
# Numerical Simulation of Oil and Water Displacements in Petroleum Reservoirs Using a Non-Linear Two-Point Flux Approximation Method Coupled to a Modified Flow Oriented Formulation Using a Sequential Implicit Procedure

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**Abstract.** The numerical modeling of the multiphase and multicomponent flow in oil reservoirs is very complex and demands the development of robust and efficient computational tools. Two important issues whenever designing numerical formulations for the modeling such problems are the so-called Grid Orientation Effect (GOE), which is the dependence of the numerical solution with the spatial alignment of the computational grid, and the non-monotonicity of the solution generally obtained when highly anisotropic reservoir rocks or distorted meshes are addressed. The GOE effect is linked to the anisotropic distribution of the truncation error in the numerical approximation of the transport term. The uneven amount of anisotropic numerical diffusion introduced in each direction may trigger a nonlinear process in which the error may grow exponentially, particularly when the displaced fluid (e.g. heavy oil) is much less mobile than the displacing fluid (e.g. water). Due to the GOE, different and, usually, wrong solutions may occur considering grids with different spatial orientations with respect to the direction of the fluid flow. Besides, if the mesh is non k-orthogonal, classical Linear Two-Point Flux Approximation (L-TPFA) methods, may not even converge to the proper solution. Even the use of more robust linear Multipoint Flux Approximation (MPFA) methods may produce pressure fields with spurious oscillations for highly anisotropic media or distorted meshes. Therefore, we propose a truly multidimensional cell centered finite-volume method to simulate oil and water displacements in heterogeneous and anisotropic petroleum reservoirs. The sequential implicit procedure is used to handle the coupling between the pressure and the saturation equations. The elliptic pressure equation is discretized by a positivity preserving Non-Linear Two-Point Flux Approximation (NL-TPFA) using harmonic averaging points located on cell edges. A variation of the first order Modified Flow-Oriented Scheme (M-FOS) is used to implicitly solve the non-linear hyperbolic saturation equation. Adaptive weights tune the formulation multidimensionality according to the grid in this scheme. This strategy is used to reduce the occurrence of Grid Orientation Effects (GOE). In order to verify the accuracy and robustness of our formulation, we test it against classic benchmarks available in literature. In the future, we intend to extend our formulation to higher-order, to improve front resolution and further reducing GOE.

**Keywords:** Reservoir Simulation, Grid Orientation Effects (GOE), Non-Linear Two Point Flux Approximation (NL-TPFA), Modified Flow Oriented Scheme (M-FOS), Sequential Implicit Procedure.

# 1 Introduction

The numerical simulation of fluid flow and transport in highly heterogeneous and anisotropic oil and gas poses a great challenge for modern numerical algorithms. Usually, the industrial software programs approximate the governing equations by Two Point Flux Approximation (TPFA) to estimate the diffusive fluxes and the First Order Upwind Scheme (FOUS) to estimate the advective fluxes. Although this strategy is computationally efficient and simple to implement, although it is unable to properly deals with anisotropic full permeability tensors and complex geometries. In addition, this method promotes excessive spreading of the saturation fronts and is deeply sensitivity to the Grid Orientation Effect (GOE), which is related to the hard dependence of the numerical solution on the orientation of grid lines.

In order to diminish these problems it is required a robust formulation able to correctly express the anisotropy and heterogeneity for the fluxes. In this work, we proposed the adoption of a Non-Linear Finite Volume Positive–Preserving (NLFV–PP) scheme developed by [1] to solve the elliptic equation and a Modified Flow Oriented Scheme (M-FOS) developed by [2] to solve the hyperbolic equation.

