Analysis of high order multilayered functionally graded composite beams: a numerical approach

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Abstract. Functionally graded materials (FGMs) are multiphase composites whose properties vary continuously in one or more directions, obeying a function that better suits the purpose of the structure. In the present work, the study of FGMs is integrated to multilayered composite beams with varying properties per layer. The continuity of the material properties from the FGM mode mitigates the stress peaks that occur in the interface between layers with different orientations. A unified high-order theory for beams incorporating FGMs is developed and applied to the Equivalent Single Layer theory approach to analyze single and multilayered beams using any order finite element method. Examples of simply supported beams are analyzed and compared to the results presented in the literature. The proposed formulation presented results in agreement with the literature.

Keywords: Composite materials, Functionally graded materials, Equivalent Single Layer theory, High-order beam theory, Unified beam theory

1 Introduction

Multilayered beams, of which laminate composite beams are an example, grant more optimal use of the stiffness properties of the materials. However, the interface between layers with very different properties may suffer from large stress jumps, damaging the structure's performance. Further development in engineering led to the creation of functionally graded materials (FGMs), in which the material properties vary continuously through the thickness of the beams. This concept has found many applications like thermal and corrosion barriers, medical implants, and lightweight armor material with high-ballistic efficiency [1].

The classical beam theory (CBT) developed by Euler-Bernoulli famously neglects shear effects, making it unsuitable for analysis of thick FG beams in which those effects are very prominent. In 1921, Timoshenko proposed what became known as the first-order shear deformation theory (FSDT) or Timoshenko beam theory (TBT), introducing a linear shear distribution. Albeit an improvement, this theory requires correction factors due to not satisfying the zero shear stress conditions at the top and bottom surfaces of the beam. Higher-order theories were developed to overcome these deficiencies in the CBT and TBT theories to allow an arbitrary shear strain distribution across the beam thickness [2].

In the present work, a unified multilayered beam theory is developed and paired with the distribution of material properties through the thickness seen in FGMs. The consideration of FGM is carried out by considering a continuous power function. For the numerical approach, the finite element method is used. Results are compared to analytical and experimental values from the literature for single and multilayered simply supported beams.

2 Analytical formulation

Consider a system of coordinates where x, y and z are taken along the beam's length, width and height respectively. The axial and transverse displacements (u and w, respectively) are described as functions of x and z only as shown in eqs. (1) and (2).

$$u(x,z) = u_0(x) - z\frac{dw(x)}{dx} + f(z)\phi(x) \tag{1}$$

$$w(x,z) = w(x) \tag{2}$$

In these equations, u_0 is the midplane axial displacement; $-dw/dx$ and ϕ are the bending and shear rotations of a given cross section respectively; and $f(z)$ is a shape function determining the distribution of shear strain and shear stress through the height of the beam. This function must be chosen so shear stress and shear strain at the top and bottom surfaces of the beam are null. Such is the case of eq. 3, which is a third order polynomial proposed by Reddy [3].

$$f(z) = z\left(1 + \frac{4z^2}{3h^2}\right) \tag{3}$$

The axial and shear strains are given by

$$\varepsilon_x = \frac{du_0}{dx} - z\frac{d^2w(x)}{dx^2} + f(z)\frac{d\phi(x)}{dx} \tag{4}$$

$$\gamma_{xz} = \frac{df(z)}{dz}\phi = f'(z)\phi \tag{5}$$

Consider the kth layer of the beam. Let there be a coordinate system $z^{(k)}$ through the layer's height, with origin in its middle. $E_b^{(k)}$ and $E_t^{(k)}$ are the bottom and top limiting values of Young's modulus for the given layer, and $h^{(k)}$ is its height.

The normal and shear stresses are per-layer functions tied to the strains for each layer via Hooke's law. Young's modulus is made to vary through the thickness of each layer k of the multilayered beam obeying the power law presented by Thai and Vo [4] and shown in eq. 6.

$$E^{(k)}(z^{(k)}) = E_b^{(k)}\left[1 + \left(\frac{E_t^{(k)}}{E_b^{(k)}} - 1\right)\left(\frac{1}{2} + \frac{z^{(k)}}{h^{(k)}}\right)^p\right] \tag{6}$$

where p is the power law index which governs the volume fraction gradation. With that, and given constant Poisson ratio $\nu^{(k)}$, the stress-strain relations are given by

$$\sigma_x^{(k)} = E^{(k)}\varepsilon_x \tag{7}$$

$$\sigma_{xz}^{(k)} = \frac{E^{(k)}}{2\left(1+\nu^{(k)}\right)}\gamma_{xz} = G^{(k)}\gamma_{xz} \tag{8}$$

