Analytical-numerical study of natural frequencies and mode shapes of interconnected acoustic cavities due to the influence of several significant parameters

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Abstract. Nuclear power plants, industrial plants, refineries, and other engineering installations, are made up of tubular circuits traversed by fluids, and the dynamic response of these systems excited by flows, are characterized by fluid-structure problems of great interest in engineering projects. This paper presents a comparative analytical-numerical study on the behavior of interconnected acoustic cavities, considering the influence of several significant parameters, such as the number of tubes, the boundary conditions and section changes of the circuit. As a response to the analysis, natural frequencies and mode shapes are obtained by comparing an analytical method and numerical modeling. The analytical formulation was developed from the classic idea of the Transfer Matrix Method (TMM), considering different boundary conditions at the ends of the circuit, such as the open-open, closed-closed and closed-open system. For all cases, the fundamental equation composed by the association of several interconnected elements is obtained. All analyzes were compared with numerical simulations obtained by the Finite Element Method (FEM), with the aid of the commercial software ANSYS. The cases treated in this paper include the evaluation of the effects of these parameters, oriented to applications in tubular circuits, present in industrial plants, analyzing the behavior of the acoustic pressures and flows in these cavities, through the study of natural frequencies and mode shapes in each case. There is a good agreement of the results obtained by the Finite Element Method in comparison with the analytical solutions obtained by the Transfer Matrix Method. Thus, the study of the behavior of acoustic cavities interconnected by different methods, in addition to enabling the validation of this simple analytical formulation, allows the preliminary treatment of the problem, for the fluid-structure approach, which is a more complex issue and dependent on these analyzes.

Keywords: Interconnected acoustic cavities, Transfer Matrix Method, Finite Element Method, Tubular circuits, Natural frequencies, Mode shapes.

1 Introduction

This work presents a comparative analytical and numerical study on the behavior of interconnected acoustic cavities, considering the influence of several significant parameters, such as the number of tubes, section changes and different boundary conditions. As a response to the analyses, the natural frequencies and mode shapes, or the pressure field, are obtained by comparing an analytical method and numerical modeling, using the Finite Element Method.

Initial studies are approached by Sommerfeld (1986), with the development of mathematical expressions from the theory of diffraction, evaluating the differential equation of the wave and acoustic modes. Havelock (1940) also studies the pressure of water waves on a fixed obstacle, using the theory of diffraction from the wave equation. Morse (1948) deals with wave propagation in tubes, evaluating different boundary conditions and constrictions in several associated tubes, presenting the mathematical formulation for the analysis of acoustic modes. Lamb (1975) is an important reference for hydrodynamic studies, where wave propagation is analyzed using wave theory, tube association, different boundary conditions and fluid-structure interaction.

Zienkiewicz and Bettes (1969) use the Eulerian formulation to calculate the pressure field, and adopt the wave theory to study the fluid-structure interaction. Daneshfaraz and Kaya (2008) adopt the Transfer Matrix Method in Ocean Engineering, investigating waves in open channels, and emphasize that the results are similar to those obtained with the Characteristics Method and the Finite Differences Method. Mendes, Souza and Pedroso (2012) develop an analytical-numerical study, evaluating the effects of a constriction on the frequencies of a system composed of a structure and acoustic cavity, using the Transfer Matrix Method and the Finite Element Method, obtaining good convergence of results.

In this work, the analytical formulation was developed from the classical idea of the Transfer Matrix Method (TMM), presented in the works of Gibert (1988) and Pedroso (1986, 1992). The system will consist of two connected tubes and considering the following boundary conditions: open-open, closed-closed and closed-open system. In all cases, the fundamental equation composed by the association of the interconnected elements is obtained, determining system frequencies and mode shapes. All analyzes were compared with numerical simulations obtained by the Finite Element Method (FEM), with the aid of commercial ANSYS software.

The cases dealt with in this article include the evaluation of the effects of these parameters, aimed at applications in tubular circuits, present in industrial plants, analyzing the behavior of acoustic pressures and flows in these cavities, through the study of natural frequencies and mode shapes in each case. There is a good agreement of the results obtained by the Finite Element Method in comparison with the analytical solutions obtained by the Transfer Matrix Method.