The NLFV-PP scheme provides accurate results even when tackling anisotropic permeability tensors and it's able to deal with any star-shaped polygonal meshes, furthermore, it respects the linearity preserving criterion. Aiming to use the NLFV-PP, first it is required construct the one-sided fluxes on each cell independently, afterward integrate the two one-sided fluxes on both sides. Then, to recover a unique flux expression we express the cell edge fluxes as a convex combination of the one-sided fluxes. On other hand, to obtain the weights which are used to multiply the one-sided fluxes we just impose the flux continuity over the edges. These weights are positive and dependent on the unknown pressures and so the final scheme becomes a Non-Linear TPFA.

In the literature it is common the use of flow-oriented schemes (FOSs) aiming to reduce the grid orientation dependence. We can quote some works related to the context of petroleum reservoir simulation such as [3], [4], [5] and [2]. The M-FOS is able to reduce the GOE mainly in non-orthogonal grids. This scheme is characterized by the use of a convex combination of the water fractional fluxes values following the approximate flow orientation throughout the computational domain.

As far as the authors know, it's the first time that is proposed a non-linear scheme to solve the elliptic equation associated to a truly multidimensional scheme to the hyperbolic equation. Besides, our framework is able to deal with the Implicit Pressure and Explicit Saturation (IMPES) and with the Sequential Implicit (SEQ) strategies. In the IMPES the pressure is solved implicitly while the saturation is solved explicitly and in the SEQ both, pressure and saturation, are solved implicitly. Finally, we present a comparison between the classic upwind and the M-FOS using both, IMPES and SEQ, procedures.

# 2 Mathematical Formulation

In this section, we will describe the basic governing equations for the two-phase flow of water and oil in petroleum reservoirs. We adopted incompressible fluid flow and rock, in the absence of capillarity, gravity and thermal effects. Besides, we consider that the reservoir rock is fully saturated by water and oil and by using the mass conservation equation and the Darcy's law [2] we obtain the pressure and saturation equations, respectively, as follows:

$$\nabla.\vec{v} = Q \quad with \quad \vec{v} = -\lambda \bar{K} \nabla P \tag{1}$$

$$\phi \frac{\partial (S_i)}{\partial t} = -\nabla.\vec{F}(S_i) + Q_i \quad with \quad \vec{F}(S_i) = f_i \vec{v} \tag{2}$$

In Eq.1, $Q$ means the total fluid production or injection specific rate with $Q_i = q_i/\rho_i$ with $Q = Q_w + Qo$. The absolute rock permeability is represented by $\bar{K}$. In addition $\nabla P$ is the pressure gradient. Besides, $\lambda$ is the total mobility which is given by $\lambda = \lambda_w + \lambda_o$. In Eq.2 $\vec{F}(S_i)$ is the advective flux function and $f_i$ is the fractional flux function which can be expressed by $f_i = \lambda_i/\lambda$. Furthermore, $\phi$, $\rho_i$ and $S_i$ represents the rock porosity, density and saturation of each phase i (i=w for water and i=o for oil), respectively. To complete the mathematical model adequate boundary and initial conditions are required, such as:

$$\begin{aligned} p(\vec{x},t) &= g_D \quad on \quad \Gamma_D \times [0,t] \\ \vec{v} \cdot \vec{n} &= g_N \quad on \quad \Gamma_N \times [0,t] \\ S_w(\vec{x},t) &= \bar{S}_w \quad on \quad \Gamma_I \times [0,t] \ or \ \Gamma_D \times [0,t] \\ S_w(\vec{x},0) &= \bar{S}_w^0 \quad on \quad \Omega \times t^0 \end{aligned} \tag{3}$$

where $\Gamma_I$ represent the injector wells and $\Gamma_N$, $\Gamma_D$ are, in this order, the Neumann and Dirichlet conditions. Besides, and mean the prescribed pressures and fluxes respectively. $\bar{S}_w$ is the prescribed water saturation attributed at the injection wells, and in addition $\bar{S}_w^0$ is the initial water saturation distribution.