In order to derive the equations of equilibrium, the variational principle is used. It is imposed that the variations of internal and external energy within the volume of the body equals zero. The internal energy variation δU is described in terms of the stresses and strains through the body's volume Ω (eq. 9), whereas for the external energy variation, the distributed load $q(x)$ through the length L of the beam is considered (eq. 10).

$$\delta U = \int_\Omega \sigma_x\delta\varepsilon_x + \sigma_{xz}\delta\gamma_{xz}\, dV \tag{9}$$

$$\delta V = -\int_L q(x)\delta w\, dx \tag{10}$$

Equations (1) and (4)-(5) are substituted into the sum of eqs. (9)-(10) and the order of the derivatives in the variational terms are reduced so only δu_0, δw, $\delta(\frac{dw}{dx})$ and $\delta\phi$ remain. By collecting the coefficients of those terms, the governing equations and boundary conditions are obtained in terms of stress resultants through the surface of cross section S presented below. Per the equivalent single layer (ESL) theory, the resultants are the sum of the integrals across each of the n layers of the beam. This method dramatically reduces the computational cost of the theory as the order of the problem isn't increased for beams with many layers.

$$\langle N, M, \hat{M}, \hat{V}\rangle = \sum_{k=1}^{n} \int_S \langle \sigma_x^{(k)} \varepsilon_x, z\sigma_x^{(k)}\varepsilon_x, f(z)\sigma_x^{(k)}\varepsilon_x, \sigma_{xz}^{(k)} f'(z)\gamma_{xz}\rangle \, dA \tag{11}$$

The governing equations are

$$\frac{dN}{dx} = 0 \tag{12}$$

$$\frac{d^2M}{dx^2} + q = 0 \tag{13}$$

$$\frac{d\hat{M}}{dx} - \hat{V} = 0 \tag{14}$$

The boundary conditions are: N or u_0 prescribed; M or dw/dx prescribed; $\hat{M}$ or ϕ prescribed; dM/dx or w prescribed.

The stress resultants can also be written as a function of the displacement components and of constant values that must be integrated through the cross section of the beam, as shown in eqs. (15)-(16).

$$\begin{Bmatrix} N \\ M \\ \hat{M} \end{Bmatrix} = \begin{bmatrix} A_0 & A_1 & B_0 \\ A_1 & A_2 & B_1 \\ B_0 & B_1 & D_0 \end{bmatrix} \begin{Bmatrix} u_0' \\ -\frac{d^2w}{dx^2} \\ \frac{d\phi}{dx} \end{Bmatrix} \tag{15}$$

$$\hat{V} = \hat{V}_0 \phi \tag{16}$$

where

$$\langle A_0, A_1, A_2, B_0, B_1, D_0\rangle = \sum_{k=1}^{n} \int_S E^{(k)} \langle 1, z, z^2, f(z), zf(z), [f(z)]^2\rangle \, dA \tag{17}$$

$$\hat{V}_0 = \sum_{k=1}^{n} \int_S G^{(k)} [f'(z)]^2 \, dA \tag{18}$$

3 Numerical formulation

3.1 Weak formulation

By the method of weak formulation, the governing eqs. (12)-(14) are multiplied each by weight functions μ, ω and ψ which are related, respectively, to the displacement quantities u_0, w and ϕ, and the expressions are integrated across length L, corresponding to interval (x_a, x_b). With the aid of the chain rule, the degree of the derivatives is weakened, boundary conditions are removed from the integral, and the expressions in eqs. (15)-(16) replace the stress resultants.

$$0 = \int_L \mu \left(\frac{dN}{dx} \right) dx = A_0 \int_L \frac{d\mu}{dx} \frac{du_0}{dx} dx - B_1 \int_L \frac{d\mu}{dx} \frac{d^2 w}{dx^2} dx + B_0 \int_L \frac{d\mu}{dx} \frac{d\phi}{dx} dx - \mu N \Big|_{x_a}^{x_b} \tag{19}$$

$$\begin{aligned} 0 = &\int_L \omega \left(\frac{d^2 M}{dx^2} + q \right) dx = \\ &- A_1 \int_L \frac{d\omega}{dx} \frac{du_0}{dx} dx + A_2 \int_L \frac{d\omega}{dx} \frac{d^2 w}{dx^2} dx - B_1 \int_L \frac{d\omega}{dx} \frac{d\phi}{dx} dx - \int_L \omega q \, dx - \omega \frac{dM}{dx} \Big|_{x_a}^{x_b} + \frac{d\omega}{dx} M \Big|_{x_a}^{x_b} \end{aligned} \tag{20}$$