2 Mathematical formulation

The configuration presented in this work is composed by the junction of two cavities (two tubes) as indicated in Figure 1. The model is defined by two tubes with different lengths and sections, called L_1 and L_2, H_1 and H_2, respectively. Three situations will be evaluated for the boundary conditions at the ends, such as the open-open, closed-closed and closed-open system.

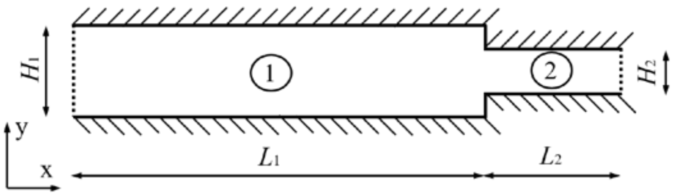

Figure 1. Schematic drawing for the configuration of tubes under study.

According to the mathematical formulation presented by Gibert (1988) and Pedroso (1986, 1992), using the solutions of the wave equation for a one-dimensional acoustic cavity, the Transfer Matrix is obtained, presented by solving the equations below:

$$p\left(\frac{\omega}{c}\right)^2+\frac{\partial^2 p}{\partial x^2}=0\rightarrow\begin{cases}p(x)=A\cos\frac{\omega x}{c}+B\,\text{sen}\frac{\omega x}{c}\\ q(x)=\frac{iS}{c}\left(-A\,\text{sen}\frac{\omega x}{c}+B\cos\frac{\omega x}{c}\right)\end{cases}\rightarrow\begin{bmatrix}p_S\\ q_S\end{bmatrix}=\underbrace{\begin{bmatrix}\cos\theta & \frac{c}{iS}\text{sen}\,\theta\\ -\frac{iS}{c}\text{sen}\,\theta & \cos\theta\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}p_E\\ q_E\end{bmatrix} \quad (1)$$

where p_E and p_S are the inlet and outlet pressures of the acoustic cavity; q_E and q_S, the acoustic flows; c, the propagation velocity of the wave; ω, the system frequencies; θ the compressibility parameter given by $\theta=\omega L/c$; S, the section area; and **A**, the system transfer matrix. It is important to point out that the matrix system presented in (1) represents the configuration for a one-dimensional cavity.

To associate two or more tubes, the following operation is necessary:

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \underbrace{\mathbf{A_2 A_1}}_{\mathbf{A}} \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \rightarrow \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \frac{c}{i H_2}\operatorname{sen}\theta_2 \\ -\frac{i H_2}{c}\operatorname{sen}\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \frac{c}{i H_1}\operatorname{sen}\theta_1 \\ -\frac{i H_1}{c}\operatorname{sen}\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \tag{2}$$

When performing the multiplication of matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ you have the following matrix system:

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \frac{H_1}{H_2}\operatorname{sen}\theta_1 \operatorname{sen}\theta_2 & -\frac{i c}{H_1}\operatorname{sen}\theta_1 \cos\theta_2 - \frac{i c}{H_2}\cos\theta_1 \operatorname{sen}\theta_2 \\ -\frac{i H_1}{c}\operatorname{sen}\theta_1 \cos\theta_2 - \frac{i H_2}{c}\cos\theta_1 \operatorname{sen}\theta_2 & \cos\theta_1 \cos\theta_2 - \frac{H_2}{H_1}\operatorname{sen}\theta_1 \operatorname{sen}\theta_2 \end{bmatrix} \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \tag{3}$$

In which sub-indices 1 and 2 of transfer matrix **A** are associated with tubes 1 and 2, respectively, indicated in Figure 1. The sub-indices 0 and 2 of the acoustic pressures and flows indicate the beginning (left side) and end (side right) of the model. When applying the boundary conditions for the open-open (AA), closed-closed (FF) and closed-open (FA) system in equation (3), the following relations are respectively obtained:

$$\text{AA} \rightarrow \begin{bmatrix} 0 \\ q_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} 0 \\ q_0 \end{bmatrix} \qquad \text{FF} \rightarrow \begin{bmatrix} p_2 \\ 0 \end{bmatrix} = \mathbf{A} \begin{bmatrix} p_0 \\ 0 \end{bmatrix} \qquad \text{FA} \rightarrow \begin{bmatrix} 0 \\ q_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} p_0 \\ 0 \end{bmatrix} \tag{4}$$

