until

$$F_{int} - F_{ext} \cong 0 \tag{3}$$

Where F_{int} is the internal force, a function of displacement, and F_{ext} is the external force.

Now, the nonlinear problem of the mechanic system illustrated in Fig. 2 can be solved, through this procedure, where $F_{int} = K(\varphi - \varphi_0)$, $F_{ext} = PLsin\varphi$ and $F_{int} - F_{ext}$ is the unbalanced force. The following methodology described in Tab. 3, used in this study, is based primarily on the iterative-incremental solution of Eq. (3).

Table 3. Solution strategy nonlinear through the Standard Newton-Raphson method

1. Initial configuration: φ_0
2. Incremental iterative loop (external loop)
   Increases F
3. Iterative cycle (internal loop, Newton-Raphson iteration)
   a. It calculates the second differential of Π
   b. It calculates the unbalanced load vector $F_{int} - F_{ext}$ (first differential of Π)
   c. It checks the convergence: $F_{int} - F_{ext} < tolerance$
      If true, it returns to item 2
   d. If false, it updates displacement (φ)
      $\varphi = \varphi - (d\Pi/d\varphi)/(d^2\Pi/d\varphi^2)$
      It returns to item 3

4 Unstable-symmetric bifurcation system

Figure 5 shows a system consisting of a vertical bar with compression load F at the upper end associated to a linear spring of stiffness K.

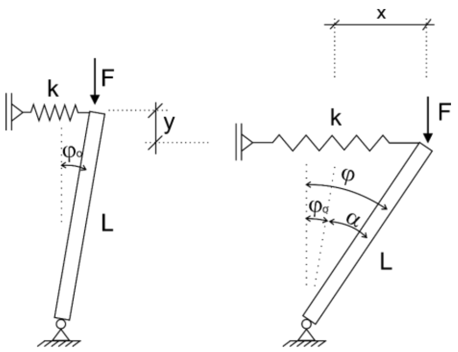
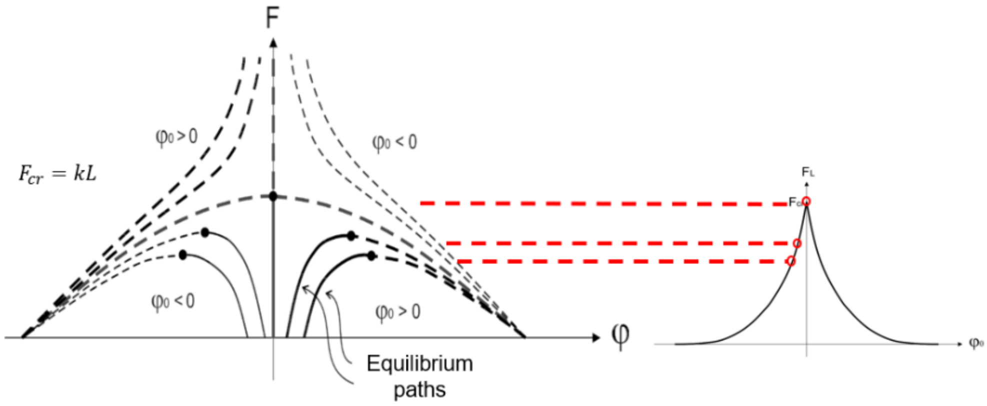


Figure 5. Rigid bar with linear spring

The analysis of this system shows that: for $F < KL$, the equilibrium is stable, but when $F > KL$, the path is the postcritical and it is always unstable, with symmetric bifurcation. This case is known as unstable-symmetric bifurcation system, according to Allen and Bulson [5]. The imperfections in this system are treated similarly to the case of the rigid bar with circular spring (Fig. 2). Figure 6 shows this behavior.



(a) (b)

Figure 6. Nonlinear solution of the system of with imperfections and variation of limit load

In this system, is verified that starting from $F = 0$ (when $\varphi_0 > 0$), the critical path is stable until it reaches a maximum value represented by F_L, from which it becomes unstable. The load is called buckling load or limit load of the imperfect structure and the corresponding point is called a limit point (limit load point). From the limit point there is instability with the strains growing indefinitely (Fig. 6 (a)). This process of loss of stability is called dynamic jump. As it increases φ_0 imperfection of the system, the limit load becomes lower (Fig. 6 (b)). Figure 7 shows a practical example with behavior similar to that shown in Fig. 6 (see Gonçalves and Batista [6]).

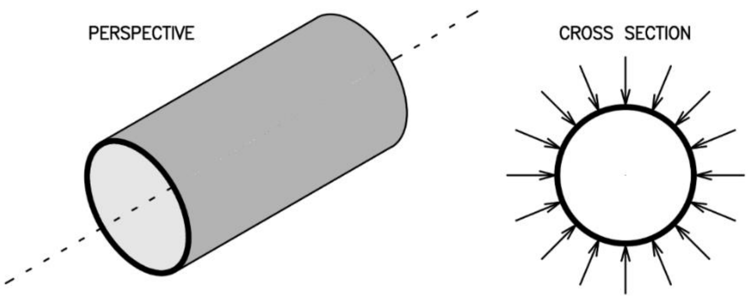


Figure 7. Stiffened cylinder under compression

5 System with no bifurcation

The structure shown in Fig. 8(a) consisting of two rigid bars freely hinged to each other and with two supports: support C attached to a linear spring of stiffness K. Both bars have initial φ_0 slope.