$$\begin{aligned} 0 = &\int_L \psi \left(\frac{d\hat{M}}{dx} - \hat{V} \right) dx = \\ &B_0 \int_L \frac{d\psi}{dx} \frac{du_0}{dx} dx - B_1 \int_L \frac{d\psi}{dx} \frac{d^2 w}{dx^2} dx + D_0 \int_L \frac{d\psi}{dx} \frac{d\phi}{dx} dx + \hat{V} \int_L \psi \phi \, dx - \psi \hat{M} \Big|_{x_a}^{x_b} \end{aligned} \tag{21}$$

3.2 Finite element

Consider a finite element with m nodes. From eqs. (19)-(21), it can be seen that the quantity w is also present in its first derivative form and thus requires an approximating polynomial of continuity C_1. Hermite polynomials $H_i(x)$ of degree $2m - 1$ are picked.

$$w \approx \sum_{i=1}^{m} H_{2i-1}(x) t_i + H_{2i}(x) r_i \tag{22}$$

where t_i is the transverse displacement and r_i is the flexural rotation for each node. For the quantities u_0 and ϕ, polynomials of continuity C_0 are sufficient, so Lagrange polynomials are picked. However, additional nodes are placed between the existing ones to keep order consistency between the polynomials as much as possible. The existence of additional nodes provides a node quantity and degree of the Lagrange approximation of $2m - 1$ and $2m - 2$, respectively.

$$[u_0, \phi] \approx \sum_{i=1}^{m} L_{2i-1}(x)[a_i, s_i] + \sum_{i=1}^{m-1} L_{2i}(x)[a_i^*, s_i^*] \tag{23}$$

where a_i and s_i are the axial displacement and shear rotation values in the original nodes, and a_i^* and s_i^* are those quantities evaluated in the new intermediary nodes where transverse displacement and flexural rotation are not evaluated.

With eqs. (22)-(23) substituting the displacement quantities as well as the corresponding weight functions, eqs. (19)-(21) are rearranged into the following system:

$$\begin{bmatrix} [K_{11}] & [K_{12}] & [K_{13}] \\ [K_{21}] & [K_{22}] & [K_{23}] \\ [K_{31}] & [K_{32}] & [K_{33}] \end{bmatrix} \begin{Bmatrix} \langle a \rangle \\ \langle d \rangle \\ \langle s \rangle \end{Bmatrix} = \begin{Bmatrix} \langle 0 \rangle \\ \langle q \rangle \\ \langle 0 \rangle \end{Bmatrix} + \begin{Bmatrix} \langle F_1 \rangle \\ \langle F_2 \rangle \\ \langle F_3 \rangle \end{Bmatrix} \tag{24}$$

where

$$K_{11_{ij}} = A_0 \int_L \frac{dL_i}{dx} \frac{dL_j}{dx} dx \tag{25}$$

$$K_{12_{ij}} = -A_1 \int_L \frac{dL_i}{dx} \frac{d^2 H_j}{dx^2} dx \tag{26}$$

$$K_{13_{ij}} = B_0 \int_L \frac{dL_i}{dx}\frac{dL_j}{dx}\, dx \tag{27}$$

$$K_{21_{ij}} = -A_1 \int_L \frac{d^2H_i}{dx}\frac{dL_j}{dx}\, dx \tag{28}$$

$$K_{22_{ij}} = A_2 \int_L \frac{d\,2H_i}{dx}\frac{d^2H_j}{dx}\, dx \tag{29}$$

$$K_{23_{ij}} = -B_1 \int_L \frac{dL_i}{dx}\frac{d^2H_j}{dx}\, dx \tag{30}$$

$$K_{31_{ij}} = B_0 \int_L \frac{dL_i}{dx}\frac{dL_j}{dx}\, dx \tag{31}$$

$$K_{32_{ij}} = -B_1 \int_L \frac{dL_i}{dx}\frac{d^2H_j}{dx}\, dx \tag{32}$$

$$K_{33_{ij}} = D_0 \int_L \frac{dL_i}{dx}\frac{dL_j}{dx}\, dx + \hat{V}_0 \int_L L_i L_j\, dx \tag{33}$$

$$d_i = \begin{cases} t_{\frac{i+1}{2}},\ i \mod 2 = 0 \\ r_{\frac{i}{2}},\ i \mod 2 \neq 0 \end{cases} \tag{34}$$

$$q_i = \int L_i q(x)\, dx \tag{35}$$

$$F_{1_i} = L_i(x_b)N(x_b) - L_i(x_a)N(x_a) \tag{36}$$

$$F_{2_i} = H_i(x_b)M'(x_b) - H_i(x_a)M'(x_a) - H_i'(x_b)M(x_b) + H_i'(x_a)M(x_a) \tag{37}$$

$$F_{3_i} = L_i(x_b)\hat{M}(x_b) - L_i(x_a)\hat{M}(x_a) \tag{38}$$

In each component, the indices that accompany Lagrange polynomials are of order $2m-1$, whereas those tied to Hermite polynomials are of order $2m$. Therefore, the full system presented in eq. (24) is of order $6m-2$.