The natural frequencies of the open-open system are obtained through the first relation in (4):

$$-\frac{i c}{H_1}\operatorname{sen}\frac{\omega L_1}{c}\cos\frac{\omega L_2}{c} - \frac{i c}{H_2}\cos\frac{\omega L_1}{c}\operatorname{sen}\frac{\omega L_2}{c} = 0 \tag{5}$$

Similarly, for the closed-closed and closed-open systems, the frequencies are obtained through the second and third relations in (4), respectively:

$$-\frac{i H_1}{c}\operatorname{sen}\frac{\omega L_1}{c}\cos\frac{\omega L_2}{c} - \frac{i H_2}{c}\cos\frac{\omega L_1}{c}\operatorname{sen}\frac{\omega L_2}{c} = 0 \qquad \cos\frac{\omega L_1}{c}\cos\frac{\omega L_2}{c} - \frac{H_1}{H_2}\operatorname{sen}\frac{\omega L_1}{c}\operatorname{sen}\frac{\omega L_2}{c} = 0 \tag{6}$$

The final frequencies are the result of the composition in both directions, obtained as follows:

$$f_{ij} = \sqrt{f_i^{\,2} + f_j^{\,2}},\ i = 0,1,2...,\ j = 0, 1, 2,... \tag{7}$$

To obtain the modal deformations of the two-tube system, the pressure fields for each tube are calculated and then graphs are drawn with the adopted numerical values. For tubes 1 and 2, comprised in the sections $0 \le x_1 \le L_1$ and $0 \le x_2 \le L_2$, have:

$$\begin{bmatrix} p_2(x) \\ q_2(x) \end{bmatrix} = \begin{bmatrix} \cos\theta_{x_1} \cos\theta_{x_2} - \frac{H_1}{H_2}\operatorname{sen}\theta_{x_1} \operatorname{sen}\theta_{x_2} & -\frac{i c}{H_1}\operatorname{sen}\theta_{x_1} \cos\theta_{x_2} - \frac{i c}{H_2}\cos\theta_{x_1} \operatorname{sen}\theta_{x_2} \\ -\frac{i H_1}{c}\operatorname{sen}\theta_{x_1} \cos\theta_{x_2} - \frac{i H_2}{c}\cos\theta_{x_1} \operatorname{sen}\theta_{x_2} & \cos\theta_{x_1} \cos\theta_{x_2} - \frac{H_2}{H_1}\operatorname{sen}\theta_{x_1} \operatorname{sen}\theta_{x_2} \end{bmatrix} \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \tag{8}$$

For the open-open system, the pressure p_0 is null, and in the closed-closed and closed-open systems, the acoustic flow q_0 is equal to zero in the expression in (8).

3 Results

As shown in Figure 1, the model analyzed in this work is composed of two connected tubes, one with a longer length L_1, height of section H_1, and another tube of shorter length L_2 and height of section H_2. Three different section variations will be considered in the connection between these two tubes, defined by the relation $\alpha = H_2/H_1$, with numerical values of α equal to 0,1, 0,5 and 1,0. Three different boundary conditions at the ends will also be evaluated, being the open-open, closed-closed and closed-open system. The numerical data for the physical and geometric parameters used in the problem are: L_1 = 4,5 m; L_2 = 1,5 m; H_1 = 1 m; H_2 = 0,1; 0,5 e 1,0 m; c = 1500 m/s; ρ_f = 1000 kg/m³. Where ρ_f is the density of water, which was the considered fluid. For numerical modeling, mesh was discretized by flat quadrangular fluid elements (FLUID29), from the ANSYS code library. The number of finite elements of the meshes for each model is: 1860 for α = 0,1; 2100 for α = 0,5; and 2400 for α = 1,0.

Table 1 shows the comparison of frequencies by analytical and numerical calculations for the open-open system for the first two modes obtained, in addition to the value of the compressibility parameter. Figure 2 compares the first two modes obtained by TMM and FEM for the open-open system and considering the three geometric relationships (α=0,1; 0,5 e 1,0). The pressure fields of the longitudinal profile $\varphi(x)$ are normalized.