## 3 Numerical Formulation

By integrating the Eqs.1 and 2 on each control volume of computational domain and using the mean value and Gauss's divergence theorem, we obtain the elliptic pressure equation and hyperbolic saturation equation, respectively:

$$\sum_{IJ\in\Gamma_k} \vec{v}_{IJ}\cdot\vec{N}_{IJ} = \bar{Q}_{w(k)}\Omega_k \tag{4}$$

$$\bar{\phi}_k \frac{\Delta\bar{S}_{w(k)}}{\Delta t}\Omega_k = -\sum_{IJ\in\Gamma_k} (f_w)_{IJ}\vec{v}_{IJ}\cdot\vec{N}_{IJ} + \bar{Q}_{w(k)}\Omega_k \tag{5}$$

where $\bar{S}_{w(k)}$ and $\bar{\phi}_k$ are the representative mean values of the water saturation and porosity, respectively, to a generic control volume $(k)$. $\bar{Q}_{w(k)}$ is sink/source term. Besides, $(f_w)_{IJ}$ and $\vec{v}_{IJ}$ are, respectively, the fractional flux and the Darcy's velocity associated to edge $IJ$.

In this work, we use the sequential implicit (SEQ) strategy to solve the pressure and saturation equations in a segregated procedure. In the SEQ strategy, known an initial saturation distribution, we calculate the total mobility throughout the domain and we compute both the unknown pressure field and the unknown saturation field implicitly. After that, the new saturation field is used to update the mobilities which is take into account when the pressure field is evaluated. This process happens until the final time of the simulation.

### 3.1 Implicit Pressure Discretization Using NL-TPFA

This method was developed in the work of [1], in which it's used the harmonic points strategy originally inspired in the work of [6] and [7]. This scheme guarantees monotone solutions on arbitrary anisotropic permeability tensors and any star-shaped polygonal meshes. To solve the non-linear system, we adopted the Picard scheme because it preserves solution non-negativity on each iteration. The slow convergence behavior of the Picard scheme drove us to use the Anderson acceleration (AA) strategy.

### 3.2 Implicit Saturation Discretization Using FOUM

Considering a backward Euler and First Order Upwind Method (FOUM), we can rewrite Eq.5 in matrix form such as:

$$S_w^n + \Delta t\,[G]\,f_w^{n+1} - S_w^{n+1} = 0 \tag{6}$$

where $S_w^{n+1}$ and $f_w^{n+1}$ represent, respectively, the vectors containing the water saturation and water fractional flux in each grid cell at the $n+1$ timestep. Moreover, $G$ is a matrix that computes the fluxes and source/sink terms, which entries are given by:

$$G_{kk} = \frac{1}{\phi_k\Omega_k}\left[\bar{Q}_k\Omega_k - \sum_{IJ\in\Gamma_k}\chi\left(\vec{v}_{IJ}\cdot\vec{N}_{IJ}\right)\right]$$

$$G_{ki} = (\chi-1)\,\frac{\vec{v}_{IJ}\cdot\vec{N}_{IJ}}{\phi_k\Omega_k}, \quad IJ = \Gamma_k\cap\Gamma_i \tag{7}$$

$$\chi = \begin{cases} 1, \text{ if k-th mesh cell is the CV upwind to } \vec{v}_{IJ}\cdot\vec{N}_{IJ} \\ 0, \text{ if i-th mesh cell is the CV upwind to } \vec{v}_{IJ}\cdot\vec{N}_{IJ} \end{cases}$$