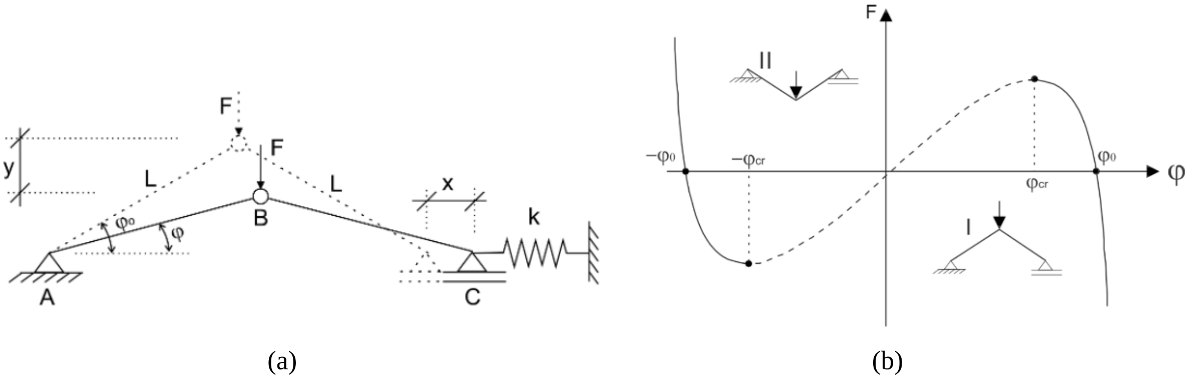


Figure 8. System with no bifurcation

Comments will be made only for $\varphi > 0$. By increasing load F from zero, the angle φ, which for the system without loading is φ_0, will decrease until it reaches a critical point φ_{cr}. An infinitesimal increment greater than this value there will be a sudden change in system configuration, passing the configuration I to configuration II (Fig. 8(b)). This abrupt configuration changing is called a dynamic jump.

6 Systems with multiple degrees of freedom

In this study, using the energy criterion of stability, a structural model with two degrees of freedom, consisting of three weightless rigid bars (length L each). The bars are freely hinged to each other . At the lower end there is a pinned support and at the top a roller support. The model is supported by two linear springs with stiffness K and it is subject to a compressive vertical load. All concepts that were seen for systems with single degree of freedom are valid for systems with multiple degrees of freedom. Thus, only a compact representation of this system of equations for two degrees of freedom, as represented on Fig. 9, will be presented.

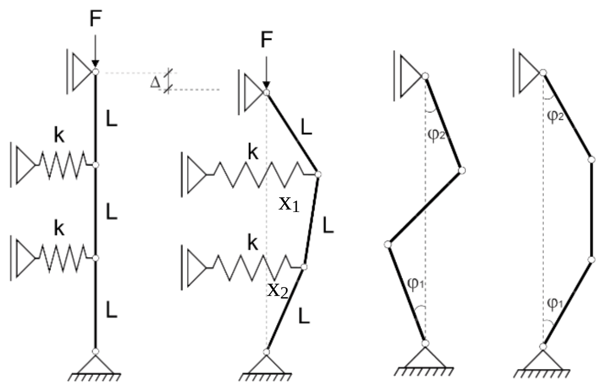


Figure 9. Proposed structure for analysis and configuration mechanism after it loses stability

For structural system shown, Eq. (4) defines the potential energy from values x_1, x_2 and Δ, as follows

$$U(\varphi_1, \varphi_2) = \frac{kx_1^2}{2} + \frac{kx_2^2}{2} \tag{4}$$

$$V(\varphi_1, \varphi_2) = -F\Delta \tag{5}$$

Thus, as in Souza [7],

$$\Pi(\varphi_1,\varphi_2)=\frac{kL^2}{2}(sin^2\varphi_1+sin^2\varphi_2)-FL\left[3-cos\varphi_1-cos\varphi_2-\sqrt{1-(sin\varphi_1-sin\varphi_2)^2}\right] \tag{6}$$

Now, using the Principle of Stationary Potential Energy11, the pair of equations of nonlinear equilibrium becomes:

$$\frac{d\Pi(\varphi_1,\varphi_2)}{d\varphi_1}=0 \tag{7}$$

Thus,

$$kL^2sin\varphi_1cos\varphi_1-FL\left[sin\varphi_1+\frac{(sin\varphi_1-sin\varphi_2)cos\varphi_1}{1-(sin\varphi_1-sin\varphi_2)^2}\right]=0 \tag{8}$$

And

$$\frac{d\Pi(\varphi_1,\varphi_2)}{d\varphi_2}=0 \tag{9}$$

Thus,

$$kL^2sin\varphi_2cos\varphi_2-FL\left[sin\varphi_2+\frac{(sin\varphi_2-sin\varphi_1)cos\varphi_2}{1-(sin\varphi_2-sin\varphi_1)^2}\right]=0 \tag{10}$$

So, the procedure follows as seen to single-degree-of-freedom systems.

7 Conclusions

This article aims to present relevant concepts and terminology of the theory of structural stability. To achieve it, mechanical models with rigid weightless bars associated with circular or linear springs were used. By studying the stability of these simple models all the concepts and behaviors commonly used to study the stability of real structures have been discussed, such as plane frames, arches and others. This text is an important teaching aid in this subject, for students, also serving as a pedagogical support to teachers in the basic disciplines and professional courses in civil and mechanical engineering, not familiar the geometric nonlinear analysis.

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