Table 1. Comparison of frequencies obtained by analytical and numerical calculations for the open-open system.

f_{ij} (Hz)	α = 0,1			α = 0,5			α = 1,0		
Mode (ij)	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$
1 (1,0)	91,46	91,23	0,36	111,67	111,09	0,45	125,00	125,00	0,50
2 (2,0)	250,00	250,03	1,00	250,00	250,03	1,00	250,00	250,03	1,00

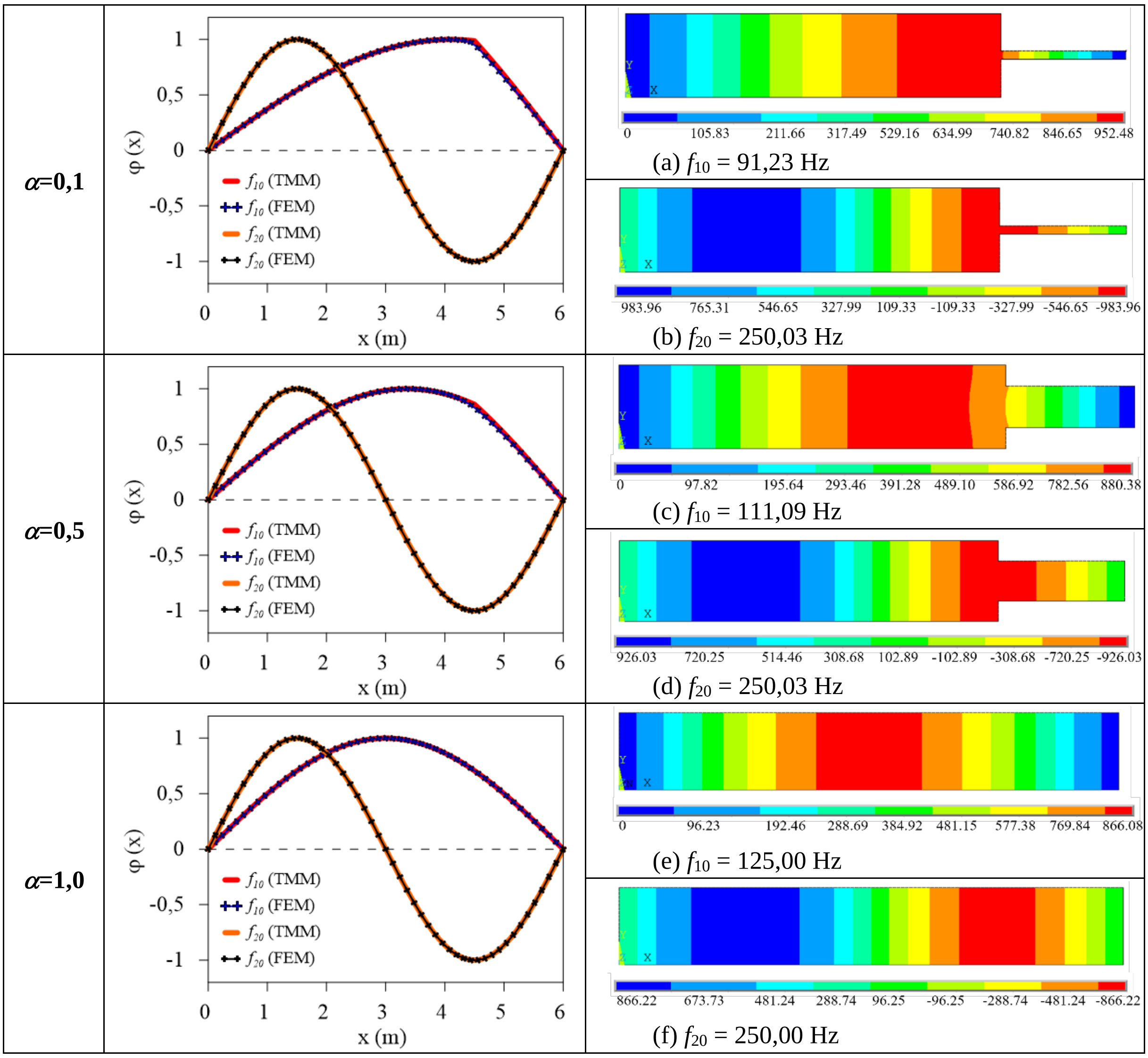

Figure 2. Comparison of the first two modes obtained by TMM and FEM for the open-open system and considering the three geometric relationships (α = 0,1; 0,5 e 1,0).