In Eq. 7, $\bar{Q}$ is the mean source/sink term corresponding to $kth$ cell. In order to solve implicitly the non-linear saturation equation the classic Newton-Raphson method was adopted, following the works of [8], [9] and [10]. So, in the ith iteration, the Eq. 6 becomes:

$$S_w^n + \Delta t\,[G]\,f_w^{n+1}\Big|^i - S_w^{n+1}\Big|^i = r^i \tag{8}$$

The convergence is achieved when the $L_\infty$ norm of the residue, $r$ is below or equal to a prescribe tolerance. Aiming to compute the saturation vector at $(i+1)th$ iteration Eq. 8 can be expressed as:

$$S_w^{n+1}\Big|^{i+1} = S_w^{n+1}\Big|^i - J\left(S_w^{n+1}\Big|^i\right)^{-1} r^i \quad with \quad J\left(S_w^{n+1}\right) = \Delta t\,[G]\left(I\left[\dot{f}_w^{n+1}\right]\right) - I \tag{9}$$

in which I is the identity matrix and $\dot{f}_w^{n+1}$ is a vector of partial derivates of water fractional flux in relation to saturation in each grid cell. Until the convergence criterium is reached, the saturation vector computed by Eq. 9 is used as input to the Eq. 8 in the $(i+1)th$ iteration. In addition, if the quantity of required iterations to reach the convergence criterium is bigger than the chosen value, it's possible to decrease the timestep by half of the presviously timestep and execute this whole procedure again, this prevents the iteration process from being extending too far.

### 3.3 Finite Volume Discretization of the Saturation Equation

In this section, we describe the Modified Flow Oriented Scheme (M-FOS) developed by [2] and extended here in an implicit fashion. This method influences the way fractional flux is calculated. When it's used classic upwind the fractional flux only depends on the information of the upstream control volume, however when it's used the M-FOS method the fractional flux becomes a composition of the contribution of many control volumes. The Fig.1 represents the median dual iteration region obtained by connecting the centroid of the cells, surrounding a general vertex J, and the midpoint of the edges sharing this common vertex.

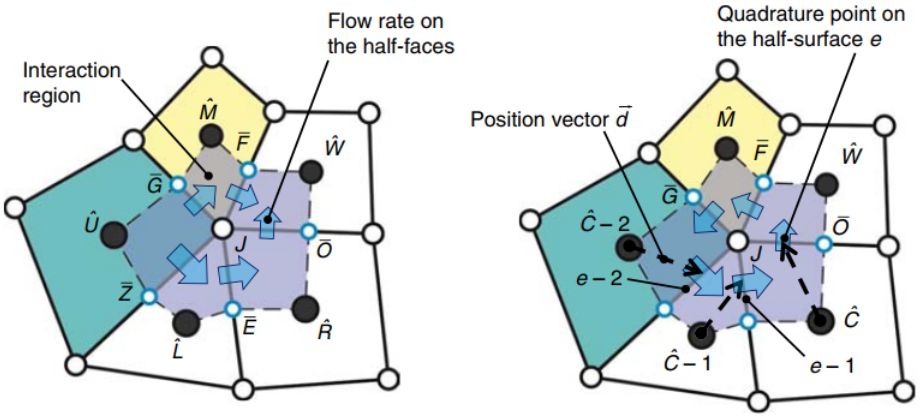



Figure 1. Fractional flow composed of the contributions of the control volumes. Adapted from [2].

Aiming to obtain the equation that computes the flow-oriented fractional flux is required to consider the number of upstream half-edges associated with the one evaluated. Therefore, the equation that calculates the fractional flux on edge in a generical edge is denoted as follows:

$$f_{w_i} = \frac{1}{\left(1 - \prod_{j=1}^{N_{HE}} w_j\right)} \left[ (1 - w_i)\, f_{w_{\hat{C}}} + \sum_{k=1}^{\max(1, N_{UHE})} \left( \prod_{m=0}^{k-1} w_{i-m} \right) (1 - w_{i-k})\, f_{w_{\hat{C}-k}} \right] \tag{10}$$

where $N_{HE}$ and $N_{UHE}$ are, respectively, the number of half edges and the number of upstream half edges. $w_{\overline{JE}}$ is an adaptative normalized weight which varies between 0 and 1. If $w_{\overline{JE}}$ is equal to zero it means that there's no contribution of any upstream edge on the calculation of fractional flux at edge $JE$, that is, the standard upwind formulation is recovered in this case.