4 Results and discussion

Two numerical examples are presented to analyze the accuracy of the present theory in predicting the displacement responses of simply supported FG beams under uniform load $q(x) = q_0$. The results are compared to theoretical and experimental results from the literature.

4.1 Single layer beams

A single layer of Al/Al_2O_3 beam composed of aluminum (as metal) and alumina (as ceramic) is considered. The bottom surface is purely composed of metal ($E_b = 70GPa$) and the ceramic content gradually increases up to the top surface, which is purely ceramic ($E_t = 380GPa$). Poisson modulus $\nu = 0.3$ is considered for the entire beam. For analysis of the axial displacement, the following dimensionless form is used:

$$\bar{u}(0,z) = \frac{100E_b h^3}{q_0 L^4} u(0,z) \tag{39}$$

The power law index values $p = 0$ (pure metal beam), $p = 0.5$, $p = 1$ and $p = 10$ are analyzed and the beam is given very low span-to-depth ratio $L/h = 2$. In this analysis, a single 5 node element for Hermite's approximation and 11 node for Lagrange's approximation was used.

Fig. 1 shows excellent agreement between the present work results and those presented by the theoretical formulation of Thai and Vo's [4].

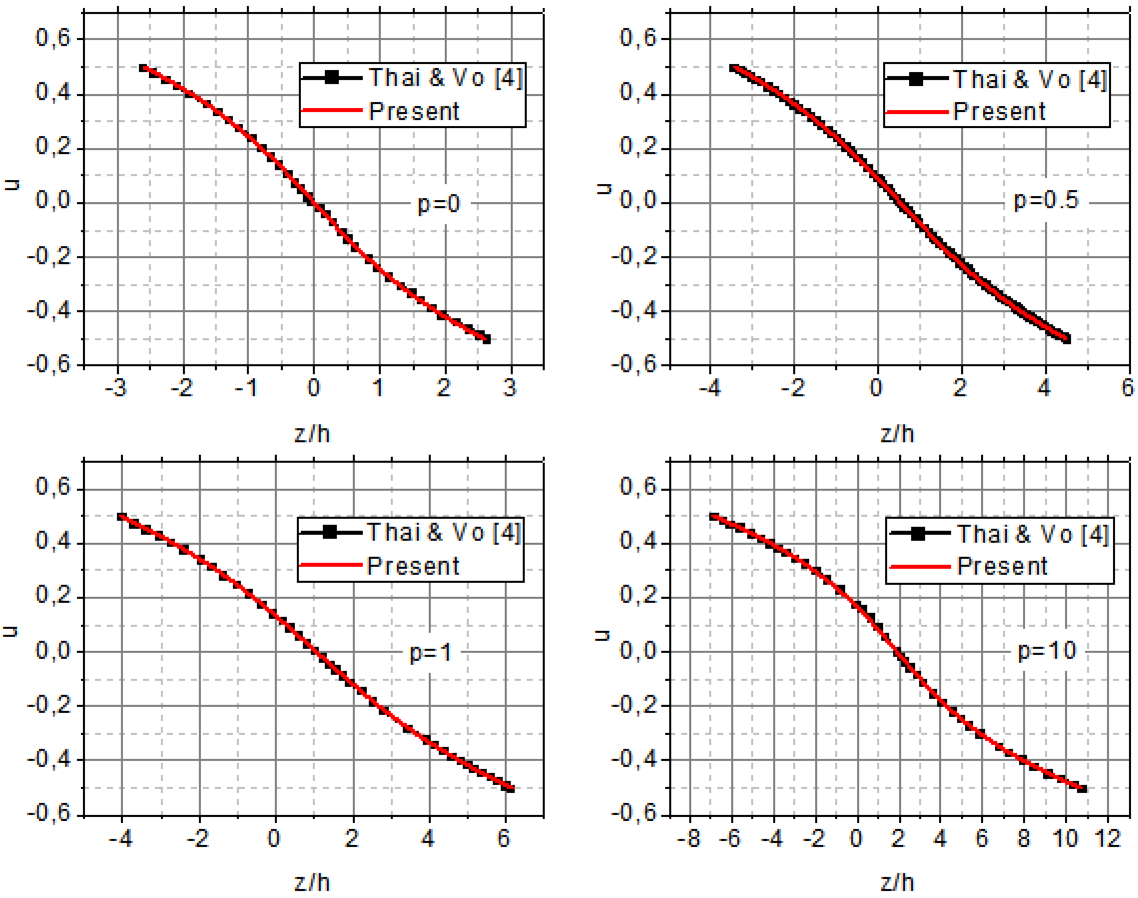

Figure 1. Variation of nondimensional axial displacement $\bar{u}(0, z)$ across the depth of FG beams under uniform load ($L/h = 2$).

4.2 Multilayered beams

Kapuria et al. [5] present various results from experimental tests with multilayered composite beams. Two Al/SiC beam models are considered: a three-layer and a five-layer beam, with each layer 3 mm and 2 mm thick respectively. Both beams have length 125 mm and width 15 mm and receive a transverse