Analogously to the previous case, Table 2 shows the comparison of frequencies by analytical and numerical calculations of the closed-closed system for the first two modes obtained, in addition to the values of the compressibility parameter $\theta = \omega L/c$. Figure 3 presents a comparison of the first two modes obtained by the TMM and FEM of the closed-closed system, considering the three geometric relationships, and the pressure fields $\varphi(x)$ are normalized. The unit of pressures indicated in the graph legends is Pa.

Table 2. Comparison of frequencies by analytical and numerical calculations for the closed-closed system.

f_{ij} (Hz)	$\alpha = 0{,}1$			$\alpha = 0{,}5$			$\alpha = 1{,}0$		
Mode (ij)	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$
1 (1,0)	158,54	158,11	0,63	138,33	137,58	0,55	125,00	125,00	0,50
2 (2,0)	250,00	243,02	0,97	250,00	246,06	0,98	250,00	250,03	1,00

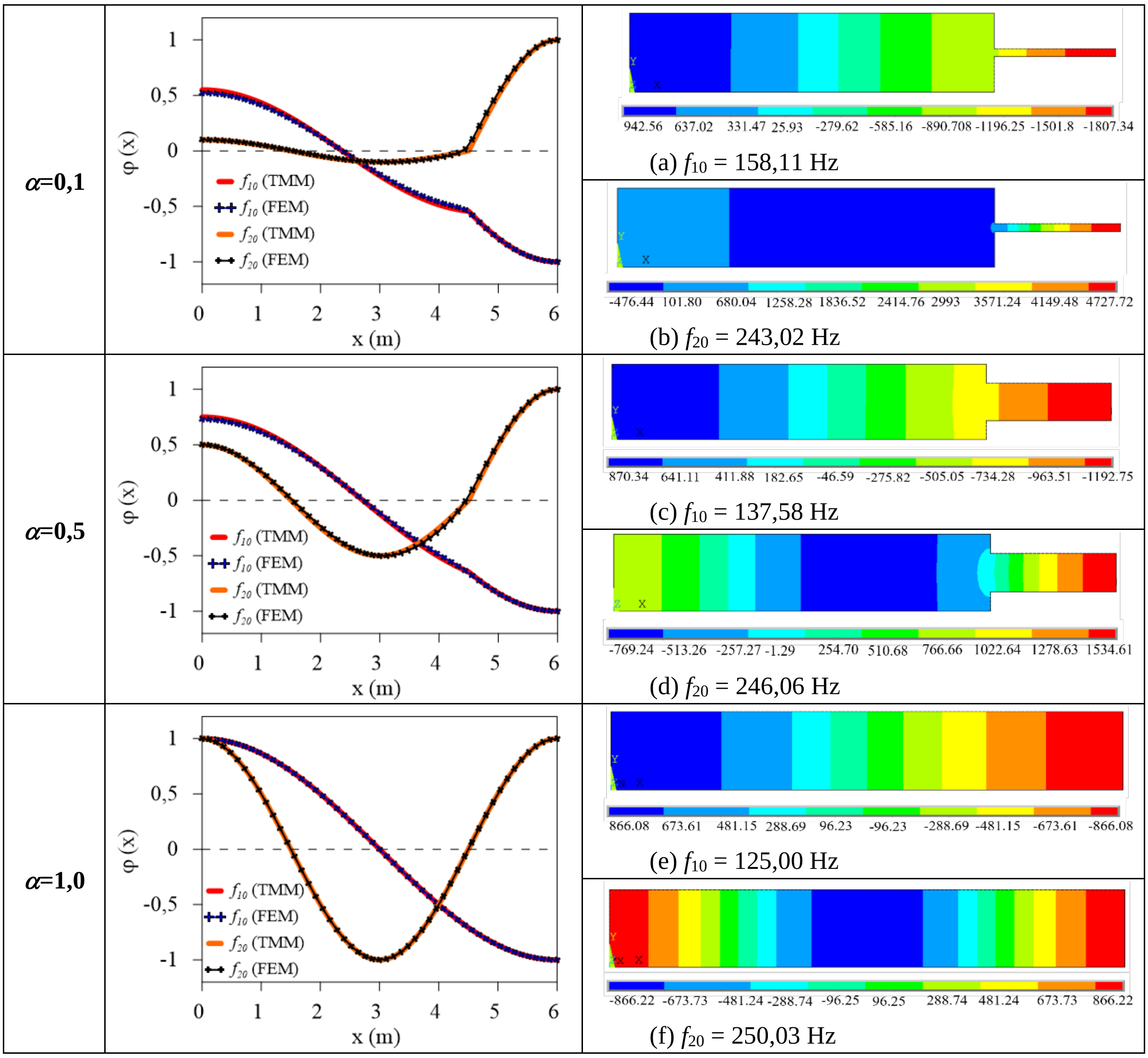

Figure 3. Comparison of the first two modes obtained by TMM and FEM for the closed-closed system and considering the three geometric relationships (α = 0,1; 0,5 e 1,0).

Finally, Table 3 shows the comparison of frequencies by analytical and numerical calculations of the closed-open system for the first two modes obtained and the values of θ. Similar to the previous cases, Figure 4 compares the first two modes for the closed-open system.

Table 3. Comparison of frequencies by analytical and numerical calculations for the closed-open system.

f_{ij} (Hz)	$\alpha = 0,1$			$\alpha = 0,5$			$\alpha = 1,0$		
Mode (ij)	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$	**TMM**	**FEM**	$\omega L/c$
1 (1,0)	27,56	27,11	0,11	51,82	51,21	0,21	62,50	62,50	0,25
2 (2,0)	169,60	169,55	0,68	179,10	178,90	0,72	187,50	187,51	0,75