Whenever it's used a "truly" multidimensional formulation, we need to define the local flow-rate ratio ($\Lambda_e$) as follows:

$$\Lambda_e = \frac{\vec{V}_{e-1} \cdot \vec{N}_{e-1}}{\vec{V}_e \cdot \vec{N}_e} \tag{11}$$

Aiming to calculate the adaptative normalized weight ($w$), two common approaches come up, in which ($w$) is a function of $\Lambda$: tight multidimensional upstream (TMU) and smooth multidimensional upstream (SMU), which are defined by Eq. 12.

$$w_{\bar{e}}^{TMU} = \min\left(1, \beta \Lambda_e\right) \quad and \quad w_{\bar{e}}^{SMU} = \frac{\Lambda_e}{1 + \Lambda_e} \tag{12}$$

The parameter $\beta$ present in Eq. 12 is, in general, unitary, such as used by [3], [5] and [11]. According to [5] and [11], the choose of the type of weighting function strongly influences the diminish of GOE for some benchmarks problems.

## 4 Numerical Experiment: Homogeneous – Three – Wells Test Case With Rotated Meshes

This problem was proposed by [5] and we have used it to analyze the GOE in the saturation front when the computational grid is rotated. The domain is delimitated by a square $[-0.5; 0.5]^2$ rotated by $\alpha = 30^\circ$ that is defined with the horizontal line. Furthermore, the domain has been discretized with a $51 \times 51$ uniform quadrilateral grid and initially is fully saturated with oil.

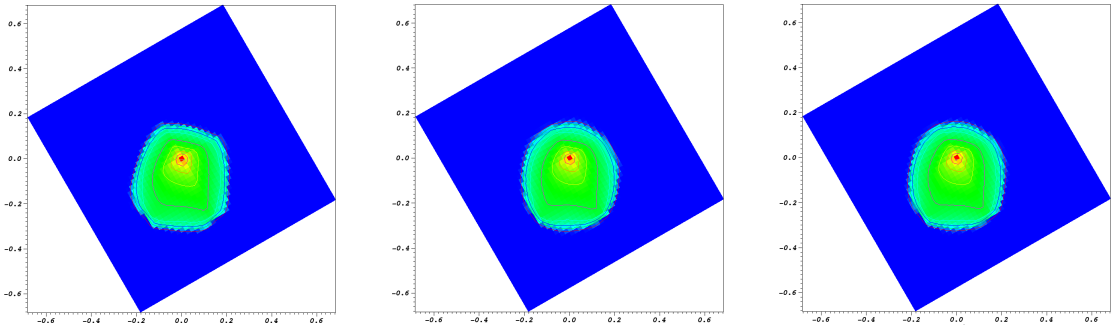
The production wells are symmetrically positioned at the points given by the coordinates $[+0.3\cos(30^\circ), -0.3\sin(30^\circ)]$ and the injection well is placed at the domain's center. The permeability tensor is defined by $K_1 = I$ and $K_2 = 10^{-6}I$, $I$ is the second order identity tensor. The porosity is constant throughout the domain. The viscosity ration between the two phases is designated by $M = \mu_o/\mu_w$ = 100.

In this problem, the water and oil relative permeabilities are defined by the Brooks-Corey model where $k_{rw} = (S_w)^4$ and $k_{ro} = (1 - S_w)^2$, in this order. The pressure at the productions wells are equal to zero $(\bar{p}_{pord} = 0)$. The water saturation and water flow-rate are, respectively, defined by $\bar{S}_{w,inj} = 1$ and $\bar{Q}_{w,inj} = 1$.

Based on the symmetry of the location of the wells we know the saturation front should be ideally the same even if the computational grid rotates by the angles previously defined. Besides, the watercut curves corresponding to both producer wells should be exactly the same. Any difference between these curves is a prove of GOE occurrence. Our analysis begins by comparing our results with those reported by [5]. In this paper, the authors use the NL-TPFA to solve the pressure equation and the flow-oriented variants TMU and SMU both with first order accuracy to solve the saturation equation. Average weighting based on the volumes that sharing a common face is used to approximation of the total mobility.