point load in the middle of the length. Each layer has constant material properties, but the metal-ceramic ratio linearly changes from bottom to topmost layers from 100-0 to 60-40. The effective Young modulus value for each layer is given by eq. (40) as a function of the metallic ($E_m = 67,0\ GPa$) and ceramic ($E_c = 302,0\ GPa$) properties, metal fraction per layer $V_m^{(k)}$, and an experimentally determined value $q = 91.6\ GPa$. The Poisson ratio is taken as $\nu = 0.3$, which is close enough to the average of the ratios of both materials.

$$E^{(k)} = \frac{V_m^{(k)} E_m \left(q + E_c\right) / \left(q + E_m\right) + \left(1 - V_m^{(k)}\right) E_c}{V_m^{(k)} \left(q + E_c\right) / \left(q + E_m\right) + \left(1 - V_m^{(k)}\right)} \tag{40}$$

The transverse displacement is observed in the point of application of the force for varying intensity. The authors also present a theoretical model to predict its behavior. Fig. 2 shows improved performance of the present theory in predicting the real values when the power law is set to $p = 4$.

5 Conclusions

The present theory has shown great performance in describing the behavior of very thick single-layer FG beams as well as multilayered beams with varying compositions with a very simple and computationally light

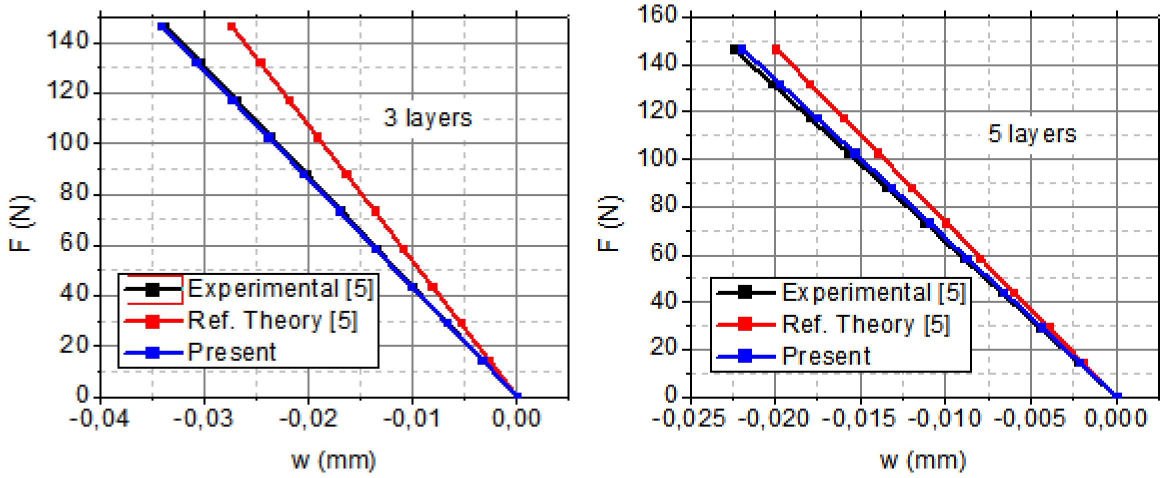

Figure 2. Experimental and theoretical values of transverse displacement for different applied point loads in multilayered composite beams.

approach. Furthermore, it is a very flexible theory that allows for the application of many different kinematics and power laws easily, so comparing its results to further examples in the literature is encouraged.

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