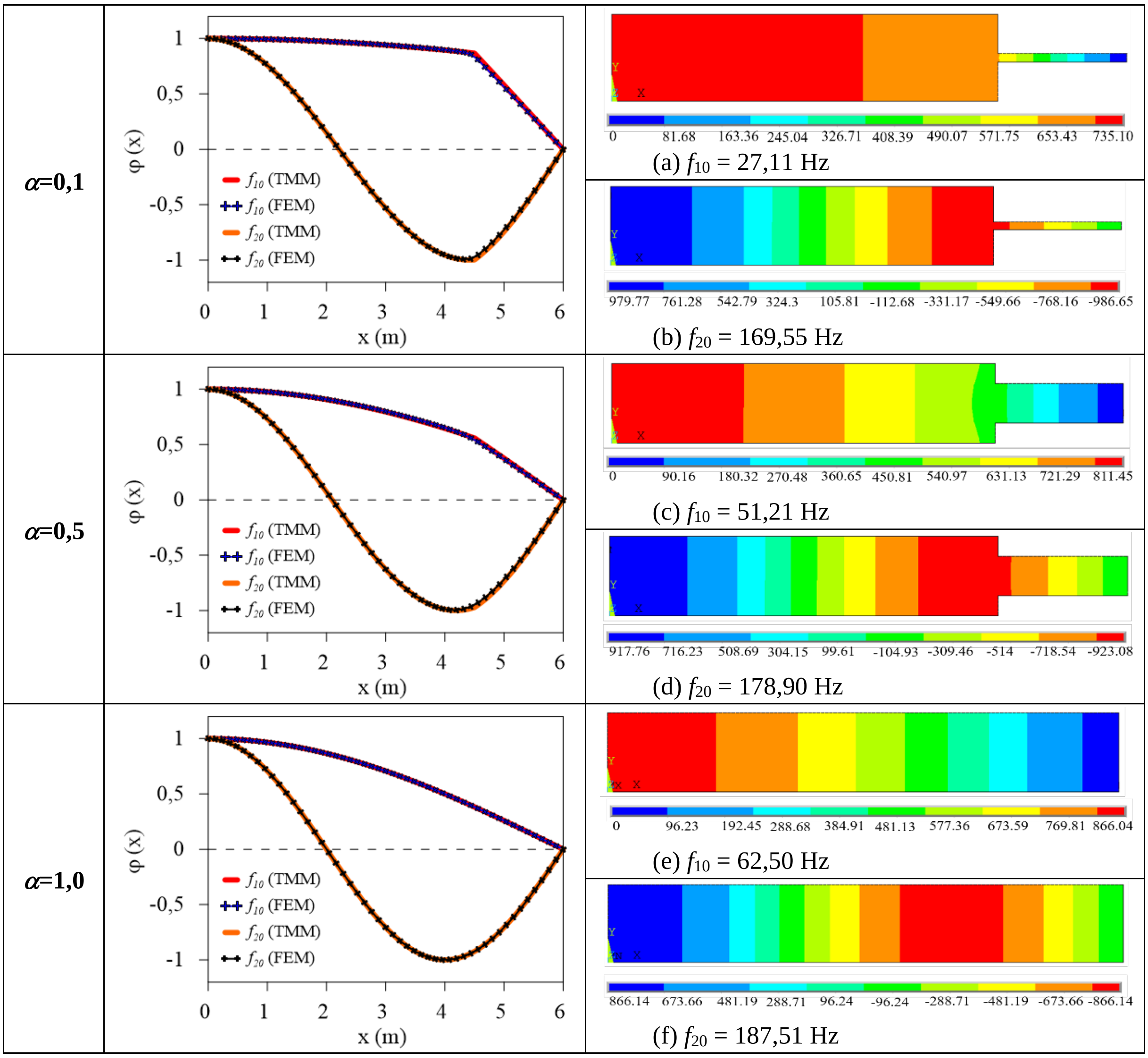

Figure 4. Comparison of the first two modes obtained by TMM and FEM for the closed-open system and considering the three geometric relationships (α = 0,1; 0,5 e 1,0).

In the pressure profiles shown in Figures 2, 3 and 4, the red- and orange-colored lines refer to the first and second modes of analytical calculation, respectively. The lines with symbols in blue and black, refer to the first and second modes of the numerical model.

It is important to point out that all the obtained modes refer to an uncoupled acoustic cavity, being a necessary preliminary step for the solution of fluid-structure interaction problems. Considering the results presented in terms of frequencies, mode shapes (pressure profile and in two-dimensional format), some observations can be made.

There is a good convergence of results, with values of frequencies and mode shapes for plane wave modes, comparing the results obtained between the analytical and numerical methods, observing small divergences, which can be explained by the two-dimensional effect detected in the model numerical, not being foreseen in the solution of the analytical calculation, which is developed in a one-dimensional way.

The two-dimensional modal deformed in Figures 2, 3 and 4 show the effects of plane waves (straight lanes) and in the constriction zones, the distorted lanes are identified in the first modes with geometric relations α = 0,1 and 0,5, and with $\theta < 1$. The modes with the ratio α = 0,1 and 0,5 are characterized as a Helmholtz resonator, in which the larger acoustic cavity is compressible ($\omega L_1/c \geq 1$), and the smaller cavity behaves like an incompressible column ($\omega L_2/c << 1$), due to the increase in inertial effects from constriction (MENDES, SOUZA, PEDROSO, 2012).