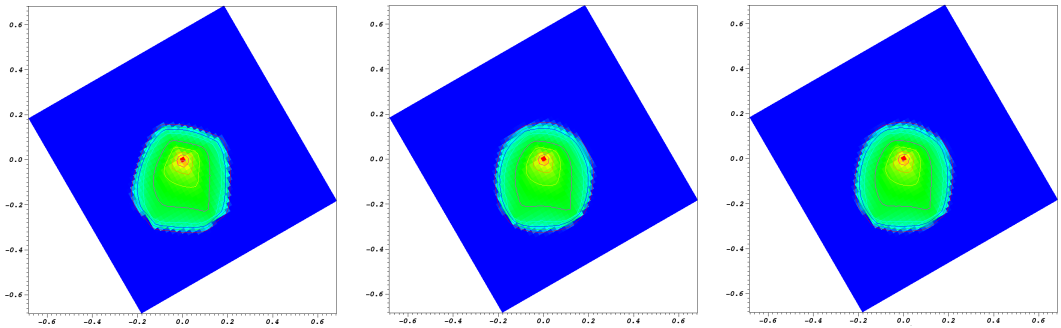
In Fig.2, we present the saturation fields for the domain rotated by $\alpha = 30^\circ$ using the IMPES strategy, in which the pressure method was the Multi-Point Flux Approximation D (MPFA-D). This figure was obtained from [12]. In Fig.3, we show our results using the SEQ strategy, in which the NL-TPFA was adopted as pressure method. The saturation method was the same in Fig.2 and Fig.3 and is denoted in the subtitle of the figures.

In this case, there is no symmetry of the flow with respect to the grid orientation because of the rotation of the domain. So, it is expected the occurrence of the GOE and consequently are expected differences in the breakthrough time for each producer well as seen in Fig.4.



(a) Standard Upwind. (b) M-FOS with TMU. (c) M-FOS with SMU.

Figure 2. Saturation field using IMPES strategy obtained by [2] for domain rotated ($\theta = 30^\circ$).



(a) Standard Upwind. (b) M-FOS with TMU. (c) M-FOS with SMU.

Figure 3. Saturation field using SEQ strategy obtained by the authors for domain rotated ($\theta = 30^\circ$).

In this case, the Standard Upwind was the most sensitive scheme to the GOE, thus it is possible notice the M-FOS is able to reduce the grid orientation effect. Furthermore, the SMU approach of compute the adaptative normalized weight performed slightly better than TMU.

The results presented in Figs.2 and 3 using the MPFA-D and the NL-TPFA, respectively, were quite similar.

Although this case does not highlight all robustness of the method, the NL-TPFA is able to deal with more difficults problems such as in highly heterogeneous and anisotropic media.

In the IMPES strategy the saturation equation is calculated explicitly, thus the timestep used in the simulation requires restriction to ensure stability. In this case it was adopted a Courant Number equal to 0.3. On other hand, in the SEQ strategy the saturation equation is computed implicitly, so we used a Courant Number equal to 4. Therefore, a advantage in the SEQ strategy is that it requires less timesteps to end the simulation.

## 5 Conclusions

In this paper we presented a robust NL-TPFA method coupled with a first order modified flow oriented scheme (M-FOS) to simulate two-phase flows of water and oil in anisotropic and heterogeneous petroleum reservoirs. The NL-TPFA method was adopted to solve the pressure equation and produced good approximate solutions when compared to MPFA-D, which is also a robust scheme. The M-FOS with TMU and M-FOS with SMU used to solve the saturation equation proved to diminish the GOE of the numerical solutions. The problem included adverse mobility ratios and anisotropic and heterogeneous permeability tensors.

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