For all analyzed boundary conditions, the first modes of the system have the value $\omega L/c < 1$, which indicates that the fluid is incompressible in this mode. In the open-open system, the second mode indicates the compressibility effect throughout the system for all analyzed geometric relationships (α = 0,1, 0,5 and 1,0). For the closed-closed system, the second mode has $\omega L/c < 1$ with values close to 1 for the relations α = 0,1 and 0,5; and for α = 0,1, the entire length has fluid compressibility. Finally, for the closed-open system, the second mode still presents the incompressibility effect for all section relations (α = 0,1, 0,5 and 1,0). It is important to emphasize that the modal strains, and the pressure profiles for α = 1, serve as a reference throughout the system.

4 Conclusions

Therefore, it was possible to observe a good agreement of the results obtained by the Finite Element Method in comparison with the analytical solutions obtained by the Transfer Matrix Method, to carry out the study of frequencies and mode shapes of acoustic cavities. Numerical modeling using ANSYS software allowed the satisfactory analysis of systems composed of interconnected tubes with different boundary conditions and with the presence of a singularity, also validating the mathematical formulation arising from the Transfer Matrix Method.

Analyzes for three-dimensional cases, observing the effects of singularity are ongoing, with a greater discretization of finite elements and promoting the validation of the analytical method simultaneously. The effects of coupling a structure to the acoustic cavity will be made with the aim of evaluating the behavior of the fluid-structure interaction, since the results of this work were restricted to the study of uncoupled acoustic cavities, which refers to a preliminary step of an analysis more robust.

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